Upper bounds on 2 and 3-rainbow Domination Number of Sierpiński Graphs*

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Abstract

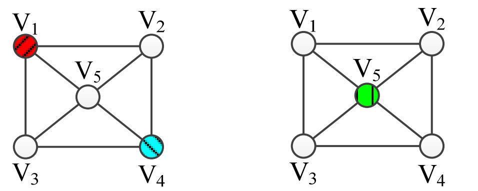
The k-rainbow domination is a variant of the classical domination problem in graphs and is defined as follows. Given an undirected graph G = (V, E), we have a set of k colors and we assign an arbitrary subset of these colors to each vertex of G. If a vertex which is assigned an empty set, then the union of color set of its neighbors must be k colors. This assignment is called the k-rainbow dominating function of G. The minimum sum of numbers of assigned colors over all vertices of G is called the k-rainbow domination number of G. In this paper, we present upper bounds on 2 and 3-rainbow domination number of Sierpiński Graphs.

Keywords: Domination; *k*-rainbow domination; Sierpiński Graphs.

1 Introduction

The *k-rainbow domination* is a variant of classical domination [7, 8]. Let $G = (V(G), E(G))$ be a finite, simple and undirected graph, where $V(G)$ and $E(G)$ are the vertex and edge sets of G, respectively. For a vertex $v \in V(G)$, the *open neighborhood* $N(v) = \{u \in V(G) \mid (u, v) \in E(G)\}$. Let $C = \{1, 2, \ldots, k\}$ be a set of k colors, and f be a function that assign to each vertex a set of colors chosen from C, that is, $f : V(G) \to P(\{1, \ldots, k\})$. If for each vertex $v \in V(G)$ such that $f(v) = \varnothing$ we have $\cup_{u \in N(v)} f(u) = C$ then f is called a *k-rainbow dominating function* (*k*RDF) of G. The *weight*, $\omega(f)$, of a function f is defined as $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The *k-rainbow domination number* $\gamma_{rk}(G)$ of G is the minimum weight of a k-rainbow dominating function.

For example, two 2-rainbow dominations of graph G with 5 vertices are shown in Figure 1(a) and 1(b), respectively. In Figure 1(a), we assign a color set {1} to V_1 and assign a color set {2} to V_4. When $f(V_2) = \varnothing$, $\cup_{u \in N(V2)} f(u) = \{1, 2\}$. Similarly, the union of color sets of neighbors of V_3 and V_5 are both {1, 2}, respectively. In Figure 1(b), we assign a color set {1, 2} to V_5. For each vertex v in $\{V_1, V_2, V_3, V_4\}$, $f(v) = \varnothing$ and $\cup_{u \in N(v)} f(u) = \{1, 2\}$. In Figure 1(a) and 1(b), both $\omega(f) = 2$, and it is easy to verify that $\gamma_{r2}(G) = 2$.



(a) a 2RDF of G (b) another 2RDF of G

Figure 1: 2-rainbow dominations on graph G. In (a), the vertex with color set {1} and the vertex with color set {2} are filled with slashes and backslashes, respectively. In (b), the vertex with color set {1, 2} is filled with vertical line.

In [2, 3], Brešar, Henning and Rall introduced the concept of k-rainbow domination of a graph G. The application is that there are k types of guards being used, and vertices v with $f(v) = \varnothing$ must have all types of guards in their neighborhoods. In [4], Brešar and Šumenjak showed that the decision version of 2-rainbow domination of graphs is NP-complete even when restricted to chordal graphs (or bipartite graphs). They also gave the exact values of the 2-rainbow domination numbers for paths, cycles and suns. Thus, further investigations tended to study bounds on k-rainbow domination number for certain families of graphs, such as trees, grids, generalized Petersen graphs and lexicographic product of graphs. This problem is widely studied in [1-6, 11-13, 15-19]. In this article, we give upper bounds on 2 and 3-rainbow domination number of Sierpiński Graphs.

2 Preliminaries

The definition of Sierpiński graph is described as follows. Readers may refer to [9, 10, 14] for the

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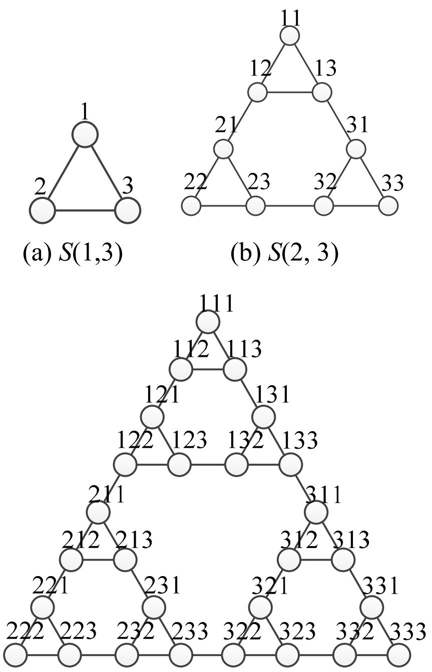
details. The vertex set of $S(n, m)$ consists of all n-tuples of integers 1, 2, …, m, for integers $n \geq 1$ and $m \geq 3$, namely $V(S(n, m)) = \{1, 2, ..., m\}^n$. Thus, let $\langle v_1, v_2, \ldots, v_n \rangle$ be the label of vertex v, denoted $l(v)$, or in the regular expression form $v_1v_2 \ldots v_n$ for short when no confusion arises. For notational convenience, we always use w to denote a substring of $v_1v_2 \ldots v_n$ and a, b, c to denote a number in $v_1v_2 \ldots v_n$, i.e., $a, b, c \in \{1, 2, ..., m\}$. Let $|w|$ be the length of a substring w, and b^i is the *Kleene closure* in regular expression, i.e., b^4 is equal to *bbbb*. For example, $l(v) = wab^{n-|w|-1}$ means that the label of v begins with prefix w, then concatenates with number a, and finally ends with $n-|w|-1$ b's.

Two distinct vertices u and v are adjacent in $S(n, m)$ if and only if $l(u) = wab^{n-|w|-1}$ and $l(v) = wba^{n-|w|-1}$ with $a \neq b$ and $0 \leq |w| \leq n-1$. Note that if $|w| = 0$, then w is a null string. Further, if $|w| = n-1$, then both $b^{n-|w|-1}$ and $a^{n-|w|-1}$ are empty. We use $S_a(n-1, m)$ (respectively, $S_w(n-|w|, m)$) to denote the subgraph induced by the vertices with prefix a (respectively, prefix w) in their labels. When $|w| = n-1$, it is obvious that $S_w(1, m)$ is a complete graph K_m. If $l(v) = a^n$ then we call v an *extreme vertex*. There are exactly m extreme vertices in $S(n, m)$. Since the label of an extreme vertex v is a^n, by definition, v has exactly $m-1$ neighbors whose labels are of the form $a^{n-1}b$ with $b \neq a$. Every non-extreme vertex v with $l(v) = wab^{n-|w|-1}$ has exactly m neighbors whose labels are of the form $wba^{n-|w|-1}$ and $wab^{n-|w|-2}c$ while $c \neq b$. Let $d(v)$ be the degree of vertex v. Thus, $d(v) = m-1$ if v is a extreme vertex, and $d(v) = m$ otherwise. Figure 2 depicts $S(1, 3)$, $S(2, 3)$ and $S(3, 3)$.



(a) $S(1,3)$ (b) $S(2, 3)$

(c) $S(3, 3)$

Figure 2: Labeled Sierpiński graphs

For notational convenience, we refer to a color set $\{1, 2, \ldots, k\}$ as $c\{1, 2, \ldots, k\}$, and call a vertex with an empty color set as a *non-color vertex*.

3 2-rainbow Dominations on Sierpiński Graphs

In this section, we proceed by giving upper bounds of the 2-rainbow domination number of Sierpiński Graphs $S(n, m)$ while $m \geq 3$. First, it is easy to check the following proposition.

Proposition 1 $\gamma_{rk}(K_m) = k$ when $k \leq m$.

Since $S(1, m)$ is isomorphic to K_m by definition, we have the following lemma.

Lemma 2 $\gamma_{rk}(S(1, m)) = k$ when $k \leq m$ and $m \geq 3$.

We give algorithm A to provide a 2RDF of $S(2, m)$ for $m \geq 3$.

Algorithm A

Input: a Sierpiński graph $S(2, m)$ for $m \geq 3$
Output: A 2RDF of $S(2, m)$
Step 1. $D \leftarrow \varnothing$
Step 2. $D \leftarrow D \cup jj$ with $c\{1\}$ for $j \in \{1, 2, \ldots, m\}$
Step 3. $D \leftarrow D \cup j1$ with $c\{2\}$ for $j \in \{2, 3, \ldots, m\}$

Lemma 3 $\gamma_{r2}(S(2, m)) \leq 2m-1$ for $m \geq 3$.

Proof. To achieve this upper bound, we give algorithm A to produce all 2RDFs of $S(2, m)$ for $m \geq 3$. There are m $S(1, m)$ in a $S(2, m)$, and $S(1, m)$ is isomorphic to K_m. By algorithm A, there are a vertex jj with $c\{1\}$ and a vertex $j1$ with $c\{2\}$ in each $S_j(1, m)$ for $j \in \{2, 3, \ldots, m\}$, thus all non-color vertices are dominated beside the ones of $S_1(1, m)$. In $S_1(1, m)$, the non-color vertex $1j$ can be dominated by the vertex 11 with $c\{1\}$ and the vertex $j1$ with $c\{2\}$ for $j \in \{2, 3, \ldots, m\}$. Finally, all non-color vertices are dominated and the weight of 2RDF of $S(2, m)$ is $2m-1$. □

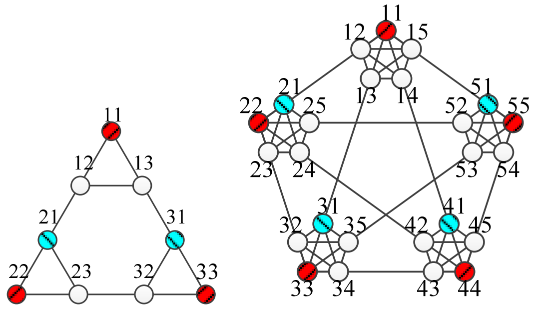
For example, Figure 2(a) and 2(b) shows 2-rainbow dominations of $S(2, 3)$ and $S(2, 5)$, respectively.

Then, we give algorithm B to provide a 2RDF of $S(3, m)$ for $m \geq 3$.

Algorithm B

Input: a Sierpiński graph $S(3, m)$ for $m \geq 3$
Output: A 2RDF of $S(3, m)$
Step 1. $D \leftarrow \varnothing$
Step 2. for $i \leftarrow 1$ to m
Step 3.1. if m is odd then

Step 3.2. $x \leftarrow (\, i + \lfloor m/2 \rfloor - 1)\ \%\ m + 1$
Step 3.3. $y \leftarrow (\, i + \lfloor m/2 \rfloor\,)\ \%\ m + 1$
Step 3.4. else
Step 3.5. $x \leftarrow i\ \%\ m + 1$
Step 3.6. $y \leftarrow (\, i + m - 2\,)\ \%\ m + 1$
Step 3.7. end if
Step 4.1. for $j \leftarrow 1$ to m
Step 4.2. if $j \neq x$ then $D \leftarrow D \cup ijx$ with $c\{1\}$
Step 4.3. end for
Step 5.1. for $j \leftarrow 1$ to m
Step 5.2. if $j \neq x$ and $j \neq y$ then $D \leftarrow D \cup ijy$ with $c\{2\}$
Step 5.3. end for
Step 6. $D \leftarrow D \cup ixx$ with $c\{2\}$
Step 7. end for



(a) a 2RDF of $S(2, 3)$ (b) a 2RDF of $S(2, 5)$

Figure 3: 2-rainbow dominations of $S(2, 3)$ and $S(2, 5)$. The vertex with $c\{1\}$ and the vertex with $c\{2\}$ are filled with slashes and backslashes, respectively.

Lemma 4 $\gamma_{r2}(S(3, m)) \leq 2m^2 - 2m$ for $m \geq 3$.

Proof. There are $m \times m$ $S(1, m)$ in a $S(3, m)$.

Case 1: m is odd.

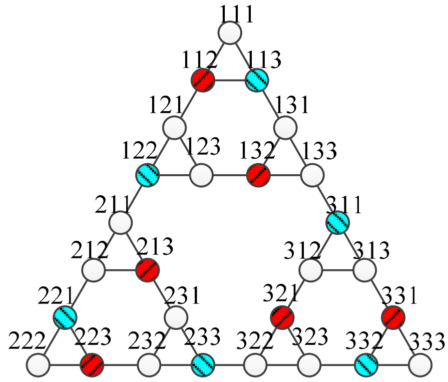
In this case, let $i \in \{1, 2, \ldots, m\}$, $x = (\, i + \lfloor m/2 \rfloor - 1)\ \%\ m + 1$, $y = (\, i + \lfloor m/2 \rfloor\,)\ \%\ m + 1$ and $j \in \{1, 2, \ldots, m\} - \{x\}$. By algorithm B, there are a vertex ijx with $c\{1\}$ and a vertex ijy with $c\{2\}$ in each $S_{ij}(1, m)$, thus all non-color vertices in $S_{ij}(1, m)$ are dominated. In $S_{ix}(1, m)$, each non-color vertex ixj is dominated by the vertex ixx with $c\{2\}$ and the vertex ijx with $c\{1\}$. In $S_{iy}(1, m)$, each non-color vertex iyz is dominated by the vertex ixx with $c\{1\}$ and the vertex izy with $c\{2\}$ while $z \in \{1, 2, \ldots, m\} - \{x, y\}$, and the non-color vertex iyy is dominated by the vertex ixx with $c\{1\}$ and the vertex yii with $c\{2\}$.

Case 2: m is even.

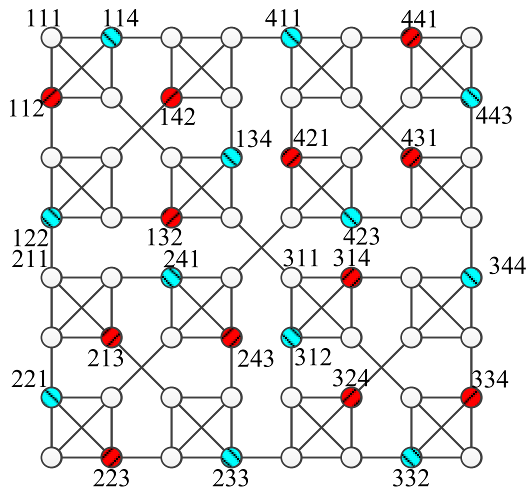
In this case, let $i \in \{1, 2, \ldots, m\}$, $x = i\ \%\ m + 1$, $y = (\, i + m - 2\,)\ \%\ m + 1$ and $j \in \{1, 2, \ldots, m\} - \{x\}$. Since the remaining parts of algorithm B are the same in case 1 and case 2, the proof can be verified by a similar way of above case.

Finally, all non-color vertices are dominated in both cases and the weight of 2RDF of $S(3, m)$ is $2m^2 - 2m$. □

For example, Figure 3(a) and 3(b) shows 2-rainbow dominations of $S(3, 3)$ and $S(3, 4)$, respectively.



(a) a 2RDF of $S(3, 3)$



(b) a 2RDF of $S(3, 4)$

Figure 4: 2-rainbow dominations of $S(3, 3)$ and $S(3, 4)$. The vertex with $c\{1\}$ and the vertex with $c\{2\}$ are filled with slashes and backslashes, respectively.

Lemma 5 $\gamma_{rk}(S(n, m)) \leq m \times \gamma_{rk}(S(n-1, m))$.

Proof. By the definition of Sierpiński graph, there are m $S(n-1, m)$ in a $S(n, m)$. We can apply a kRDF with $\gamma_{r2}(S(n-1, m))$ on each $S(n-1, m)$ to obtain a kRDF with $\omega(f) = m \times \gamma_{rk}(S(n-1, m))$, which will be the upper bound of $\gamma_{rk}(S(n, m))$, on $S(n, m)$. □

By applying Lemma 5 recursively on Lemma 4, we have the following lemma.

Lemma 6 $\gamma_{r2}(S(n, m)) \leq 2(m-1)m^{n-2}$ for $n \geq 4$.

We summarize Lemmas 2, 3, 4 and 6 as follows.

Theorem 7 Some upper bounds of $\gamma_{r2}(S(n, m))$ are shown as below, which $m \geq 3$.

$$\gamma_{r2}(S(n,m)) \leq \begin{cases} 2, & \text{if } n = 1, \\ 2m-1, & \text{if } n = 2, \\ 2(m-1)m^{n-2}, & \text{otherwise.} \end{cases}$$

4 3-rainbow Dominations on Sierpiński Graphs

In this section, we proceed by giving upper bounds of the 3-rainbow domination number of Sierpiński Graphs *S(n, m)* while $m \geq 3$. First, We give algorithm C to provide a 3RDF of $S(2, m)$ for $m \geq 3$.

Algorithm C

Input: a Sierpiński graph $S(2, m)$ for $m \geq 3$
Output: A 3RDF of $S(2, m)$
Step 1. $D \leftarrow \varnothing$
Step 2. $D \leftarrow D \cup$ 11 with $c\{1\} \cup$ 32 with $c\{1\}$
Step 3. $D \leftarrow D \cup j2$ with $c\{1\}$ for $j \in \{4, 5, \ldots, m\}$
Step 4. $D \leftarrow D \cup$ 13 with $c\{2\} \cup$ 22 with $c\{2\}$
Step 5. $D \leftarrow D \cup j3$ with $c\{2\}$ for $j \in \{4, 5, \ldots, m\}$
Step 6. $D \leftarrow D \cup$ 21 with $c\{3\} \cup$ 33 with $c\{3\}$
Step 7. $D \leftarrow D \cup j1$ with $c\{3\}$ for $j \in \{4, 5, \ldots, m\}$

Lemma 8 $\gamma_{r3}(S(2, m)) \leq 3m - 3$ for $m \geq 3$.

Proof. There are m $S(1, m)$ in a $S(2, m)$. When m = 3, it is easy to verify that algorithm C provide a 3RDF of $S(2, 3)$ by Figure 5 (a). When $m \geq 4$, there are a vertex $j2$ with $c\{1\}$, a vertex $j3$ with $c\{2\}$ and a vertex $j1$ with $c\{3\}$ in each $S_j(1, m)$ for $j \in \{4, 5, \ldots, m\}$, thus all non-color vertices are dominated beside the ones of $S_1(1, m)$, $S_2(1, m)$ and $S_3(1, m)$. In $S_1(1, m)$, the non-color vertex 12 can be dominated by the vertex 11 with $c\{1\}$, the vertex 13 with $c\{2\}$ and the vertex 21 with $c\{3\}$, and the other non-color vertex $1j$ can be dominated by the vertex 11 with $c\{1\}$, the vertex 13 with $c\{2\}$ and the vertex $j1$ with $c\{3\}$ while $j \in \{4, 5, \ldots, m\}$. The non-color vertices in $S_2(1, m)$ and $S_3(1, m)$ can be dominated by the similar way. Finally, all non-color vertices are dominated and the weight of 3RDF of $S(2, m)$ is $3m - 3$. □

For example, Figure 5(a) and 5(b) shows 2-rainbow dominations of $S(2, 3)$ and $S(2, 6)$, respectively.

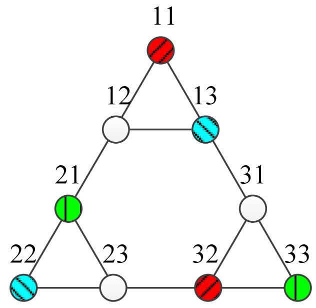
By applying Lemma 5 recursively on Lemma 8, we have the following lemma.

Lemma 9 $\gamma_{r3}(S(n, m)) \leq 3(m-1)m^{n-2}$ for n ≥ 3.

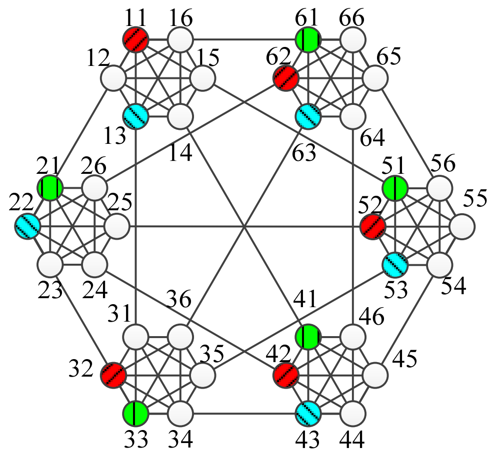
We summarize Lemmas 2, 8 and 9 as follows.

Theorem 10 Some upper bounds of $\gamma_{r3}(S(n, m))$ are shown as below, which $m \geq 3$.

$$\gamma_{r3}(S(n,m)) \leq \begin{cases} 3, & \text{if } n = 1, \\ 3(m-1)m^{n-2}, & \text{otherwise.} \end{cases}$$



(a) a 3RDF of $S(2, 3)$



(b) a 3RDF of $S(2, 6)$

Figure 5: 3-rainbow dominations of $S(2, 3)$ and $S(2, 6)$. The vertex with $c\{1\}$, the vertex with $c\{2\}$ and the vertex with $c\{3\}$ are filled with slashes, backslashes and vertical line, respectively.

5 Conclusion

In this paper, we study 2 and 3-rainbow domination on Sierpiński Graphs. Then, we design some algorithms to produce 2RDFs and 3RDFs of $S(n, m)$, and the weights of these rainbow domination functions are upper bounds on $\gamma_{rk}(S(n, m))$. As a future work, we will try to prove the lower bounds on $\gamma_{rk}(S(n, m))$.

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