Rainbow Domination in Generalized de Bruijn Digraphs *

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Abstract

In this paper, we are concerned with the k-rainbow domination problem on generalized de Bruijn digraphs. We give an upper bound and a lower bound for the k-rainbow domination number in generalized de Bruijn digraphs $G_B(n,d)$. We also show that $\gamma_{rk}(G_B(n,d)) = k$ if and only if $\alpha \leqslant 1$, where $n = d+\alpha$ and $\gamma_{rk}(G_B(n,d))$ is the k-rainbow domination number of $G_B(n,d)$.

1 Introduction

A *dominating set* in a digraph $D = (V, A)$ is a set $S \subseteq V$ such that, for every $v \in V \setminus S$, there is an arc $(u, v) \in A$ with $u \in S$. The domination number of D, denoted by $\gamma(D)$, is defined as the minimal cardinality of a dominating set of D. The concept of rainbow domination was introduced by Breŝar, Henning, and Rall [4]. Let $G = (V, E)$ be a graph and $v \in V(G)$. Hereafter, $V(G)$ and $E(G)$ are simply written as V and E, respectively. Let f be a function that assigns to each vertex a subset of colors chosen from the set $\mathscr{C} = \{1, 2, \ldots, k\}$; that is, $f : V \longrightarrow \mathcal{P}(\mathscr{C})$, where $\mathcal{P}(\mathscr{C})$ is the set of all subsets of $\mathscr{C}$. If $\bigcup_{u\in N(v)} f(u) = \mathscr{C}$ for each vertex $v \in V$ with $f(v) = \emptyset$, then f is called a *k-rainbow dominating function* (kRDF for short) of G, where $N(v) = \{u \in V | uv \in E\}$. The *weight* of f is defined as $w(f) = \sum_{v\in V} |f(v)|$. Given a graph G, the minimum weight of a kRDF is called the *k-rainbow domination number* of G, denoted by $\gamma_{rk}(G)$. When $w(f) = \gamma_{rk}(G)$, we say that f is a γ_{rk}-function and the set S with $f(v) \neq \emptyset$ for every $v \in S$ is called a *γ_{rk}-set*. Notice that the ordinary domination problem is the 1-rainbow domination problem.

Breŝar and Ŝumenjak [3] studied the 2-rainbow domination problem in generalized Petersen graphs and proved that the 2-rainbow domination problem is NP-complete in general graphs. In [5], Chang, Wu, and Zhu proved that the k-rainbow domination problem is NP-complete for any value of k and provided a linear-time algorithm for the k-rainbow domination problem in trees.

In this paper, we are concerned with the k-rainbow domination problem in digraphs $D = (V, A)$. That is, the requirement on $f(v) = \emptyset$ is changed to

$$\bigcup_{u\in N^-(v)} f(u) = \{1, \ldots, k\},$$

where $N^-(u) = \{v | (v, u) \in A\}$ is the *in-neighborhood* of u.

We give an example in Figure 1. In Figure 1, we can see that v_0, v_1, v_3 and v_4 is the in-neighborhood of v_2, and we give $v_2 \to \{1, 2, 3\}$, so that v_1, v_3, and v_4 is the in-neighborhood with 3 colors, so v_2 is a 3-rainbow dominating set.

The generalized de Bruijn digraph $G_B(n, d)$ has vertex set $V(G_B(n, d)) = \{0, 1, \ldots, n-1\}$ and arc set $A(G_B(n, d)) = \{(x, y) | y \equiv dx+i \pmod n, 0 \leqslant i \leqslant d-1\}$, where n and d are two positive integers with $n \geqslant d \geqslant 2$. In particular, if $n = d^m$, then $G_B(n, d)$ is the de Bruijn digraph $B(d, m)$.

The generalized de Bruijn digraphs have been widely studied as interconnection network topologies because of various good properties [3, 4]. For notational convenience, sometimes we simply

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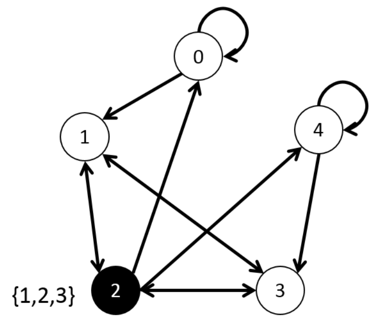

Figure 1: 3-rainbow domination in digraphs.

write G_B instead of $G_B(n,d)$ if n and d are explicit from the context.

For generalized de Bruijn digraphs, their Hamiltonian property [6], digameter [8], connectivity [9], absorbant [13], domination [10] and twin domination [11, 14] have been studied. Also, several structural objects such as spanning trees, eulerian tours [12], closed walks [16] and small cycles [7] have been counted. Recently, various k-tuple domination problems in de Bruijn digraphs have also been studied [1, 2, 18, 15]. In [17], Tian and Xu further investigated the distance for these digraphs.

2 The bounds of k-rainbow domination numbers of generalized de Bruijn digraphs

In this section we propose a lower bound and an upper bound of the k-rainbow domination number for generalized de Bruijn digraphs. We also show that the proposed bounds are tight.

Proposition 1. *For any graph G of n vertices, if $k \geqslant n$, then $\gamma_{rk}(G) = n$.*

By Proposition 1, hereafter, we assume that $k < n$.

Theorem 2. $\gamma_{rk}(G_B(n,d)) \geqslant \max\{k, \left\lceil \frac{kn}{d+k} \right\rceil\}$.

Proof. It is clear that $\gamma_{rk}(G_B(n,d)) \geqslant k$. Let f be a kRDF of digraph D and let S be the set of vertices with $f(v) = \emptyset$ for $v \in V$. Then obviously $w(f) \geqslant |V \setminus S|$. Moreover, since every vertex v with $f(v) = \emptyset$ must have all k colors in its in-neighborhood, it follows that $d \cdot w(f) \geqslant |S| \cdot k$. Therefore, we have

$$\begin{aligned} kw(f) &\geqslant k|V \setminus S| \\ &\geqslant k|V| - k|S| \\ &\geqslant kn - d \cdot w(f). \end{aligned}$$

After rearranging, we have $w(f) \geqslant \left\lceil \frac{kn}{d+k} \right\rceil$. If f is a γ_{rk}-function, then the theorem follows. □

Theorem 3. $\gamma_{rk}(G_B(n,d)) \leqslant k\lceil \frac{n}{d} \rceil$.

Proof. For $G_B(n,d)$, let $S = \{0, 1, \ldots, \lceil \frac{n}{d} \rceil - 1\}$. It is easy to verify that S is a dominating set of G_B and $|S| = \lceil \frac{n}{d} \rceil$. By setting $f(v) = \{1, \ldots, k\}$, this results in S to be a k-rainbow dominating set. Thus $\gamma_{rk}(G_B(n,d)) \leqslant k\lceil \frac{n}{d} \rceil$. □

3 The class of generalized de Bruijn digraphs with $\gamma_{rk}(G_B(n,d)) = k$

In this section, we investigate the class of generalized de Bruijn digraphs with $\gamma_{rk}(G_B(n,d)) = k$.

Lemma 4. *If $n = d$, then $\gamma_{rk}(G_B(n,d)) = k$.*

Proof. By Theorem 1, $\gamma_{rk}(G_B(n,d)) \geqslant \max\{k, \left\lceil \frac{kn}{d+k} \right\rceil\} \geqslant k$. When $n = d$, by Theorem 3, $\gamma_{rk}(G_B(n,d)) \leqslant k\lceil \frac{n}{d} \rceil = k$. Thus the lemma follows. □

Theorem 5. *For a digraph D, there is a γ_{rk}-function f with $w(f) = k$ if and only if there exists a set $S \subseteq V$ with $|S| \leqslant k$ such that $V \setminus S \subseteq N^+(v)$ for each $v \in S$, where $N^+(v) = \{u|(v,u) \in A\}$ is the* out-neighborhood *of v.*

Proof. Assume that there exists a set $S \subseteq V$ with $|S| \leqslant k$ such that $V \setminus S \subseteq N^+(v)$ for each $v \in S$. Let f be a kRDF of D such that the colors assigned to the vertices in S form a partition of $\mathscr{C}$. It is clear that f is a γ_{rk}-function with $w(f) = k$. Now we consider the other direction. Suppose to the contrary that there is a vertex, say v, in S which does not dominate all vertices in $V \setminus S$. Let $u \in V \setminus S$ be a vertex which is not in $N^+(v)$ and $f(u) = \emptyset$. By definition, $\bigcup_{x \in N^-(u)} f(x) = \{1, \ldots, k\}$. This further implies that $w(f) > k$, a contradiction. This completes the proof. □

Lemma 6. *If $n = d+1$, then $\gamma_{rk}(G_B(n,d)) = k$ for $k \geqslant 2$.*

Proof. It is clear that $N^+(0) = \{0, \ldots, n-2\}$ and $N^+(n-1) = \{1, \ldots, n-1\}$ in $G_B(n,d)$ when $n = d+1$. Let $S = \{0, n-1\}$. It is easy to verify that $V \setminus S \subseteq N^+(v)$ for every $v \in S$. Thus, by Theorem 5, $\gamma_{rk}(G_B(n,d)) = k$ when $k \geqslant 2$. This completes the proof. □

Lemma 7. *If f is a γ_{rk}-function of $G_B(n,d)$ with $w(f) = k$ and S is its corresponding γ_{rk}-set, then there is a subset S' of S in which all vertices in S' and in $V \setminus S'$ are consecutive.*

Proof. By Theorem 5, we have $V \setminus S \subseteq N^+(v)$ for each $v \in S$. If the vertices in $V \setminus S$ are not consecutive, then the missing vertices must be in S. Let M be the set of missing vertices. Now let $S' = S \setminus M$. By the consecutive property of the vertices in $N^+(v)$ for each $v \in V$ in $G_B(n,d)$, all vertices in S' are consecutive. This further implies that all vertices in $V \setminus S'$ are also consectutive. This completes the proof. □

Corollary 8. *If f is a γ_{rk}-function of $G_B(n,d)$ with $w(f) = k$, then there is γ_{rk}-set in which all vertices are consecutive.*

Proof. By Lemma 7, there is a set S' in which all vertices are consecutive. Clearly, the property $V \setminus S' \subseteq N^+(v)$ for each $v \in S'$ still holds. By Theorem 5, S' is a γ_{rk}-set. This completes the proof. □

By Corollary 8, in the following, we assume that all the vertices in the γ_{rk}-set S of $G_B(n,d)$ are consecutive. We also assume that the set $V \setminus S$ contains all the vertices x with $s \leqslant x \leqslant t$. Moreover, let $n = d + \alpha$ with $\alpha \geqslant 0$ in $G_B(n,d)$, and let $N_0(x) = d \cdot x$ and $N_{d-1}(x) = d \cdot x + d - 1$.

Lemma 9. *If f is a γ_{rk}-function of $G_B(n,d)$ with $w(f) = k$ and S is its corresponding γ_{rk}-set, then both $|S|$ and $|V \setminus S|$ are less than d.*

Proof. First, we consider the cardinality of $V \setminus S$. By Theorem 5, we have $|V \setminus S| \leqslant d$; for otherwise, every vertex $v \in S$ has $|N^+(v)| > d$, a contradiction. Now we consider the case where $|V \setminus S| = d$. By Theorem 5 again, $V \setminus S \subseteq N^+(v)$ for every $v \in S$. This further implies that $u \notin N^-(v)$ and $v \notin N^-(u)$ for any two vertices $u, v \in S$. Furthermore, there is also no self-loop vertex in S. Since $|V \setminus S| = d$, it follows that $N_0(u+1) = N_{d-1}(u)+1$. Note that both $u+1$ and $N_0(u+1)$ are in S. This implies that there is an arc from $u+1$ to $N_0(u+1)$, a contradiction. Therefore, we have $|V \setminus S| < d$.

Now we consider the cardinality of S. If $|S| > d$, then, by Theorem 5, $|N^-(v)| > d$ for each $v \in V \setminus S$, a contradiction. Now we consider the case where $|S| = d$. By using a similar argument as above, we can find that there exists an arc from vertex u to $N_0(u)$, where both u and $N_0(u)$ are in $V \setminus S$. This contradicts the assumption that $N^-(N_0(u)) = S$. This completes the proof. □

Lemma 10. *If f is a γ_{rk}-function of $G_B(n,d)$ with $w(f) = k$ and S is its corresponding γ_{rk}-set, then $|V \setminus S| > \alpha$.*

Proof. By Lemma 9, we have $d - 1 \geqslant |S|$. Accordingly, we have the following derivation:

$$\begin{aligned} |V \setminus S| &= |V| - |S| \\ &\geqslant |V| - (d-1) \\ &= d + \alpha - d + 1 \\ &= \alpha + 1. \end{aligned}$$

This completes the proof. □

Theorem 11. $\gamma_{rk}(G_B(n,d)) = k$ *if and only if* $\alpha \leqslant 1$.

Proof. By Lemmas 4 and 6, if $\alpha \leqslant 1$, i.e., $n = d$ or $d+1$, then $\gamma_{rk}(G_B(n,d)) = k$. Now we consider the other direction.

Suppose to the contrary that there exists a γ_{rk}-function f of $G_B(n,d)$ with $w(f) = k$ and $n \geqslant d+2$, namely $\alpha \geqslant 2$. Let S be the γ_{rk}-set and $M = V \setminus S$ in which s and t are the smallest and largest, respectively, numbers in M. For simplicity, we only consider the case where M contains all the vertices x with $s \leqslant x \leqslant t$. The case where M contains $V \setminus \{x | s < x < t\}$ can be handled similarly.

It is clear that there exist vertices $u, v \in S$ such that $N_0(u) = s$ and $N_{d-1}(v) = t$. If $v + 1$ is in S, then $N_0(v+1) = t + 1$. Since $M \subseteq N^+(v+1)$, vertex t is also in $N^+(v+1)$. This implies that $n = d$, a contradiction. Thus $v = s - 1$. By using a similar argument, we can find that $u = t + 1$.

By definition, we have

$$\begin{aligned} N_{d-1}(u) &\equiv s + d - 1 \pmod{n} \\ &\equiv s - \alpha - 1 \pmod{n}. \end{aligned}$$

By Lemma 10, it follows that $s - \alpha - 1 \pmod{n}$ is in S. Similarly, $N_{d-1}(u) \equiv s - 2 \cdot \alpha - 1 \pmod{n}$ which is also in S. Thus $s - x \cdot \alpha - 1 \pmod{n}$ for $1 \leqslant x \leqslant |S|$ are all in S. By Lemma 10 again, all $s - x \cdot \alpha - 1 \pmod{n}$ for $1 \leqslant x \leqslant |S|$ are distinct. This implies that $\alpha = 1$. This contradicts the assumption that $\alpha \geqslant 2$. This completes the proof. □

4 Conclusion

In this paper, we discuss the k-rainbow domination problem in generalized de Bruijn digraphs.

We present an upper bound and a lower bound on the k-rainbow domination number for generalized de Bruijn digraphs. We also show the proposed lower bound is tight. As a future study, we have the following conjecture: $\lceil \frac{kn}{d+1} \rceil \leqslant \gamma_{rk}(G_B(n,d)) \leqslant k\lceil \frac{n}{d+1} \rceil$.

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