Approximation Algorithms on Consistent Dynamic Map Labeling

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Abstract

We consider the dynamic map labeling problem: given a set of rectangular labels on the map, the goal is to appropriately select visible ranges for all the labels such that no two consistent labels overlap at every scale and the sum of total visible ranges is maximized. We propose approximation algorithms for several variants of this problem. For the simple ARO problem, we provide a $3c \log n$*-approximation algorithm for the unit-width rectangular labels if there is a* c*-approximation algorithm for unit-width label placement problem in the plane; and a randomized polynomial-time* $O(\log n \log\log n)$*-approximation algorithm for arbitrary rectangular labels. For the general ARO problem, we prove that it is NP-complete even for congruent square labels with equal selectable scale range. Moreover, we contribute* 9*-approximation algorithms for both arbitrary square labels and unit-width rectangular labels, and a* 5*-approximation algorithm for congruent square labels.*

1 Introduction

Online maps have been widely used in recent years, especially on portable devices. Such geographical visualization systems provide user-interactive operations such as continuous zooming. Thus, the interfaces provide to a new model in map labeling problems. Been *et al.* [3] initiated an interesting consistent dynamic map labeling problem whose objective is to maximize the sum of total visible ranges, each of which corresponds to the consistent interval of scales at which the label is visible; in other words, the aim is to maximize the number of consistent labels at every scale. In contrast with the static map labeling problem, the dynamic map labeling problem can be considered a traditional map labeling by incorporating *scale* as an additional dimension. During zooming in and out operations on the map, the labeling is regarded as a function of the zoom scale and the map area.

Several desiderata [2] are provided by Been *et al.* to define this problem. We adopt all desiderata to our problem. Labels are selected to display at each scale and labels should be visible continuously without intersection. Moreover, labels could change their sizes as a function during monotonic zooming at some specific scale. In order to maintain the consistence of notations, we also follow the definition by Been *et al.*'s work [2, 3], and define *active* (visible) range to be a continuous interval lying between the maximum scale and the minimum scale where labels could be exactly displayed. Our goal is to maximize the number of consistent labels at every scale, and thus we maximize the sum of total *active* ranges to achieve this goal. The detailed problem definition is described in the following.

Problem Definition. Given a set of n extrusions $\mathcal{E}$, and each extrusion $E \in \mathcal{E}$ with an open interval $(s_E, S_E) \subseteq (0, S_{\max})$, which we call *selectable range*, among the scale s. Note that $S_{\max}$ is an universal maximum scale for all extrusions. The goal is to compute a suitable active range $(a_E, A_E) \subseteq (s_E, S_E)$, for each E (see Figure 1). Actually, when an extrusion E intersects a horizontal plane at s, it forms a cross-section. We say that this cross-section is a label L. Here, we consider invariant point placements with axis-aligned rectangular labels, in which labels always map to

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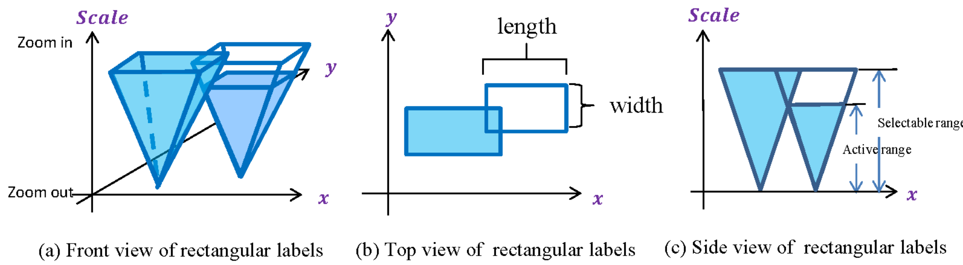

Figure 1: Two unit-width rectangular labels with selectable ranges and active ranges

the same location, so labels do not slide and rotate.

According to [3], we consider two models in this problem—*general* and *simple*. The general active range optimization (ARO) problem is to choose the active ranges (a_E, A_E) so as to maximize the sum of total active ranges. For the simple ARO problem, it is a variant in which the active range are restricted so that a label is never deselected when zooming in. That is, the active range of a selected extrusion $E \in \mathcal{E}$ is $(0, A_E) \subseteq (0, S_{\max})$.

Moreover, we consider two types of dilation cases in this paper—*proportional dilation* and *constant dilation*. We say that labels have *proportional dilation* if their sizes could change with scale proportionally. In contrast, if the sizes of labels are fixed at every scale, we say that labels have *constant dilation*. For the simple ARO problem with proportional dilation, because we consider rectangular labels, the shapes of extrusions are in fact rectangular pyramids. Let $\pi(s)$ be the hyperplane at scale s. Also let the width and length of the rectangular label $E \cap \pi(s)$ of an (pyramid) extrusion E at scale s be functions $w_E(s) = \frac{s}{S_{\max}} w_E$ and $l_E(s) = \frac{s}{S_{\max}} l_E$, respectively, where w_E and l_E are the width and length of E, respectively, at scale $S_{\max}$. Then, for the general ARO problem with constant dilation, the shapes of extrusions are rectangular prisms. Let width and length be $w_E(s) = w_E$ and $l_E(s) = l_E$, respectively, where $s \in (s_E, S_E)$, because the sizes of all labels are fixed at every scale. In addition, we say that E and $E' \in \mathcal{E}$ intersect at scale s, if and only if $s \subset (s_E, S_E) \cap (s_{E'}, S_{E'})$, $|x_E - x_{E'}| \leq \frac{1}{2}(l_E(s) + l_{E'}(s))$ and $|y_E - y_{E'}| \leq \frac{1}{2}(w_E(s) + w_{E'}(s))$ are satisfied, where (x_E, y_E) is the central point of a pyramid extrusion E.

Accordingly, our goal is to compute a set of pairwise disjoint truncated extrusions $\mathcal{T} = \{T_E : (a_E, A_E) \mid E \in \mathcal{E}\}$, where T_E is the truncated extrusion of E, so as to maximize the sum of total active range height $\mathcal{H}(\mathcal{T}) = \sum_{E \in \mathcal{E}} |A_E - a_E|$.

Previous Work. Map labeling is an important application [9] and a popular research topic during the past three decades [16]. The labeling problems which were proposed before dynamic labeling problems are mostly static labeling problems [3]. There are various settings for static labeling problems [10] and they have been shown to be NP-hard [11]. One of major topics and its typical goal is to select and place labels without intersection and its objective is to maximize the total number of labels. Agarwal *et al.* presented a PTAS for the unit-width rectangular label placement problem and a $\log n$-approximation algorithm for the arbitrary rectangle case [1]; Berman *et al.* [5] improved the latter result and obtained a $\lceil \log_k n \rceil$-factor algorithm for any integer constant $k \geq 2$. Then, Chan [7, 8] improved the running time of these algorithms. Chalermsook and Chuzhoy [6] showed an $O(\log^{d-2} n \log\log n)$-approximation algorithm for the maximum independent set of rectangles where rectangles are d-dimensional hyperrectangles.

In addition, there have been a few studies on dynamic labeling. Poon and Shin [15] developed an algorithm for labeling points that precomputes a hierarchical data structure for a number of specific scales. For dynamic map labeling problems, Been *et al.* [2] proposed several consistency desiderata and presented several algorithms for one-dimensional (1D) and two-dimensional (2D) labeling problems [3]. Note that labels in 1D problems are open intervals; labels in 2D problems are open rectangles. They showed NP-completeness of the general 1D ARO problem with "constant" dilation with square extrusions of distinct sizes, and the simple 2D ARO problem with "proportional" dilation with congruent square cone extrusions. They focused on dynamic label selection, i.e., assuming a 1-position model for label

Problem	Extrusion Shape	Approximation Ratio	Time Complexity
Simple ARO	unit-width rectangular pyramids	$6 \log n$	$O(n \log^2 n)$
		$3\frac{k+1}{k} \log n$	$O(n \log^2 n + n \bigtriangleup^{k-1} \log n)$
	rectangular pyramids	$O(\log n \log \log n)$	Polynomial
General ARO	unit-width rectangular prisms	9	$O(n \log^3 n)$
	arbitrary square prisms	9	$O(n \log^3 n)$
	congruent square prisms	5	$O(n \log^3 n)$

Table 1: Summary of our approximation results

placement. Moreover, Gemsa *et al.* [12] provided a FPTAS for general sliding models of the 1D dynamic map labeling problem. Since dynamic map labeling is still a new research topic, there are still many unsolved problems. Yap [17] summarized some open problems.

Our contribution. In this paper, we consider simple ARO with proportional dilation and general ARO with constant dilation. We design a list of approximation algorithms as shown in Table 1. Moreover, we also prove that the general ARO problem with constant dilation is NP-complete.

2 Approximation for the simple ARO problem

In this section, we investigate the simple ARO problem with proportional dilation and present two approximation algorithms for a given set $\mathcal{E}$ of axis-aligned rectangular pyramids, where the intersection of a pyramid E with the horizontal plane at scale s is a rectangular label whose width and length are $\frac{s}{S_{max}} w_E$ and $\frac{s}{S_{max}} l_E$, respectively. First, we explore the simple ARO problem for an input set of unit-width rectangular pyramids, in which the rectangular label of a pyramid at scale $S_{\max}$ is associated with a given uniform width and an arbitrary length. In particular, we propose a $3c \log n$-approximation algorithm for this problem, where c is an approximation factor for the unit-width rectangular label placement problem in the plane. The best known-to-date approximation ratio for this two-dimensional label placement problem is $c = \frac{k+1}{k}$, derived by Agarwal *et al.* [1] and Chan [8], for any integer $k \geq 1$. Subsequently, we extend the technique to the arbitrary rectangular pyramid case and obtain an expected $O(\log n \log \log n)$-approximation algorithm.

2.1 Approximation for unit-width rectangular pyramids

Given a set $\mathcal{E}$ of n unit-width rectangular pyramids for the simple ARO problem with proportional dilation, where the uniform label width of each pyramid at scale s is $\frac{s}{S_{max}} w$, the objective is to select a set of truncated pyramids such that they are pairwise disjoint and the total sum of their active range height is maximized. Note that the maximum unit-width label placement problem at scale s can be approximated well by using the famous shifting technique [13]. However, the major challenge is that a feasible label placement at each scale cannot be merged into a feasible solution for the ARO problem; that is, an integrated solution may cause inconsistent active range for a pyramid, even if an optimal label placement can be derived at each scale s.

The rationale behind the proposed approach is described as follows. We divide the scale into $(\log n + 1)$ heights for $(\log n + 1)$ restricted simple ARO problems such that in each of the problems, every rectangular pyramid has an upper bound on the selectable range that cannot exceed $s_j = S_{\max}/2^{\log n - j + 1}$, $1 \leq j \leq (\log n + 1)$, where $s_0 = 0$ and the $(\log n + 1)$'th scale $s_{\log n + 1}$ is in fact the universal maximum scale $S_{\max}$. Then, for each restricted simple ARO problem, we devise a good approximation solution $\mathcal{S}$ for the unit-width rectangular label placement problem in the hyperplane at scale s_j, and select the whole rectangular pyramids whose labels at scale s_j are selected in $\mathcal{S}$. That is, we take the complete selectable ranges $(0, s_j)$ of those pyramids in $\mathcal{S}$ as their active ranges. Finally, we choose the largest approximation solution among all the $(\log n + 1)$ restricted ARO problems and analyze its ratio.

First, we recall the unit-width rectangular label placement problem in the plane. Agarwal *et al.* [1] presented a $\frac{k+1}{k}$-approximation algorithm based on the shifting technique [13]. For ease of exposition, we refer to Agarwal *et al.*'s method and use a simple 2-approximation algorithm to derive a label

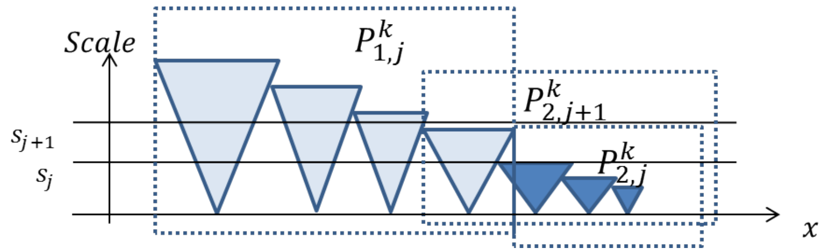

Figure 2: Side view of an illustration of $P^k_{1,j}$, $P^k_{2,j}$ and $P^k_{2,j+1}$

placement solution M_j in the hyperplane at scale s_j. We draw a set of horizontal lines from top to bottom of y-axis using an incremental approach, i.e., from $y_{\max} = \max\{y_E\}$ to $y_{min} = \min\{y_E\}$ of y-axis. The separation between two horizontal lines is larger than the uniform width, i.e., $\frac{s_j}{S_{max}}w$, and the lines that do not intersect any labels, if any, are skipped. For each line ℓ^j_k at scale s_j, where $1 \le k \le m_j \le n$, let a subset of labels intersected by line ℓ^j_k be denoted by R^j_k. The lines are drawn such that the next two properties hold: every line ℓ^j_k intersects at least one unit-width rectangular label, i.e., $R^j_k \neq \emptyset$, and each label is intersected by exactly one line. Hence, we have $\sum_{k=1}^{m_j} |R^j_k| = n$, where $|R^j_k|$ is the cardinality of R^j_k, i.e., the number of labels that are intersected by ℓ^j_k. Moreover, for every line ℓ^j_k, $R^j_k \cap R^j_i = \emptyset$, when $k+1 < i \le m_j$ and $1 \le i < k-1$. Subsequently, we apply a greedy algorithm to compute a *one-dimensional* maximum independent set, denoted by M^j_k, for each subset R^j_k of labels that are intersected by line ℓ^k_j. The greedy strategy, which takes $O(|R^j_k| \log |R^j_k|)$ time, proceeds as follows: Sort the right boundaries of all labels in R^j_k by their x-axis and scan the labels from left to right. Select the label whose right boundary is the smallest, say L, into the independent set, M^j_k, and remove labels that overlap L. Repeat the argument until each label is scanned. The correctness of this simple greedy algorithm is straightforward. Then, consider two sets $M_{odd} = \{M^j_1, M^j_3, ..., M^j_{2\lceil m_j/2\rceil-1}\}$ and $M_{even} = \{M^j_2, M^j_4, ..., M^j_{2\lceil m_j/2\rceil}\}$; clearly, both of them are independent sets, i.e., feasible label placement at scale s_j. Let the larger one of M_{odd} and M_{even} be M_j, which implies $|M_j| \ge \frac{1}{2}(|M_{odd}| + |M_{even}|)$. Thus, a 2-approximation algorithm follows for the unit-width rectangular label placement problem at scale s_j.

In the restricted ARO problem for an input set of unit-width rectangular pyramids whose selectable range cannot exceed scale s_j, we select the whole pyramid E whose label at scale s_j is in M_j; that is, we set the active range of E as $A_E = s_j$, for every $E \in M_j$, and the solution at scale s_j, denoted by $\mathcal{S}_j = \{T_E : (0, A_E) \mid E \in M_j\}$, has the sum of active ranges $\mathcal{H}(\mathcal{S}_j) = \sum_{E \in M_j} A_E$. As mentioned earlier, we select the maximum among all the $(\log n + 1)$ approximation solutions at scale s_j, $1 \le j \le \log n + 1$, denoted by $\mathcal{S}$. That is $\mathcal{H}(\mathcal{S}) = \max_j\{\mathcal{H}(\mathcal{S}_j)\}$. Note that the running time of the overall algorithm is $O(n \log^2 n)$, because there are $(\log n + 1)$ restricted ARO problems.

In the following theorem, we analyze the approximation ratio of the proposed algorithm.

Theorem 1. *Given a set of n unit-width rectangular pyramids in the simple ARO problem, there exists a $6\log n$-approximation algorithm, which takes $O(n \log^2 n)$ time, for this problem.*

Proof. Given a set $\mathcal{E}$ of n unit-width rectangular pyramids, let $\mathcal{S}^* = \{T^*_E : (0, A^*_E) \mid E \in \mathcal{E}, A^*_E > 0\}$ be the optimum solution for the problem and $|\mathcal{S}^*|$ be its cardinality. The sum of active range height of $\mathcal{S}^*$ is $\mathcal{H}(\mathcal{S}^*) = \sum_{E \in \mathcal{S}^*} A^*_E$. We consider the intersection of those pyramids in $\mathcal{S}^*$ with the hyperplane $\pi(s_j)$ at scale $s_j = S_{\max}/2^{\log n - j+1}$, denoted by $\mathcal{S}^*_j$, $1 \le j \le \log n + 1$. We define two subsets $P^k_{1,j}$ and $P^k_{2,j}$ recursively as follows.

$$P^k_{1,j} = \begin{cases} \{T^*_E : (0, A^*_E) \mid A^*_E \ge s_j\}, \text{ if } k = 1; \\ \{T^*_E : (0, A^*_E) \mid A^*_E > s_j \text{ and} \\ \qquad T^*_E \in \mathcal{S}^* \setminus \{\bigcup_{k'=1}^{k-1} P^{k'}_{2,j+1}\}\}, \text{ if } k > 1; \end{cases}$$

$$P^k_{2,j} = \begin{cases} \{T^*_E : (0, A^*_E) \mid A^*_E < s_j\}, \text{ if } k = 1; \\ \{T^*_E : (0, A^*_E) \mid A^*_E \le s_j \text{ and} \\ \qquad T^*_E \in \mathcal{S}^* \setminus \{\bigcup_{k'=1}^{k-1} P^{k'}_{2,j+1}\}\}, \text{ if } k > 1; \end{cases}$$

Figure 2 illustrates the definition of $P^k_{1,j}$ and $P^k_{2,j}$. Initially, let $k = 1$, and we locate the scale

s_j such that

$$|P^1_{1,j}| \geq |\mathcal{S}^*|/2, \quad |P^1_{2,j}| \leq |\mathcal{S}^*|/2, \text{ and}$$
$$|P^1_{1,j+1}| < |\mathcal{S}^*|/2, \quad |P^1_{2,j+1}| > |\mathcal{S}^*|/2.$$

As mentioned above, because M_j is a 2-approximation solution at scale s_j, we have $2|M_j| \geq |\mathcal{S}^*_j|$, which implies

$$2|\mathcal{S}_j| \geq |\mathcal{S}^* \setminus P^1_{2,j}| = |P^1_{1,j}| \geq |P^1_{2,j}|, \text{ and}$$

$$\begin{aligned} 2\mathcal{H}(\mathcal{S}_j) = 2|\mathcal{S}_j| \times s_j \geq |P^1_{1,j}| \times s_j \\ \geq |P^1_{2,j}| \times s_j \geq \mathcal{H}(P^1_{2,j}). \end{aligned} \quad (1)$$

Because $P^1_{2,j+1} \setminus P^1_{2,j} \subseteq P^1_{1,j}$ and $2s_j = s_{j+1}$, we have

$$\begin{aligned} 2\mathcal{H}(\mathcal{S}_j) \geq |P^1_{1,j}| \times s_j \geq |P^1_{2,j+1} \setminus P^1_{2,j}| \times \frac{s_{j+1}}{2} \\ \geq \frac{1}{2}\mathcal{H}(P^1_{2,j+1} \setminus P^1_{2,j}). \end{aligned} \quad (2)$$

Based on the equations (1) and (2), clearly, $6\mathcal{H}(\mathcal{S}_j) \geq \mathcal{H}(P^1_{2,j+1})$. Then, we remove the rectangular pyramids in $P^1_{2,j+1}$ from $\mathcal{S}^*$. Since $|P^1_{2,j+1}| > |\mathcal{S}^*|/2$, more than half of pyramids in $\mathcal{S}^*$ are removed at this step. Now we set the index j for this first round to be j_1. Next, for the remaining active ranges in $\mathcal{S}^*$, we proceed to locate another scale s_j in the same fashion. Then we consider $P^2_{1,j}$ and $P^2_{2,j}$ for the new scale s_j. Similarly we remove $P^2_{2,j+1}$ from $\mathcal{S}^*$, and set this index j to be j_2. We repeat the above step until we locate the scale $s_j = S_{\max}$ satisfying the property or all the rectangular pyramids in $\mathcal{S}^*$ are removed. Suppose that the number of rounds considered in the above process in k, and we set last scale considered to be s_{j_k}. Note that for each of the above steps, more than half of remaining rectangular pyramids in $\mathcal{S}^*$ are removed. Thus, the above step repeats at most $\log n$ times, i.e., $k \leq \log n$. According to the above reasoning,

$$\begin{aligned} 6\sum_{\ell=1}^{k}\mathcal{H}(\mathcal{S}_{j_\ell}) \geq \sum_{\ell=1}^{k}\mathcal{H}(P^\ell_{2,j_\ell+1}) \geq \mathcal{H}(\mathcal{S}^*), \\ \Rightarrow \mathcal{H}(\mathcal{S}) = \max_{j=1}^{\log +1}\{\mathcal{H}(\mathcal{S}_j)\} \geq \max_{\ell=1}^{k}\{\mathcal{H}(\mathcal{S}_{j_\ell})\} \\ \geq \frac{\sum_{\ell=1}^{k}\mathcal{H}(\mathcal{S}_{j_\ell})}{\log n} \geq \frac{\mathcal{H}(\mathcal{S}^*)}{6\log n}. \end{aligned}$$

We remark that two special cases need to be addressed. Assume the first scale we reach is $s_1 = S_{\max}/n$ such that $|P^1_{1,1}| < |P^1_{2,1}|$. In this case, we skip the rectangular pyramids in $P^1_{2,1}$, and we conduct the similar analysis on the remaining rectangular pyramids in $\mathcal{S}^*$, as mentioned earlier. Moreover, $\mathcal{H}(P^1_{2,1}) \leq n \times \frac{S_{\max}}{n} = S_{\max} \leq \mathcal{H}(\mathcal{S}_{\log n+1}) \leq \mathcal{H}(\mathcal{S})$. Thus, we have

$$\begin{aligned} (6(\log n - 1) + 1)\mathcal{H}(\mathcal{S}) \geq \mathcal{H}(\mathcal{S}^*) \\ \Rightarrow \mathcal{H}(\mathcal{S}) \geq \frac{\mathcal{H}(\mathcal{S}^*)}{6\log n - 5} \geq \frac{\mathcal{H}(\mathcal{S}^*)}{6\log n}. \end{aligned}$$

In addition, consider the other case that the first scale we reach is $s_1 = S_{\max}$ such that $|P^1_{1,\log n+1}| \geq |P^1_{2,\log n+1}|$. Then clearly,

$$\begin{aligned} &2|\mathcal{S}_{\log n+1}| \geq |P^1_{1,\log n+1}| \geq |P^1_{2,\log n+1}| \\ &\Rightarrow 2\mathcal{H}(\mathcal{S}_{\log n+1}) \geq \mathcal{H}(P^1_{1,\log n+1}) \geq \mathcal{H}(P^1_{2,\log n+1}) \\ &\Rightarrow 4\mathcal{H}(\mathcal{S}) \geq 4\mathcal{H}(\mathcal{S}_{\log n+1}) \\ &\qquad \geq \mathcal{H}(P^1_{1,\log n+1}) + \mathcal{H}(P^1_{2,\log n+1}) \\ &\qquad = \mathcal{H}(\mathcal{S}^*), \end{aligned}$$

which implies a better approximation ratio for $\mathcal{S}$. The proof is complete. □

We can show that the analysis of this approximation ratio is in fact tight, whose detail is omitted.

Theorem 2. *The analysis of the $6\log n$-approximation algorithm is tight.*

Moreover, if there is a c-approximation algorithm for the unit-width label placement problem in the plane, the approximation factor can be improved to $3c\log n$, according to the equations (1) and (2). The currently best ratio, obtained by Agarwal *et al.* [1] and Chan [8], is $\frac{k+1}{k}$, for any integer $k \geq 1$. Therefore, the unit-width case of the simple ARO problem can be approximated with a $\frac{3\log n(k+1)}{k}$-approximation factor, for any integer $k \geq 1$, though, in $O(n\log^2 n + n\Delta^{k-1}\log n)$ time [8], where Δ is the maximum intersection number of rectangular pyramids. We remark that the tight example, as described above, can be applied to $3c\log n$-factor approximation algorithm as well.

Corollary 1. *The simple ARO problem with unit-width rectangular pyramids can be approximated with $3c\log n$-factor, where c is an approximation ratio for the unit-width label placement problem in the plane.*

3 Complexity and approximation for general ARO problems

In this section, we first prove that the general ARO problem with constant dilation for a given set of axis-aligned congruent square prisms of equal selectable scale range (or height, for short) is NP-complete. We then proceed to present a greedy algorithm that yields constant-factor approximation for the general ARO problem with constant dilation for an input set of general axis-aligned square prisms of equal height, or for an input set of axis-aligned unit-width rectangular prisms of equal height.

3.1 Complexity of general ARO for congruent square prisms

Been *et al.* [2] showed the NP-completeness of the simple 2D ARO problem with "proportional" dilation for congruent square cone extrusions. Here, we show that even the general ARO problem with "constant dilation" for congruent square prisms is also NP-complete, whose proof uses a reduction from the known NP-complete problem, the planar 3SAT problem [14]. The detail of its proof is in Appendix.

Theorem 3. *The general ARO problem with constant dilation is NP-complete. That is, given a set $\mathcal{E}$ of axis-aligned congruent square prisms of equal selectable range of height and a real number $K > 0$, it is NP-complete to decide whether there is a set of pairwise disjoint truncated prisms $\mathcal{T}$ from the prisms in $\mathcal{E}$ such that $\mathcal{H}(\mathcal{T}) \geq K$. Moreover, the problem remains NP-complete even when restricted to instances where not all the spans on the scale dimension of the input prisms are the same.*

3.2 Approximation for general square prisms

Given a set $\mathcal{E}$ of general axis-aligned square prisms of equal height, we propose a 9-approximation algorithm for such a general ARO problem with constant dilation. Our algorithm runs in a greedy fashion as follows. We greedily select a subset of prisms $\mathcal{S}$ from $\mathcal{E}$, and we take their complete selectable ranges as their active ranges in our solution. First we select a prism E with the smallest base area from $\mathcal{E}$, i.e., the smallest square, and put it into $\mathcal{S}$. Then we discard E and the other prisms intersecting E from $\mathcal{E}$. We repeat this step on the current set of $\mathcal{E}$ until $\mathcal{E}$ becomes empty. We then show in the following theorem that the sum of active ranges of $\mathcal{S}$ is a constant-factor approximation to the optimal solution $\mathcal{S}^*$.

Theorem 4. *Given a set $\mathcal{E}$ of general axis-aligned square prisms of equal height, there is a 9-approximation algorithm which takes $O(n \log^3 n)$ time for such a general ARO problem.*

Proof. We first let h be the common height of all prisms in $\mathcal{E}$. Consider E to be an element selected into the set $\mathcal{S}$ in our algorithm. Then let N_E be the discarded set of prisms due to selection of E in our algorithm. We divide N_E into eight groups, each of which contains at least one corner point of E as E has the smallest base area comparing to the prisms in N_E and all prisms have the same selectable range of height. Let G_p be the group of prisms in N_E containing a corner point p of E. And let $\mathcal{T}_p^*$ be the set of truncated prisms in $\mathcal{S}^*$ from the prisms in the group G_p. Since all prisms in G_p contain the single point p and have the same height h, $\mathcal{H}(\mathcal{T}_p^*) \leq h$. As prism E has eight corner points, the prisms in N_E can contribute active ranges of total height at most $8h$ to the optimal solution $\mathcal{S}^*$. In addition, E contributes an active range of height at most h in the optimal solution $\mathcal{S}^*$. Hence $\mathcal{H}(\mathcal{S}) \geq \frac{1}{9}\mathcal{H}(\mathcal{S}^*)$.

Moreover, the $O(n \log^3 n)$ running time can be easily attained by making use of three-dimensional segment tree. Hence, we complete the proof. □

In fact, the approximation factor 9 for our algorithm is tight in the worst case, whose proof detail is in Appendix.

3.3 Approximation for unit-width rectangular prisms

Given a set $\mathcal{E}$ of axis-aligned unit-width rectangular prisms of equal height, we can obtain a 9-approximation algorithm for such a general ARO problem with constant dilation using the same strategy as in the previous section. In the algorithm, again we iteratively select the prism of the smallest base area among the available ones. Hence, we have the following theorem.

Theorem 5. *Given a set $\mathcal{E}$ of axis-aligned unit rectangular prisms of equal height, there is a 9-approximation algorithm which takes $O(n \log^3 n)$ time for such a general ARO problem.*

3.4 Approximation for congruent square prisms

Given a set $\mathcal{E}$ of axis-aligned congruent square prisms of equal selectable range of height, we propose a 5-approximation algorithm for such a general ARO problem with constant dilation. Been *et al.* [3] propose a greedy algorithm which has 4-approximation ratio, but our running time which takes $O(n \log^3 n)$ is better than theirs. Our algorithm again runs in a greedy fashion, which is described as follows. We greedily select a set of prisms $\mathcal{S}$ from $\mathcal{E}$, and we take their complete selectable ranges as their active ranges in our solution. We process the prisms in the order of increasing x-coordinates. First we select a prism E with the smallest x-coordinate from $\mathcal{E}$ and put it into $\mathcal{S}$. Then we discard E and the other prisms intersecting E from $\mathcal{E}$. We repeat this step on the current set of $\mathcal{E}$ until $\mathcal{E}$ becomes empty. We then show in the following theorem that the sum of active ranges of $\mathcal{S}$ is a constant-factor approximation to the optimal solution $\mathcal{S}^*$.

Theorem 6. *Given a set $\mathcal{E}$ of axis-aligned congruent square prisms of equal selectable of height, there is a* 5*-approximation algorithm which takes* $O(n \log^3 n)$ *time for such a general ARO problem.*

Proof. We first let h be the common height of all prisms in $\mathcal{E}$. Consider E to be an element selected into the set $\mathcal{S}$ in our algorithm. Then let N_E be the discarded set of prisms due to selection of E in our algorithm. According to the design of our algorithm, E is the leftmost one along the x-axis comparing to the prisms in N_E . Since all prisms are congruent, each prism in N_E contains at least one corner point of the right face f of E in the direction of x-axis. We divide N_E into four groups, each of which contains prisms enclosing one corner point of face f. Let G_p be the group of prisms in N_E containing a corner point p of E. By the same reasoning as in the proof of Theorem 4, the prisms in G_p contribute active ranges of total height at most h to the optimal solution $\mathcal{S}^*$. Thus the prisms in the four groups of N_E contribute active ranges of total height at most $4h$ to the optimal solution $\mathcal{S}^*$. In addition, E contributes an active range of height at most h in the optimal solution $\mathcal{S}^*$. Hence $\mathcal{H}(\mathcal{S}) \geq \frac{1}{5}\mathcal{H}(\mathcal{S}^*)$.

Moreover, the $O(n \log^3 n)$ running time can be easily attained by making use of three-dimensional segment tree. Hence, we complete the proof. $\square$

The approximation factor 5 for our algorithm is, in fact, tight in the worst case, whose proof detail is in Appendix.

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Appendix

A. Proof of Theorem 3

Proof. Clearly, the problem is in NP. To show its hardness, we reduce the planar 3SAT problem [14] to our problem. The input instance for the *planar 3SAT problem* is a set $\{x_1, x_2, \ldots, x_n\}$ of n variables, and a Boolean expression $\Phi = c_1 \wedge c_2 \wedge \ldots \wedge c_m$ of m clauses, where each clause consists of exactly three literals, such that the variable clause graph of the input instance is planar. The *planar 3SAT problem* asks for whether there exists a truth assignment to the variables so that the Boolean expression Φ is satisfied. In the following, we will describe our polynomial-time reduction. In our construction, each prism in $\mathcal{E}$ is a prism with unit-square base and with height h, and not all prisms have the same span along the scale dimension.

Variable gadgets. The gadget of a variable x consists of a horizontal chain G_x of $4m$ pairs of congruent square prisms, where every four consecutive pairs of square prisms are dedicated for connecting to one literal of a clause in Φ. Every pair of square prisms intersects and locates either at the span $[0, h]$, called at *down location*, or at the span $[h/2, 3h/2]$, called at *up location*, on the scale dimension (see Figure 4).

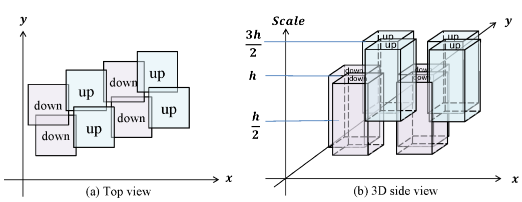


Figure 3: Top view and side view of the variable gadget

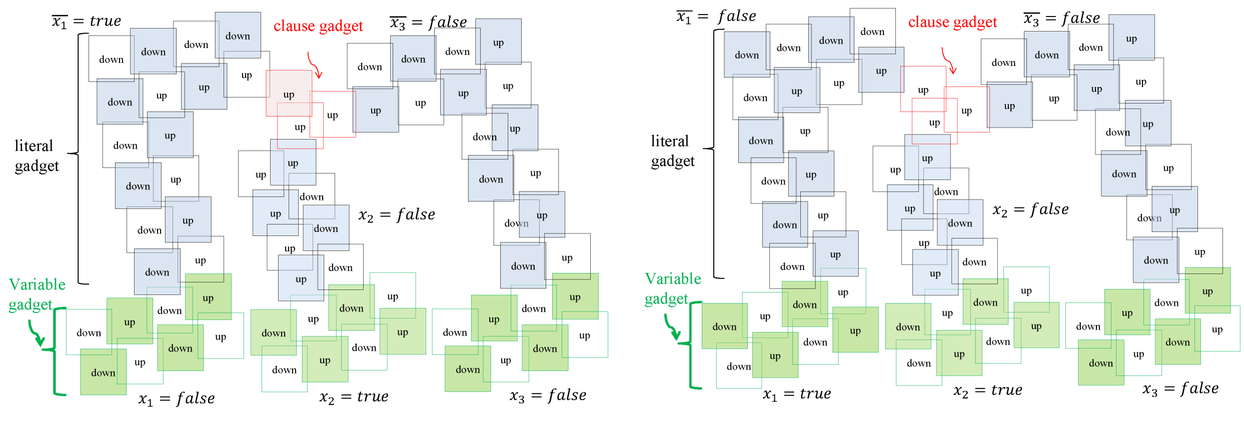
Along the chain of the variable gadget, the prism pairs jump up and down alternately. We observe that every pair of overlapping square prisms along the chain of a variable gadget can contribute at most h in total to the final solution. We let variable x be true corresponding to that the upper prism of first prism pair along G_x is selected; the lower prism of second prism pair along G_x is selected; the upper prism of third prism pair along G_x is selected; etc. On the other hand, variable x being false corresponds to the selection of the remaining prisms in contrast to the selected prisms for variable x being true. It is not hard to see that either of these two solutions for prisms in G_x are the best possible. Whatever variable x is true or false, G_x contributes $4m$ full prisms, and thus total height $4mh$ units, to the final solution $\mathcal{H}(\mathcal{T})$.

Literal gadgets. A literal gadget connects a variable gadget to a clause gadgets. Again the gadget G_λ of a literal λ consists of a chain of square prism pairs such that all its prims locate at up or down positions. However, the literal gadget consists of a vertical part and a horizontal part, if λ corresponds to the left or right literal of the corresponding clause gadget. See Figure 4 for an example.

Suppose that λ is positive, say being x, and its dedicated chain of four consecutive square pairs from the corresponding variable gadgets is $G_{x,\lambda}$; then we connect G_λ to the two middle pairs of square prisms of $G_{x,\lambda}$ (see Figure 4). Otherwise, suppose that λ is negative, say being $\overline{x}$; then we connect G_λ to the two rightmost pairs of square prisms of $G_{x,\lambda}$. The literal gadget G_λ propagates with square prisms at up or down locations in the fashion as shown in Figure 4. When a literal gadget needs to turn left or right, we have to modify the location of one square prism at the turning corner from up to down or vice versa so that the propagation of up and down prisms can proceed. If a literal λ is true, the prism that connects the clause gadget is not selected into the set $\mathcal{T}$; otherwise, if λ is false, then the prism which connects the clause gadget is selected into the set $\mathcal{T}$. For a literal λ, let n_λ be the number of square prism pairs in G_λ. Then, λ contributes n_λ full prisms, and thus $n_\lambda h$ units of height, to the final solution $\mathcal{H}(\mathcal{T})$, no matter whether literal λ is true or false.

Clause gadgets. One prism of the ending square prism pair of a literal gadget connects to a clause gadget. A gadget G_c for clause c consists of three mutually intersecting square prisms all at up locations (see Figure 4(a)). Thus the gadget G_c can contribute at most one full square prism, and thus h units of height, to the final solution $\mathcal{H}(\mathcal{T})$.

Equivalence proof. The variable, literal, and clause gadgets form the set $\mathcal{E}$ of all input prisms representing Φ. It remains for the proof to set the threshold K such that Φ is satisfiable iff $\mathcal{H}(\mathcal{T}) \geq K$. All variable gadgets contribute $4mn$ full prisms, and thus $4mnh$ units of height, to the



(a) (b)

Figure 4: (a)The gadget G_c for the clause $c = (\overline{x_1} \vee x_2 \vee \overline{x_3})$ when c is true; (b)The gadget G_c for the clause $c = (\overline{x_1} \vee x_2 \vee \overline{x_3})$ when c is false.

final solution $\mathcal{H}(\mathcal{T})$. On the other hand, all literal gadgets contribute $\sum_{\lambda \in \text{lit}(\Phi)} n_\lambda$ full prisms, and thus $h(\sum_{\lambda \in \text{lit}(\Phi)} n_\lambda)$ units of height, to the final solution $\mathcal{H}(\mathcal{T})$, where $\text{lit}(\Phi)$ is the set of literals in clauses of formula Φ.

Since at least one literal of a clause c is true if and only if a clause gadget G_c can contribute one full square prism, and thus h units of height, to the final solution $\mathcal{H}(\mathcal{T})$. If all literals of clause c are false, then G_c contributes zero prism, and thus zero units, to $\mathcal{H}(\mathcal{T})$ (see Figure 4(b)). Hence we conclude that Φ is satisfiable, *i.e.*, all clauses are satisfied, if and only if $\mathcal{H}(\mathcal{T}) \geq K$, where

$$K = h(4mn + \sum_{\lambda \in \text{lit}(\Phi)} n_\lambda + m).$$

This completes the NP-hardness proof. □

B. Tight example for Section 3.2

We provide a tight example for the 9-approximation algorithm. Let n axis-aligned square prisms of equal selectable range of height h be divided into nine groups G_p, $n \geq 9$, $1 \leq p \leq 9$, where each of the groups has $\frac{n}{9}$ prisms. In addition, the length of the prism in G_1 is one and the length of other prisms is two. Let the central point of each prism bottom in G_1 is $(4k-2, 2, h)$, in G_2 is $(4k-3, 1, \varepsilon h)$, in G_3 is $(4k-1, 1, \varepsilon h)$, in G_4 is $(4k-3, 3, \varepsilon h)$, in G_5 is $(4k-1, 3, \varepsilon h)$, in G_6 is $(4k-3, 1, (2-\varepsilon)h)$, in G_7 is $(4k-1, 1, (2-\varepsilon)h)$, in G_8 is $(4k-3, 3, (2-\varepsilon)h)$, and G_9 is $(4k-1, 3, (2-\varepsilon)h)$, where $1 \leq k \leq \lceil \frac{n}{9} \rceil$ (see Figure 5). Obviously, the optimum solution is $\mathcal{H}(\mathcal{S}^*) = \frac{8nh}{9} + (1-2\varepsilon)\frac{nh}{9} = \frac{(9-2\varepsilon)nh}{9}$ and our solution is $\mathcal{H}(\mathcal{S}) = \frac{nh}{9}$. As a result, when ε is sufficiently small, the 9-approximation ratio is proved.

C. Tight example for Section 3.4

Figure 6 illustrates a tight example for the 5-approximation algorithm. Let n axis-aligned congruent square prisms of equal selectable range of height h be equally divided into five groups G_p, $n \geq 5$, $1 \leq p \leq 5$. In addition, the length of the prism in each group is one. Let the central point of each prism bottom in G_1 is $(2k, 1, h)$, in G_2 is $(2k+0.5, 1.49, \varepsilon h)$, in G_3 is $(2k+0.5, 0.51, \varepsilon h)$, in G_4 is $(2k+0.5, 1.49, (2-\varepsilon)h)$ and in G_5 is $(2k+0.5, 0.51, (2-\varepsilon)h)$, where $1 \leq k \leq \lceil \frac{n}{5} \rceil$. Obviously, the optimum solution is $\mathcal{H}(\mathcal{S}^*) = \frac{4nh}{5} + (1-2\varepsilon)\frac{nh}{5} = \frac{(5-2\varepsilon)nh}{5}$ and our solution is $\mathcal{H}(\mathcal{S}) = \frac{nh}{5}$. As a result, when ε is sufficiently small, the 5-approximation ratio is proved.

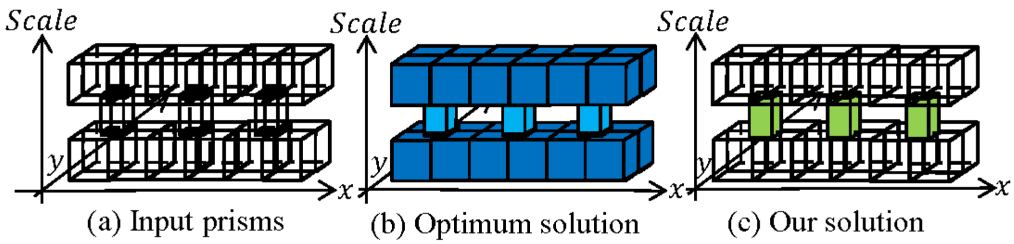


Figure 5: An illustration for the tight example of the 9-approximation algorithm

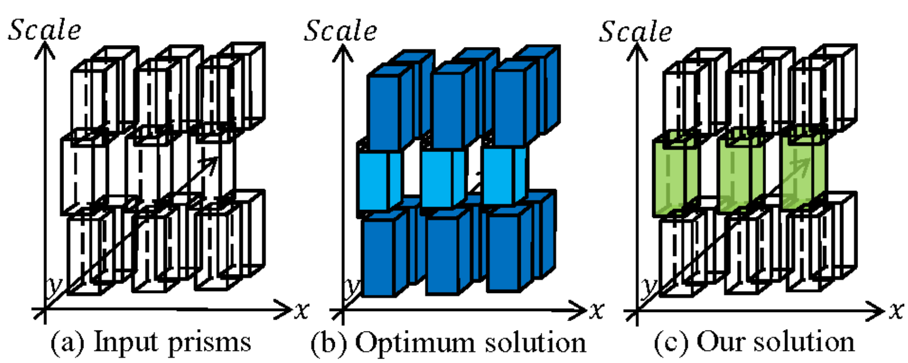


Figure 6: An illustration for the tight example of the 5-approximation algorithm