Cube Connected Crossed Cube: A New Interconnection Topology

Hong-Chun Hsu*, Shin-Pei Hsu
Department of Medical Informatics
Tzu Chi University, Hualien, Taiwan 970, R.O.C.

Abstract

An interconnection network plays a critical role of a multi-computer system because the system performance is deeply dependent on network latency, fault tolerance and throughput. There are a lot of mutually conflicting requisites in designing the topology of interconnection networks. It is nearly impossible to design a network which is optimum in all perspectives. Therefore, designing new interconnection networks is still an attractive research. In this paper, we introduce a new interconnection topology, Cube Connected Crossed Cube(QCC). This topology is suitable for the design of massively parallel systems with thousands of processors. An interesting property of this network is the low vertex degree, which enhances the VLSI design and fabrication of the system. The QCC can emulate the cube-connected machine with no significant degradation of performance but with a much more compact structure. The diameter and the optimal connectivity (equal to the degree) of QCC are proposed in this paper. *Keywords*: Interconnection networks, Hypercube, Crossed Cube, Cube Connected Crossed Cube, connectivity, diameter.

1 Introduction

It is important to design parallel computers using an interconnection network topology that can scale up to a large number of processors and that is capable of sustaining fast communication and data sharing among processors. There are a lot of mutually conflicting requisites in designing the topology of interconnection networks. It is nearly impossible to design a network which is optimum in all perspectives. Therefore, designing new interconnection networks is still an attractive research. This investigation will continue for decades since parallel and distributed computers are the main solution for the computational problems that will defy human beings in the twenty-first century.

Hierarchical interconnection networks have attracted considerable attention in the research for multiprocessor systems[13, 16, 18, 20, 25]. A hierarchical design approach allows the network to be constructed incrementally, starting from one or more essential modules. Hierarchical interconnection networks are intuitively appealing when massively parallel processors are to be connected. The Hypercube [21] is one of the most widely used topologies because it has charming properties such as strong connectivity, recursive interconnection, simple routing, a good edge complexity, and embedding of various interconnection networks. A variety of Hypercube based hierarchical interconnection networks such Crossed Cube [6, 7], Enhanced Cube [22], Extended Cube [16], Folded Cube [1], Generalized Hypercube [2], Möbius Cube [5], Twisted Cube [9, 12], Hierarchical Hypercube [18], Hierarchical Crossed cube [17] and so on have been proposed.

The Crossed Cube was first proposed by Efe, and has attracted much attention in literatures [4, 8, 10, 14, 15, 23]. An n-dimensional Crossed Cube, denoted as CQ_n, is obtained by crossing some edges in an n-dimensional Hypercube. The Crossed Cube has the same vertex and edge complexity as the Hypercube but only about half of diameter, wide diameter, and fault-diameter as the Hypercube with the same dimension. Average distance between vertices is smaller and it can simulate a Hypercube through dilation 2 embedding. The basic properties, optimal routing and broadcasting algorithms of the Crossed Cube are all developed.

Motivated by the architecture of hierarchical Interconnection network and many noticeable fea-

*Correspondence to: Associative Professor Hong-Chun Hsu, Department of Medical Informatics, Tzu Chi University, Hualien, Taiwan 970, R.O.C. e-mail: hchsu@mail.tcu.edu.tw.

tures of Hypercube and Crossed Cube, we propose a new interconnection network, called the Cube Connected Crossed Cube (hereafter abbreviated as QCC), in this paper. Theoretical network properties such as the diameter and connectivity will be discussed in detail. The paper is organized as follows: next section introduces the necessary preliminaries and the formal definition of Cube Connected Crossed Cube. In section 3, we study the diameter and connectivity of QCC. Finally, we give a conclusion.

2 Preliminary

For the graph definition and notation, we follow [3]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. A path is a sequence of adjacent vertices, written as $\langle v_0, v_1, v_2, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except possibly $v_0 = v_m$. For this path, v_0 and v_m are called *end vertices*, and v_i, $1 \le i \le m-1$, is called *internal vertex*. For convenience, we use (u, v)-path to denote the path with end vertices u and v. We also write the path $\langle v_0, P, v_m\rangle$ or $P(v_0, v_m)$, where $P = \langle v_0, v_1 \ldots, v_m\rangle$. The *length* of a path P, $len(P)$, is the number of edges in P. n (u, v)-paths $P_1, P_2, \ldots, P_n$ are said to be *internally disjoint* if they have no common internal vertices. A *vertex cut* of G is a set of vertices $S \subseteq G$ such that $G - S$ is disconnected. The *connectivity* of G, $\kappa(G)$, is the minimum cardinality over all vertex cuts of G. If $\kappa(G) = n$ then G is n-connected. By Menger's Theorem, if $\kappa(G) = n$ then there exist n internally disjoint (u, v)-paths over all pair of vertices $u, v \in V(G)$; and for any set $\{w_1, w_2, \ldots, w_n\}$ of vertices in $G - \{u\}$ there are n disjoint (except u) (u, w_i)-paths in G. The *distance* between two distinct vertices u and v of G, denoted by $d_G(u, v)$, is the length of the shortest (u, v)-path of G. The *diameter* of G, denoted by $Diam(G)$, is defined as $Diam(G) = max\{d_G(u, v) \mid u, v \in V(G)\}$.

In this section, we will introduce the formal definition of the Cube Connected Crossed Cubes. To define the Cube Connected Crossed Cubes, we need the definitions of Hypercubes[21] and Crossed Cubes[6]. The Cube Connected Crossed Cube is designed based on k-dimensional Hypercube and n-dimensional Crossed Cube with $k = 2^n$.

Definition 1 *The k-dimensional Hypercube, Q_k, is a graph $G = (V, E)$ with the vertex set $V = \{b_{k-1}b_{k-2}\ldots b_1b_0 \mid b_i \in \{0, 1\}$ for all $0 \le i \le k-1\}$ and the edge set $E = \{(u, v) \mid u$ and v differs exactly one bit$\}$.*

To define Crossed Cubes, as the proposed by Efe [6], the notion so called "pair related" relation is introduced.

Definition 2 *Let $R = \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. Two dibit binary strings $u = u_1u_0$ and $v = v_1v_0$ are pair related, denoted as $u \sim v$, if and only if $(u, v) \in R$.*

The following is the recursive definition of the n-dimensional Crossed Cube CQ_n.

Definition 3 *[6] The Crossed Cube CQ_1 is a complete graph with two vertices labelled by 0 and 1, respectively. For $n \ge 2$, an n-dimensional Crossed Cube CQ_n consists of two $(n-1)$-dimensional sub-Crossed Cubes, CQ^0_{n-1} and CQ^1_{n-1}, and a perfect matching between the vertices of CQ^0_{n-1} and CQ^1_{n-1} according to the following rule:*

Let $V(CQ^0_{n-1}) = \{0u_{n-2}u_{n-3}\cdots u_0 : u_i = 0$ or $1\}$ and $V(CQ^1_{n-1}) = \{1v_{n-2}v_{n-3}\cdots v_0 : v_i = 0$ or $1\}$. The vertex $u = 0u_{n-2}u_{n-3}\cdots u_0 \in V(CQ^0_{n-1})$ and the vertex $v = 1v_{n-2}v_{n-3}\cdots v_0 \in V(CQ^1_{n-1})$ are adjacent in CQ_n if and only if

(1) *$u_{n-2} = v_{n-2}$ if n is even, and*

(2) *$(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R$, for $0 \le i < \lfloor \frac{n-1}{2} \rfloor$.*

For convenience, let $Dec(b)$ and $Bit_i(b)$ denote the decimal number and the i-th bit of the binary string b, respectively. Moreover, let $f_i(b) = 1$ if $Bit_i(b) = 0$ and $f_i(b) = -1$ otherwise. We are now ready to define the Cube Connected Crossed Cubes.

Definition 4 *Given two positive integers $n \ge 1$ and $k = 2^n$. A Cube Connected Crossed Cube, $QCC(k, n)$, is a graph $G = (V, E)$ on 2^{k+n} vertices, where $V = \{b_{k+n-1}b_{k+n-2}\cdots\ b_1b_0 | b_i \in \{0, 1\}, 0 \le i \le k+n-1\}$ and $E = E_{int} \bigcup E_{ext}$. The label of a vertex u is divided into two parts, say $u = u_Xu_Y$, where $u_X = u_{k+n-1}u_{k+n-2}\cdots u_n$ and $u_Y = u_{n-1}u_{n-2}\cdots u_0$. The set of edges E is the union of two sets E_{int} and E_{ext}, which are the sets of internal and external edges, respectively, as the following equations.*

(1) $E_{int} = \{(u, v) \mid u_X = v_X$, and $(u_Y, v_Y) \in E(CQ_n)\}$ and

(2) $E_{ext} = \{(u,v) \mid u_Y = v_Y,$ *and* $v_X = u_X + f_{Dec(u_Y)}(u_X) * 2^{Dec(u_Y)}\}$.

Figures 1 and 2 illustrate the $QCC(2,1)$ and $QCC(4,2)$, respectively.

By the definition of Cube Connected Crossed Cubes, the structure of a $QCC(k,n)$ consists of two levels of hierarchy. At the lowest level of hierarchy, we have a pool of 2^{k+n} vertices. These vertices are grouped into clusters of 2^n vertices each, and the vertices in each cluster are connected to form a CQ_n. Let $CQ_n(i)$ denote the induced subgraph of $QCC(k,n)$ whose vertex set is $\{u = u_X u_Y \mid Dec(u_X) = i \text{ and } 0 \leq Dec(u_Y) \leq 2^n - 1\}$. Then, 2^k $CQ_n(i)$s are connected in a Hypercube fashion to form a $QCC(k,n)$. Clearly, edges of the $CQ_n(i)$s are called internal edges, and edges between $CQ_n(i)$s are referred to as external edges. The following properties are some observations on $QCC(k,n)$.

Property 1 *$QCC(k,n)$ is $(n+1)$-regular and has 2^{k+n} vertices for $k,n \geq 1$.*

Given a neighbor v of vertex u in $QCC(k,n)$ with that i is the left-most different bit between u and v, then v is called the *i-neighbor* of u, denoted by $v = N_i(u)$. If $i \geq n$, then $N_i(u)$ is an *external neighbor* of u in $QCC(k,n)$, denoted by $N^{ext}(u)$. Otherwise, $N_i(u)$ is an *internal neighbor* of u in $QCC(k,n)$, denoted by $N_i^{int}(u)$, when $0 \leq i \leq n-1$. Clearly, there are one external neighbor and n internal neighbors for each vertex in $QCC(k,n)$. Let $N^{int}(u) = \{N_i^{int}(u) \mid 0 \leq i < n\}$.

Lemma 1 *The 2^n vertices of $CQ_n(i)$ connect to exact 2^n $CQ_n(i')$s in $QCC(k,n)$.*

We say $CQ_n(i)$ is the h-dimensional neighbor cube of $CQ_n(i')$ if the position of different bit between i and i' is h. Let $QCC^b(k,n)$, $b \in \{0,1\}$, be the subgraph of $QCC(k,n)$ induced by the vertices which have the label with the left-most bit is b.

3 Combinatorial Properties

Theorem 1 *Let k,n be two positive integers with $k = 2^n$ and let u,v be two distinct vertices of $QCC(k,n)$ with $parity(u_X \bigoplus v_Y) = 1$. Then there exists a Hamiltonian path joining u and v.*

Theorem 2 *Let k,n be two positive integers with $k = 2^n$. Then there exists a Hamiltonian cycle in $QCC(k,n)$.*

Lemma 2 *Let k,n be two positive integers with $k = 2^n$ and let u,v be two distinct vertices of $QCC(k,n)$. Then $d(u,v) \leq 2^{n+1} - 1$.*

Theorem 3 *Let k,n be two positive integers with $k = 2^n$. The diameter $Diam(QCC(k,n)) = 2^{n+1} - 1$.*

Theorem 4 *Let k,n be two positive integers with $k = 2^n$. The connectivity $\kappa(QCC(k,n)) = n + 1$.*

Since $QCC(k,n)$ is $(n+1)$-regular and $(n+1)$-connected, $QCC(k,n)$ has optimal connectivity. Finally, we list some comparisons with Hypercube, Crossed Cube, and Hierarchical Hypercube, Cube Connected Cycle as the table 1.

4 Conclusion

In this paper, we introduce a new interconnection topology, Cube Connected Crossed Cube, $QCC(k,n)$. Herein, we study the Combinatorial Properties of $QCC(k,n)$ such as diameter and connectivity. $QCC(k,n)$ has many interesting properties such as low degree, logarithmic diameter, and maximum connectivity. Other properties, such as fault diameter, fault-tolerant hamiltonian, panconnectivity etc. will be investigated in the future.

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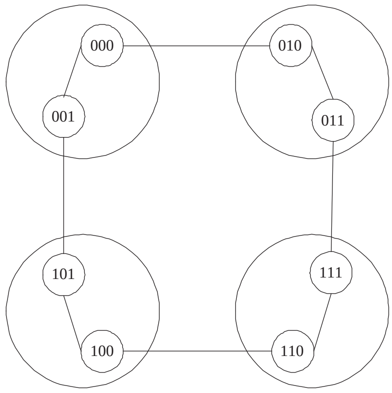

Figure 1: Illustration of $QCC(2, 1)$

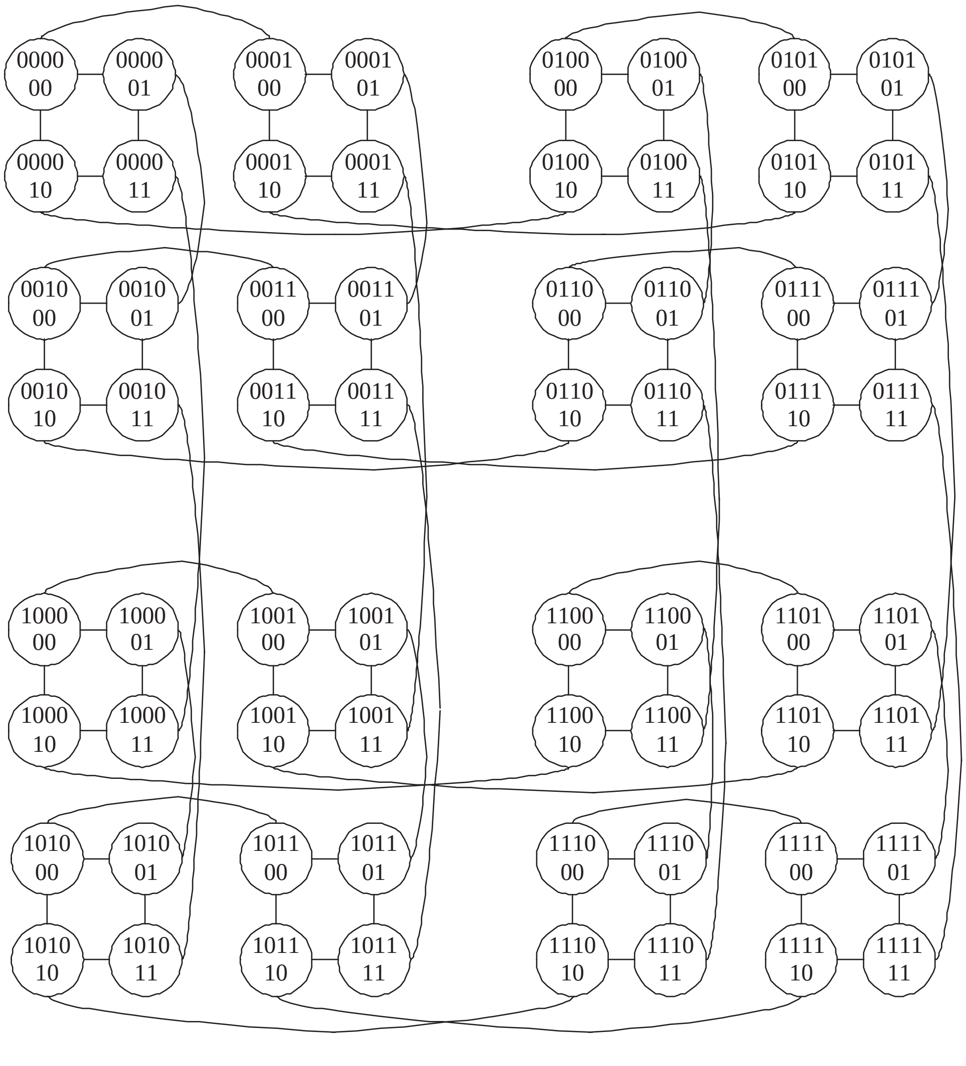

Figure 2: Illustration of $QCC(4, 2)$

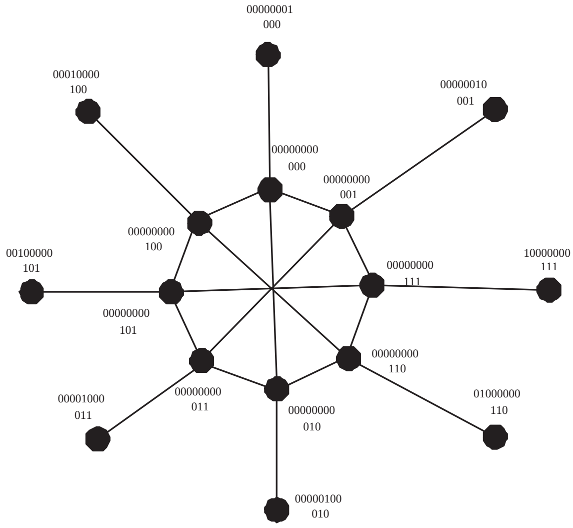

Figure 3: Illustration of $CQ(0)$ in $QCC(8,3)$

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Table 1: Compare with other hierarchical networks.

Networks	♯Vertices(N)	Degree	Connectivity	Diameter(D)
Q_{k+n}[21]	$2^{(k+n)}$	$k+n$	$k+n$	$k+n=logN$
CQ_{k+n}[6]	$2^{(k+n)}$	$k+n$	$k+n$	$\lceil \frac{k+n+1}{2} \rceil \sim \frac{\log N}{2}$
$(k+n)-HHC$[18]	$2^{(k+n)}$	$n+1$	$n+1$	$2^{n+1} < 2\log N$
$QCC(k,n)$	$2^{(k+n)}$	$n+1$	$n+1$	$2^{n+1}-1 < 2\log N$
$CCC(k)$[11]	$2^{(k+n)}$	3	3	$2^{n+1}+2^{n-1}-2 > 2\log N$

Note that $k=2^n$.

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