A short note on finding the limit of a sequence computed by a DFA *

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Abstract

We prove a folklore result regarding the limit of any sequence computed by a deterministic finite automaton. We do not know of the result, albeit folklore, being proven elsewhere.

1 Introduction

The Bolzano–Weierstrass theorem states that every bounded sequence in a finite-dimensional Euclidean space has a converging subsequence [1]. In particular, every bounded sequence of real numbers has an accumulation point. We are interested in finding an accumulation point (or the limit if it exists) of a sequence computed by a machine. It is folklore that given a deterministic finite automaton (DFA) M with outputs, $\lim_{n\to\infty} M(1^n)$ an accumulation point of the exists and can be found in linear time, where $M(1^n)$ is interpreted as a sequence of digits after the radix point and thus $M(1^n) \in [0,1]$ for each $n \in \mathbb{Z}^+$. This short note presents a complete proof, which we do not find elsewhere, of this folklore result.

A DFA with outputs is a finite state machine with an initial state and several directed arrows between states. Each directed arrow is labeled with a character. When a DFA gets an input, it will read one character at a time and follow the arrow labeled with that character until the input is exhausted. We can also assign some output to each arrow of a DFA. So, when going through an arrow, the DFA outputs.

2 Finding limits of sequences computed by DFAs

This short note proves the following folklore theorem, for which we do not know of a written proof elsewhere.

Theorem 1. *Given a deterministic finite automaton (DFA) M with at most one output character assigned to each arrow,*

$$\lim_{n\to\infty} M(1^n)$$

exists and can be found in time linear in the number of states of M*, where* $M(1^n)$ *is interpreted as a sequence of digits after the radix point and thus* $M(1^n) \in [0,1]$ *for each* $n \in \mathbb{Z}^+$.

Proof. Write m for the number of states of M. By the pigeonhole principle, there exist positive integers $h,\ k \leq m+1$ such that 1^h and 1^{h+k} lead M to the same state, say q. Denote by s the output of M on 1^k, assuming that M starts from the state q. Now, for $t \in \mathbb{Z}^+$, $M(1^{h+kt})$ equals $M(1^h)$ concatenated with t copies of s. Therefore,

$$\lim_{t\to\infty} M(1^{h+kt}) = 0.M(1^h)ss\ldots,$$

where the strings in the right-hand side above are concatenated by juxtaposition. This implies

$$\lim_{n\to\infty} M(1^n) = 0.M(1^h)ss\ldots,$$

because $M(1^q)$ is a substring of $M(1^r)$ whenever $q \leq r$.

To find positive integers $h,\ k \leq m+1$ such that 1^h and 1^{h+k} lead M to the same state, it suffices to simulate M on 1^{m+1} until a state is repeated. Once h and k are determined, it is easy to find s in $O(m)$ time.

We propose the following question: Is it true that given a deterministic pushdown automaton M with outputs and $\epsilon > 0$, an accumulation point of $\{M(1^n)\}_{n=0}^{\infty}$ can be approximated to within an additive error of ϵ?

3 Conclusions

We provide a proof for the folklore result that, given a DFA M with outputs,

$$\lim_{n\to\infty} M(1^n)$$

exists and can be found in time linear in the number of states of M. We are interested in whether a similar result holds when M is a deterministic pushdown automaton instead.

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References

[1] W. Rudin. *Principles of Mathematical Analysis*. McGraw--Hill, 3rd edition, 1976.