Complexity of the contraction mapping principle with noises

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Abstract

Chang and Lyuu [1] show that given oracle access to a finite metric space (M, d) and a contraction mapping $f: M \to M$ on it, the fixed point of f can be found in an expected number of $O(\sqrt{|M|})$ queries. We show, for each positive function $\epsilon: \mathbb{Z}^+ \to \mathbb{R}$, that their result does not generalize to the case where the distance queries may have an arbitrary error in $[\,-\epsilon(|M|), \epsilon(|M|)\,]$.

1 Introduction

A set M endowed with a distance function $d: M \times M \to \mathbb{R}$ is a metric space if the following conditions hold for all x, y, $z \in M$:

- $d(x, y) \geq 0$;
- $d(x, y) = 0$ if and only if $x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, y) + d(y, z) \geq d(x, z)$.

If there exists $\rho \in (0, 1)$ with $d(f(x), f(y)) \leq \rho d(x, y))$ for all x, $y \in M$, then we call f a contraction mapping on (M, d).

Theorem 1 ([2]). *Any contraction mapping $f: M \to M$ on a complete metric space (M, d) has a unique fixed point.*

Chang and Lyuu [1] show how to find the fixed of a contraction mapping in expected sublinear time.

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Theorem 2 ([1]). *Given oracle access to a finite metric space (M, d) and a contraction mapping $f: M \to M$ on it, the fixed point of f can be found with an expected number of $O(\sqrt{|M|})$ oracle queris, where each query may ask for $d(x, y)$ or $f(x)$ for any x, $y \in M$.*

However, the sublinear-time algorithm of Chang and Lyuu needs precise distance queries. Using Chang and Lyuu's technique [1, Theorem 11], we show that their result does not generalize to the case of imprecise distance queries.

2 Hardness of finding fixed points with imprecise queries

Let A be an algorithm for finding a fixed point with only $q = q(n) \leq 0.1n$ queries even when each distance query has an arbitrary noise in $[-\epsilon(|M|), \epsilon(|M|)]$, where $\epsilon: \mathbb{Z}^+ \to \mathbb{R}$ is a positive function. Define $g: \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n-1\}$ by

$$g(x) \stackrel{\text{def.}}{=} (x - 1 \bmod n)$$

for $x \in \{0, 1, \ldots, n-1\}$.

Denote by $\mathbf{1}$ the constant function with the value of 1. For $0 \leq i \leq n-1$,

$$X_i \stackrel{\text{def.}}{=} \begin{cases} 1, & \text{if one of the queries of } A^{g,\mathbf{1}} \text{ is } i, \\ 0, & \text{otherwise.} \end{cases}$$

As A makes no more than q queries, $\sum_{i=0}^{n-1} X_i \leq q$. By the linearity of expectation, $\sum_{i=0}^{n-1} \mathrm{E}[X_i] \leq q$. By the averaging argument, there exists $i^* \in \{0, 1, \ldots, n-1\}$ with $\mathrm{E}[X_{i^*}] \leq 0.1$. This and X_{i^*} being a Bernoulli trial imply

$$\Pr\left[\text{ one of the queries of } A^{g,\mathbf{1}} \text{ is } i^* \right] = \mathrm{E}[X_{i^*}] \leq 0.1. \quad (1)$$

For all x, $y \in \{0, 1, \ldots, n-1\}$,

$$\hat{d}\,(x, y) \stackrel{\text{def.}}{=} 1 - \frac{\epsilon}{2^{(x-i^*) \bmod n + (y-i^*) \bmod n}},$$

where $(x - i^*) \bmod n$ and $(y - i^*) \bmod n$ are taken to be in $\{0, 1, \ldots, n-1\}$. The following lemma is immediate.

Lemma 1. *For all x, $y \in \{0, 1, \ldots, n-1\}$,*

$$1 - \epsilon \leq \hat{d}(x, y) \leq 1.$$

By inequality (1) and Lemma 1,

$$\Pr\left[\text{one of the queries of } A^{g,\hat{d}+\delta} \text{ is } i^*\right] \leq 0.1$$

for some real-valued function δ with $\mathrm{Im}(\delta) \subseteq [-\epsilon, \epsilon]$.

For $x \in \{0, 1, \ldots, n-1\}$,

$$h(x) \stackrel{\text{def.}}{=} \begin{cases} x - 1 \bmod n, & \text{if } x \neq i^*, \\ i^*, & \text{otherwise.} \end{cases}$$

Lemma 2. *h is a contraction mapping on $(\{0, 1, \ldots, n-1\}, \hat{d})$.*

As g and h differ only at i^*,

$$\Pr\left[\text{one of the queries of } A^{g,\hat{d}+\delta} \text{ is } i^*\right] \leq 0.1$$

implies

$$\Pr\left[\text{one of the queries of } A^{h,\hat{d}+\delta} \text{ is } i^*\right] \leq 0.1. \quad (2)$$

By the assumption that A finds a fixed point whenever there is one even if each query has an arbitrary noise in $[-\epsilon, \epsilon]$,

$$\Pr\left[\text{one of the queries of } A^{h,\hat{d}+\delta} \text{ is } i^*\right] = 1,$$

a contradiction to inequality (2).

The following theorem summarizes our findings.

Theorem 3. *Let $\epsilon \colon \mathbb{Z}^+ \to \mathbb{R}$ be a positive function and A be any algorithm. Assume that for each finite metric space (M, d), contradiction mapping $f \colon M \to M$ on (M, d) and each function $\delta \colon M \times M \to \mathbb{R}$ with $\mathrm{Im}[\delta] \subseteq [-\epsilon(|M|), \epsilon(|M|)]$, $A^{f,d+\delta}$ finds the fixed point of f. Then the worst-case (over all f, d and δ) expected (over the random coin tosses of A) query complexity of A is $\Omega(|M|)$.*

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References

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