Improved Approximation Algorithm for the (1,2)-Partial-Terminal Steiner Tree Problem *

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Abstract

Given a complete graph $G = (V, E)$ with a metric cost function $c : E \to \mathbb{R}^+$ and two vertex subsets $R \subset V$ and $R' \subseteq R$, a partial-terminal Steiner tree *is a Steiner tree which contains all the vertices in R such that all the vertices in R' are leaves. The partial-terminal Steiner tree problem (PTSTP) is to find a partial-terminal Steiner tree with the minimum cost. The problem has been shown to be NP-hard and MAX SNP-hard, even when the edge costs are restricted in $\{1, 2\}$, namely, the (1,2)-partial-terminal Steiner tree problem (PTSTP(1,2)). In this paper, we consider PTSTP(1,2). The previous best-known approximation ratio of PTSTP(1,2) was at most 1.79. In this paper, we propose a polynomial-time approximation algorithm that improves the approximation ratio from 1.79 to 1.67.*

1 Introduction

Let $G = (V, E)$ be an undirected, simple, and finite graph, where V and E are the vertex and edge sets of G, respectively. Let $c : E \to \mathbb{R}^+$ be a cost function on edge set. For any two vertices u and v, let uv denote the edge incident with u and v. The cost function c is *metric* if for any three vertices $u, v, w \in V$, the following four conditions hold: (1) $c(uv) \geq 0$; (2) $c(uv) = 0$ if and only if $u = v$; (3) $c(uv) = c(vu)$; and (4) $c(uv)+c(vw) \geq c(uw)$ (triangle inequality). Given a connected graph $G = (V, E)$ with a metric cost function $c : E \to \mathbb{R}^+$ and a vertex subset $R \subset V$, a *Steiner tree* of G is a connected and acyclic subgraph of G which contains all vertices in R. The vertices in R are usually referred to as *terminals* and the vertices in $V \setminus R$ are *Steiner* (*or optional*) vertices. The *cost* of a Steiner tree is the sum of all edge-costs in the tree. The *Steiner tree problem* of G is to find a Steiner tree with the minimum cost.

The Steiner tree problem has been shown to be NP-hard [10]. Moreover, it has been shown to be MAX SNP-hard [2] when the edge costs are either 1 or 2. It has been studied for a long time for improving the approximation ratio achievable in polynomial time. Currently, the best-known approximation algorithm for the Steiner tree problem has an approximation ratio $\rho = \ln 4 + \epsilon < 1.39$ in general case [3]. Moreover, for the special case of the Steiner tree problem, where the edge costs are either 1 or 2, has an approximation ratio $\rho = 1.25$ [1]. The Steiner tree problem and its variants have been the focus of a great deal of research, because they have practical applications in computational biology [5] and VLSI global [9].

Given a complete graph $G = (V, E)$ with a cost function $c : E \to \mathbb{R}^+$ and two subsets $R \subset V$ and $R' \subseteq R$, a *partial-terminal Steiner tree* is a Steiner tree that contains all the vertices of R and all the vertices of R' must be leaves. The partial-terminal Steiner tree problem (PTSTP) is to find a partial-terminal Steiner tree with the minimum cost. This problem has been shown to be NP-hard and MAX SNP-hard [6] even when the edge costs are either 1 or 2. In the case that $R' = R$ (*i.e.*, all the vertices

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in R must be leaves). This problem is called the terminal Steiner tree problem. If $R' = \emptyset$, then this problem is exactly the Steiner tree problem. Currently, the best-known approximation algorithm for the partial-terminal Steiner tree problem has an approximation ratio $2\rho - \frac{\rho}{3\rho-2} - f$ in general case [6, 7], where f is a nonnegative function with range from 0 to $\rho - \frac{\rho}{3\rho-2}$. This problem is useful to some applications, such as VLSI physical design [6, 7].

In this paper, we consider the partial-terminal Steiner tree problem, where the range of the cost function is restricted to $\{1, 2\}$, namely, the $(1, 2)$-*partial-terminal Steiner tree problem* (*PTSTP*$(1, 2)$). The PTSTP(1,2) can be approximated by the algorithm presented in [7] with an approximation ratio of 1.79. In this paper, we propose an approximation algorithm for PTSTP(1,2) improving the ratio from 1.79 to 1.67. The remainder of this paper is organized as follows. Section 2 introduces the definitions and notation used throughout the paper. Our approximation algorithm is presented in Section 3. Finally, Section 4 gives some concluding remarks.

2 Preliminaries

In this paper, we consider an undirected, finite, and simple graph $G = (V, E)$, where V and E are the vertex and edge sets of G, respectively. We used uv to denote an edge with endpoints u and v. An edge $e \in E(G)$ is *incident to* any endpoint of e. The *degree* of a vertex v in G, denoted by $\deg_G(v)$, equals the number of edges incident to v. The cost on the edge $uv \in E(G)$ is denoted by $c(uv)$. Given a vertex subset $U \subseteq V(G)$, the *subgraph of G induced by U* is defined as $G[U] = (U, \{uv \in E(G) \mid u, v \in U\})$. For any vertex v in G, the neighbors of v, denoted by $N_G(v)$, are vertices adjacent to v in G. For a weighted graph G, the *cost* of G, denoted by $c(G)$, is the sum of all the edge costs in G. A *tree* T is a connected acyclic graph, and for each vertex with degree 1 is a *leaf* in T. A non-leaf vertex in T is an *internal vertex.*

3 Approximating PTSTP(1,2)

Let (G, c, R, R') be an input instance of PTSTP(1,2). Since it is easy when $|R| = 1$ or 2, we assume that $|R| \geq 3$. By the definition, all the vertices in R' must be leaves, the input graph G is assumed to be the resulting graph obtained by deleting all edges having endpoints $u, v \in R'$ from the complete graph with vertex set V. For a vertex $r' \in R'$, let $e_{r'}$ be a minimum-cost edge connecting r' to a vertex in $V(S) \setminus R'$ (*i.e.*, $c(e_{r'}) = \min_{v \in V(S) \setminus R'} c(r'v)$).

The idea of Algorithm RT is to collect the subset $R'_1 \subseteq R'$ in which each vertex in R'_1 is connected to a vertex $r \in R \setminus R'$ via an edge with cost 1. Then, all vertices in R'_1 can be deleted from R' because we do not need to do extra efforts on deciding how to connect r'. Lemma 1 show that if we can construct a partial-terminal Steiner tree on $(G, c, R \setminus R'_1, R' \setminus R'_1)$ using an α-approximation algorithm, then we can construct a partial-terminal Steiner tree T on (G, c, R, R') with the same approximation ratio α.

Algorithm 1: RT(G, c, R, R')

Input : A complete graph $G(V, E)$ with cost function $c : E \to \{1, 2\}$ and a set $R \subset V$, a set $R' \subset R$ of terminals.
Output: A set $R'_1 \subseteq R'$.
identify the set $R'_1 = \{r' \mid r' \in R'$ and r' is adjacent to a vertex $r \in R \setminus R'$ such that $c(r'r) = 1\}$.
return R'_1.

Hereafter, the notation R'_1 is used to denote the set constructed by Algorithm RT.

Lemma 1. *Let $\alpha \geq 1$ be any positive constant. If T' is a partial-terminal Steiner tree on $(G, c, R \setminus R'_1, R' \setminus R'_1)$ such that $c(T') \leq \alpha \cdot c(T'^*)$, where T'^* is an optimal partial-terminal Steiner tree on $(G, c, R \setminus R'_1, R' \setminus R'_1)$. Then, we can find in polynomial time a partial-terminal Steiner tree T on (G, c, R, R') such that $c(T) \leq \alpha \cdot c(T^*)$, where T^* is an optimal partial-terminal Steiner tree on (G, c, R, R').*

After using Algorithm RT to identify the subset R'_1 of R', we can apply the currently best-known ρ-approximation algorithm for the Steiner tree problem on $(G, c, R \setminus R'_1, R' \setminus R'_1)$ to obtain a Steiner tree S such that $c(S) \leq \rho \cdot c(S^*)$, where S^* is an optimal Steiner tree on $(G, c, R \setminus R'_1, R' \setminus R'_1)$. Because S is a Steiner tree that may not be a partial-terminal Steiner tree, S needs to be modified as a partial-terminal Steiner tree. The idea of Algorithm MODIFY_TREE_1 is to reduce the number of neighbors of each vertex $r' \in R' \setminus R'_1$ with $|N_S(r')| \geq 2$ in the tree S.

After executing Algorithm MODIFY_TREE_1 to construct a Steiner tree S_1, there may be some

Algorithm 2: MODIFY_TREE_1($S, G, c, R \setminus R'_1, R' \setminus R'_1$)

Input : A Steiner tree S on $(G, c, R \setminus R'_1, R' \setminus R'_1)$ constructed by any ρ-approximation algorithm for the Steiner tree problem.
Output: A Steiner tree S_1 on $(G, c, R \setminus R'_1, R' \setminus R'_1)$.

for *each* $r' \in R' \setminus R'_1$ *with* $|N_S(r')| \geq 2$ **do**
  for *each* $v \in N_S(r')$ **do**
    if $v \in R \setminus R'$ **then**
      delete the edge $r'v$ and connect vertex v to the vertex $u \in N_S(r') \setminus \{v\}$.
      $E(S) = (E(S) \setminus \{r'v\}) \cup \{uv\}$ and $S = (V(S), E(S))$
      if $|N_S(r')| = 1$ **then**
        break
return the Steiner tree $S_1 = (V(S), E(S))$

redundant Steiner vertices in S_1. Next, we present Algorithm MODIFY_TREE_2 to reduce the number of redundant Steiner vertices in S_1, as shown in Fig. 1. Lemma 2 shows some properties of the tree obtained by Algorithm MODIFY_TREE_2.

Lemma 2. *Suppose that S_2 is a Steiner tree obtained by Algorithm* MODIFY_TREE_2. *If $r' \in R' \setminus R'_1$ with $|N_{S_2}(r')| \geq 2$, then the following two statements hold: (1) Any Steiner vertex in S_2 has a degree of at least 3. Moreover, each vertex $v \in N_{S_2}(r')$ is also a Steiner vertex. (2) $c(S_2) \leq c(S)$.*

Corollary 1. *Let S_2 be a Steiner tree constructed using Algorithm* MODIFY_TREE_2. *Then, we have $c(S_2) \leq \rho \cdot c(S^*)$, where S^* is an optimal partial-terminal Steiner tree on $(G, c, R \setminus R'_1, R' \setminus R'_1)$.*

Theorem 1. *Algorithm* MODIFY_TREE_3 *constructs a partial-terminal Steiner tree T' on $(G, c, R \setminus R'_1, R' \setminus R'_1)$ such that $c(T') \leq 1.67 \cdot c(T'^*)$, where T'^* is an optimal partial-terminal Steiner tree on $(G, c, R \setminus R'_1, R' \setminus R'_1)$.*

Our main algorithm, namely, $\text{PTSTP}_{1,2}$, is proposed as follows.

Theorem 2. *Algorithm $PTSTP_{1,2}$ is a 1.67-approximation algorithm for PTSTP(1,2).*

Proof. After executing the Algorithm $\text{PTSTP}_{1,2}$, we obtain a partial-terminal Steiner tree T' on

Algorithm 3: MODIFY_TREE_2($S_1, G, c, R \setminus R'_1, R' \setminus R'_1$)

Input : A Steiner tree S_1 on $(G, c, R \setminus R'_1, R' \setminus R'_1)$ obtained by Algorithm MODIFY_TREE_1.
Output: A Steiner tree S_2 on $(G, c, R \setminus R'_1, R' \setminus R'_1)$.

for *each* $s \in V(S_1) \setminus R$ **do**
  if $\deg_{S_1}(s) = 1$ **then**
    delete edge sw, where $w \in N_S(s)$
    $V(S_1) = V(S_1) \setminus \{s\}$ and $E(S_1) = E(S_1) \setminus \{sw\}$
for *each* $s \in V(S_1) \setminus R$ **do**
  if $\deg_{S_1}(s) = 2$ **then**
    for edges us and vs, where $u, v \in N_{S_1}(s)$
    if $u, v \in R \setminus R'$ or $u, v \in V(S_1) \setminus R$ **then**
      delete edges in $\{us, vs\}$ and connect vertex u to vertex v
      $V(S_1) = V(S_1) \setminus \{s\}$ and $E(S_1) = (E(S_1) \setminus \{us, vs\}) \cup \{uv\}$
    else if $(u \in R \setminus R',\ v \in V(S_1) \setminus R)$ or $(u \in R' \setminus R'_1,\ v \in V(S_1) \setminus R)$ **then**
      delete edges in $\{us, vs\}$ and connect vertex u to vertex v
      $V(S_1) = V(S_1) \setminus \{s\}$ and $E(S_1) = (E(S_1) \setminus \{us, vs\}) \cup \{uv\}$
    else if $u, v \in R' \setminus R'_1$ or $(u \in R \setminus R',\ v \in R' \setminus R'_1)$ **then**
      delete edges in $\{us, vs\}$ and connect vertex u to a vertex $w \in V(S_1) \setminus R$ such that the resulting subgraph is still a tree
      $V(S_1) = V(S_1) \setminus \{s\}$ and $E(S_1) = (E(S_1) \setminus \{us, vs\}) \cup \{uw\}$.
return the Steiner tree $S_2 = (V(S_1), E(S_1))$

$(G, c, R \setminus R'_1, R' \setminus R'_1)$. By Theorem 1, $c(T') \leq 1.67 \cdot c(T'^*)$, where T'^* is an optimal partial-terminal Steiner tree on $(G, c, R \setminus R'_1, R' \setminus R'_1)$. According to Lemma 1 and the result holds. ☐

4 Conclusion

In this paper, we propose a 1.67-approximation algorithm for the partial-terminal Steiner tree problem with edge cost 1 or 2. Our algorithm improves a previous 1.79-approximation algorithm. Note that if $R' = \emptyset$, then the PTSTP(1,2) is exactly the classical Steiner tree problem STP(1,2). Our Algorithm $\text{PTSTP}_{1,2}$ can achieve the currently best-known approximation ratio for STP(1,2).

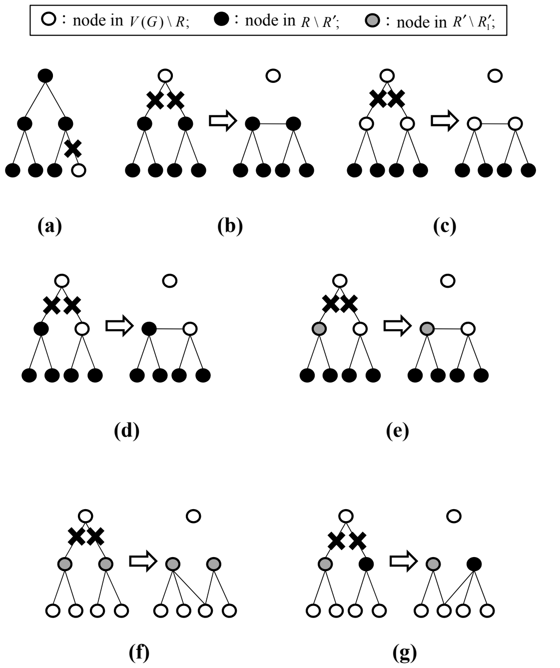

Figure 1: (a)-(g) Illustration of Algorithm Modify_Tree_2.

Algorithm 4: Modify_Tree_3($S_2, G, c, R \setminus R'_1, R' \setminus R'_1$)

Input : A Steiner tree S_2 on $(G, c, R \setminus R'_1, R' \setminus R'_1)$ obtained by Algorithm Modify_Tree_2.

Output: A partial-terminal Steiner tree T on $(G, c, R \setminus R'_1, R' \setminus R'_1)$.

if *S_2 is not a partial-terminal Steiner tree* **then**
  for *each vertex $r' \in R' \setminus R'_1$ that is not a leaf in S_2 (i.e., $|N_{S_2}(r')| \geq 2$)* **do**
    delete all the edges in $star(r')$ except the edge $e_{r'}$
    find a minimum spanning tree $T_{r'}$ of $G[N_{S_2}(r')]$
    $E(S_2) = (E(S_2) \setminus star(r')) \cup \{e_{r'}\} \cup E(T_{r'})$

return the partial-terminal Steiner tree $T = (V(S_2), E(S_2))$

Algorithm 5: $\mathrm{PTSTP}_{1,2}(G, c, R, R')$

Input : A complete graph $G(V, E)$ with cost function $c : E \to \{1, 2\}$, a set $R \subset V$, and a set $R' \subset R$ of terminals.

Output: A partial-terminal Steiner tree T on (G, c, R, R').

find the set R'_1 using Algorithm RT on (G, c, R, R').
find a Steiner tree S using a best-known approximation algorithm on $(G, c, R \setminus R'_1, R' \setminus R'_1)$.
find a Steiner tree S_1 using Algorithm Modify_Tree_1 on $(S, G, c, R \setminus R'_1, R' \setminus R'_1)$
find a Steiner tree S_2 using Algorithm Modify_Tree_2 on $(S_1, G, c, R \setminus R'_1, R' \setminus R'_1)$
find a partial-terminal Steiner tree T' using Algorithm Modify_Tree_3 on $(S_2, G, c, R \setminus R'_1, R' \setminus R'_1)$
find a partial-terminal Steiner tree T on (G, c, R, R') by using tree T' and set R'_1 according to the proof of Lemma 2.
return the partial-terminal Steiner tree T

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