Path Embedding of Various Lengths in Crossed Cubes with Faulty Vertices

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Abstract

The crossed cube is an important variant of the hypercube for parallel computing. In this paper, we investigate the path embedding of various lengths in crossed cubes with faulty vertices. More precisely, let F denote the vertex faults in an n-dimensional crossed cube, where $n \geq 5$ *and* $|F| \leq n - 3$. *Then, we show that there exists a fault-free path of length l between any two distinct fault-free vertices for each integer l satisfying* $2n - 5 \leq l \leq 2^n - |F| - 1$. *This result improves the previous one with the same number of faults.*

1. Introduction

In computer science, network topology is important and widely discussed by researchers since it is essential for parallel and distributed computation. Recently, many network topologies have been proposed. Among these topologies, the hypercube is one of the most popular topologies since it has good properties such as regularity, symmetry, small diameter, strong connectivity, recursive structure, flexible partition, and relatively low link complexity [9]. The crossed cube has a structure similar to the hypercube, including recursive structure, the same number of vertices, and the same number of edges [3]. However, the diameter of the crossed cube is only about one half of that of the hypercube [3]. The diameter is an important factor for parallel computing speed. The crossed cube has been studied extensively in the literatures: for an n-dimensional crossed cube, its diameter is $\lceil (n+1)/2 \rceil$ [3]; a $(2^n - 1)$-node complete binary tree can be embedded into the crossed cube with dilation 1 [7]; the n-wide diameter and the $(n - 1)$-fault diameter were shown to be $\lceil n/2 \rceil + 2$ [2]; the crossed cube is $(n - 2)$-fault-tolerant Hamiltonian and is $(n - 3)$-fault-tolerant Hamiltonian connected [5]. Moreover, the crossed cube is $(n - 2)$-fault-tolerant pancyclic [10]. The definition of the crossed cube will be presented in the next section.

In interconnection networks, the topological structure is represented as an undirected graph. In this paper, we interchange a node with a vertex, a link with an edge, and a network with a graph. Then, we follow the standard terminology given by Bondy and Murty [1] to describe the structure of a network. Let $G = (V, E)$ be an undirected graph, where $V = V(G)$ denotes the vertex set of G, and $E = E(G)$ denotes the edge set of G. Two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ of G are *adjacent* if $(\boldsymbol{u}, \boldsymbol{v}) \in E(G)$. A graph H is a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, H is a *spanning subgraph* of G (or H *spans* G) if $V(H) = V(G)$. A *path* P of length k, $k \geq 1$, from vertex $\boldsymbol{x}$ to vertex $\boldsymbol{y}$ in G is a sequence of distinct vertices $\langle v_1, v_2, \ldots, v_{k+1} \rangle$ such that $v_1 = \boldsymbol{x}$, $v_{k+1} = \boldsymbol{y}$, and $(v_i, v_{i+1}) \in E(G)$ for $1 \leq i \leq k$. We can write P as $\langle v_1, v_2, \ldots, v_i, Q, v_j, \ldots, v_{k+1} \rangle$ for convenience if $Q \subseteq P$ and $Q = \langle v_i, \ldots, v_j \rangle$, where $i \leq j$. We use $l(P)$ to denote the *length* of P. A *cycle* is a path with at least three vertices, and the last vertex is adjacent to the first one. For clarity, a cycle of length k, $k \geq 3$, is represented by $\langle v_1, v_2, \ldots, v_k, v_1 \rangle$. A path (or cycle) is called a *Hamiltonian path* (*or Hamiltonian cycle*) of G if it spans G. A graph G is *Hamiltonian* if it has a *Hamiltonian cycle*, and a graph G is *Hamiltonian connected* if it contains a Hamiltonian path between any pair of distinct vertices.

Since vertex faults and edge faults may happen when a network is used in practice, it is important to consider with respect to faulty networks. That is, a network is considered functional as long as there is a fault-free communication path between each pair of fault-free nodes.

In this paper, we investigate the path embedding of various lengths in crossed cubes with faulty vertices.

More precisely, let F denote the vertex faults in an n-dimensional crossed cube, where $n \geq 5$ and $|F| \leq n - 3$. Then, we show that there exists a fault-free path of length l between any two distinct fault-free vertices for each integer l satisfying $2n - 5 \leq l \leq 2^n - |F| - 1$. This result improves the previous one that admits the path lengths from $2n - 3$ to $2^n - |F| - 1$ with the same number of faults [8].

The rest of this paper is organized as follows. In Section 2, the formal definition and some properties of the crossed cube are introduced. In Section 3, we propose our main result as one theorem and show its correctness. Finally, some concluding remarks are given in Section 4.

2. The crossed cube and its properties

To define the crossed cube, we need to first introduce an additional concept "pair related".

Definition 1. [3] *Two* 2-*bit binary strings* $\boldsymbol{x} = x_2x_1$ *and* $\boldsymbol{y} = y_2y_1$ *are pair related, denoted by* $\boldsymbol{x} \sim \boldsymbol{y}$, *if and only if* $(\boldsymbol{x}, \boldsymbol{y}) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$.

The formal definition of the crossed cube is as follows.

Definition 2. [3] *The n-dimensional crossed cube* CQ_n *is recursively constructed as follows*:

(i) CQ_1 *is a complete graph with vertex set* $\{0, 1\}$.

(ii) CQ_2 *is isomorphic to a cycle of length* 4 *with vertex set* $\{00, 01, 10, 11\}$ *and edge set* $\{(00, 00), (10, 10), (01, 11), (11, 01)\}$.

(iii) *For* $n \geq 3$, *let* CQ^0_{n-1} *and* CQ^1_{n-1} *be two copies of* CQ_{n-1} with $V(CQ^0_{n-1}) = \{0u_{n-1}u_{n-2}\ldots u_1 | \ u_i = 0$ *or* 1 *for* $1 \leq i \leq n - 1\}$ *and* $V(CQ^1_{n-1}) = \{1u_{n-1}u_{n-2}\ldots u_1 | \ u_i = 0$ *or* 1 *for* $1 \leq i \leq n - 1\}$. *Then,* CQ_n *is formed by connecting* CQ^0_{n-1} *and* CQ^1_{n-1} *with* 2^{n-1} *edges so that a vertex* $\boldsymbol{u} = 0u_{n-1}u_{n-2}\ldots u_1$ *in* CQ^0_{n-1} *is connected to a vertex* $\boldsymbol{v} = 1v_{n-1}v_{n-2}\ldots v_1$ *in* CQ^1_{n-1} *if and only if* (1) $u_{n-2} = v_{n-2}$ *if n is even, and* (2) $(u_{2i}u_{2i-1}, v_{2i}v_{2i-1}) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$ *for all* $1 \leq i \leq \lfloor (n-1)/2 \rfloor$.

We depict CQ_3 and CQ_4 in Figure 1. It was proved that CQ_n is n-connected [6].

In CQ_n, a vertex $\boldsymbol{u} = u_nu_{n-1}\ldots u_1$ is said to be adjacent to a vertex $\boldsymbol{v} = v_nv_{n-1}\ldots v_1$ along the ith dimension, $0 \leq i \leq n$, if the following four conditions are all satisfied：

(i) $u_i \neq v_i$,

(ii) $u_j = v_j$ for all j, $i + 1 \leq j \leq n$,

(iii) $u_{2k}u_{2k-1} \sim v_{2k}v_{2k-1}$ for all k, $1 \leq k \leq \lfloor (i-1)/2 \rfloor$, and

(iv) $u_{i-1} = v_{i-1}$ if i is even.

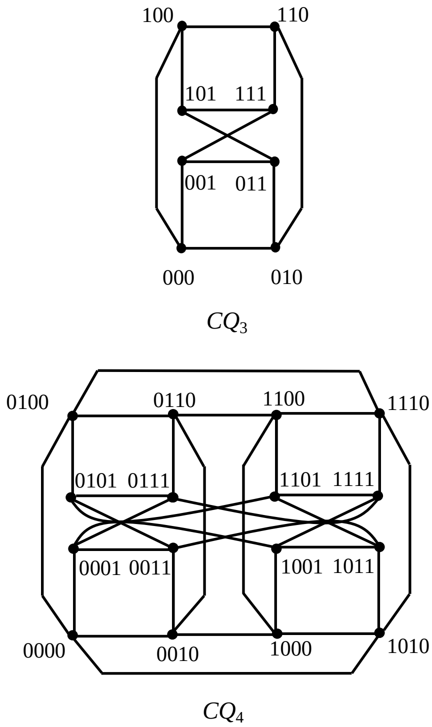

Figure 1. CQ_3 and CQ_4

A graph G is *f-fault-tolerant Hamiltonian* (respectively, *f-fault-tolerant Hamiltonian connected*) or simply *f-Hamiltonian* (respectively, *f-Hamiltonian connected*) if it remains Hamiltonian (respectively, Hamiltonian connected) after removing at most f vertices and/or edges. The following lemmas are necessary to derive our main result.

Lemma 1. [4] *If* $n \geq 3$, *for any distinct vertices* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in* CQ_n *and for any integer l with* $\lceil (n+1)/2 \rceil + 1 \leq l \leq 2^n - 1$, *there exists a path of length l between* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in* CQ_n.

Lemma 2. [5] *For any integer n,* $n \geq 3$, CQ_n *is* $(n-2)$-*fault-tolerant Hamiltonian and* $(n-3)$-*fault-tolerant Hamiltonian connected.*

3. Path embedding of various lengths with faulty vertices

The following theorem presents the main result of this paper.

Theorem 1. *Let F be a set of faulty vertices of* CQ_n, *where* $n \geq 5$ *and* $|F| \leq n - 3$. *Moreover, let* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *be any two distinct vertices of* $CQ_n - F$. *Then, there exists a path P of length l joining* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in* $CQ_n - F$ *for each integer l satisfying* $2n - 5 \leq l \leq 2^n - |F| - 1$.

Proof. We prove this theorem by induction. At first, we have the validity of the induction base on CQ_5 by brute force with a computer program. Here, we show that there exists paths of various lengths joining $\boldsymbol{x}$ and $\boldsymbol{y}$ in $CQ_n - F$ with $|F| = n - 3$ for $n \geq 6$. With a similar argument, it is easy to show the correctness of this theorem for $|F| < n - 3$. Let $F = \{v_1, v_2, \ldots, v_{n-3}\}$ be the set of faulty vertices, and let $F^* = F - \{\boldsymbol{v}_1\}$. Besides, let $F_0 = F \cap V(CQ^0_{n-1})$ and $F_1 = F \cap V(CQ^1_{n-1})$. Without loss of generality, assume $|F_1| \leq |F_0|$, and assume $\boldsymbol{x}$ is in CQ^0_{n-1} if $\boldsymbol{x}$ and $\boldsymbol{y}$ are in different subcubes. For convenience of discussion, we let $f = |F|$, $f_0 = |F_0|$, $f_1 = |F_1|$, and $f^* = |F^*|$. Consider the following five cases.

Case 1. $\{\boldsymbol{x}, \boldsymbol{y}\} \subset V(CQ^0_{n-1})$ and $f_0 = f = n - 3$ and $f_1 = 0$. The following two subcases have to be considered.

Subcase 1.1. $2n - 5 \leq l \leq 2^{n-1} + 1$. In this subcase, we can find a neighbor $(\boldsymbol{x})^n$ of $\boldsymbol{x}$ and a neighbor $(\boldsymbol{y})^n$ of $\boldsymbol{y}$ in CQ^1_{n-1}. Then, by Lemma 1, we can obtain a path S of CQ^1_{n-1} joining $(\boldsymbol{x})^n$ and $(\boldsymbol{y})^n$ with $\lceil n / 2 \rceil + 1 \leq l(S) \leq 2^{n-1} - 1$. Note that $\lceil n / 2 \rceil + 1 < 2(n - 1) - 5 = 2n - 7$ for $n \geq 6$. Then, $P = \langle \boldsymbol{x}, (\boldsymbol{x})^n, S, (\boldsymbol{y})^n, \boldsymbol{y} \rangle$ can be a path of $CQ_n - F$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2n - 5 \leq l \leq 2^{n-1} + 1$. Figure 2(a) illustrates this subcase.

Subcase 1.2. $2^{n-1} + 2 \leq l \leq 2^n - f - 1$. By Lemma 2, we can obtain a Hamiltonian path R of $CQ^0_{n-1} - F^*$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $l(R) = 2^{n-1} - f^* - 1$. We can write R as $\langle \boldsymbol{x}, R_1, \boldsymbol{a}, \boldsymbol{v}_1, \boldsymbol{b}, R_2, \boldsymbol{y} \rangle$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are adjacent to $\boldsymbol{v}_1$. Note that $\boldsymbol{x} = \boldsymbol{a}$ if $l(R_1) = 0$ and $\boldsymbol{y} = \boldsymbol{b}$ if $l(R_2) = 0$. Then, $l(R_1) + l(R_2) = 2^{n-1} - f^* - 3 = 2^{n-1} - f_0 - 2$. In CQ^1_{n-1}, we can find the neighbors $(\boldsymbol{a})^n$ and $(\boldsymbol{b})^n$ of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. By Lemma 1, there exists a path S of CQ^1_{n-1} joining $(\boldsymbol{a})^n$ and $(\boldsymbol{b})^n$ with $\lceil n / 2 \rceil + 1 \leq n - 1 \leq l(S) \leq 2^{n-1} - 1$. Then, we can set $P = \langle \boldsymbol{x}, R_1, \boldsymbol{a}, (\boldsymbol{a})^n, S, (\boldsymbol{b})^n, \boldsymbol{b}, R_2, \boldsymbol{y} \rangle$. Since $(2^{n-1} - f_0 - 2) + (n - 1) + 2 \leq 2^{n-1} + 2$ and $f_0 = f$, P can be a path of $CQ_n - F$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2^{n-1} + 2 \leq l \leq 2^n - f - 1$. Figure 2(b) illustrates this subcase.

Case 2. $\boldsymbol{x} \in V(CQ^0_{n-1})$ and $\boldsymbol{y} \in V(CQ^1_{n-1})$ and $f_0 = n - 3$ and $f_1 = 0$. The following two subcases have to be considered.

Subcase 2.1. $2n - 5 \leq l \leq 2^{n-1} + 1$. Since $\boldsymbol{x}$ has $n - 1$ neighbors in CQ^0_{n-1}, we can find a neighbor $\boldsymbol{t}$ of $\boldsymbol{x}$ with $\boldsymbol{t} \notin F_0$ and $(\boldsymbol{t})^n \neq \boldsymbol{y}$. By Lemma 1, there exists a path S of CQ^1_{n-1} joining $(\boldsymbol{t})^n$ and $\boldsymbol{y}$ with $\lceil n / 2 \rceil + 1 < 2n - 7 \leq l(S) \leq 2^{n-1} - 1$. Then, $P = \langle \boldsymbol{x}, \boldsymbol{t}, (\boldsymbol{t})^n, S, \boldsymbol{y} \rangle$ can be a path of $CQ_n - F$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2n - 5 \leq l \leq 2^{n-1} + 1$. Figure 3(a) illustrates this subcase.

Subcase 2.2. $2^{n-1} + 2 \leq l \leq 2^n - f - 1$. By induction, we have a path R of $CQ^0_{n-1} - F^*$ joining $\boldsymbol{x}$ and $\boldsymbol{v}_1$ with $2n - 7 \leq l(R) \leq 2^{n-1} - f^* - 1 = 2^{n-1} - f_0$. We can write path R as $\langle \boldsymbol{x}, R_1, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{v}_1 \rangle$, then R_1 is a path joining $\boldsymbol{x}$ and $\boldsymbol{a}$ with $2n - 10 \leq l(R_1) \leq 2^{n-1} - f_0 - 3$. Then, we have two conditions that should be considered.

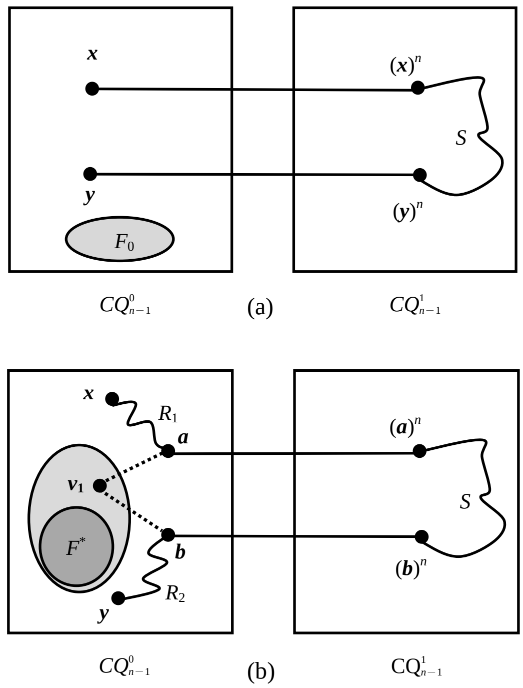

Figure 2. Illustration of Case 1

Condition 2.2.1. $(\boldsymbol{c})^n \neq \boldsymbol{y}$. Let $R^* = \langle \boldsymbol{x}, R_1, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle$, then we have $2n - 8 \leq l(R^*) \leq 2^{n-1} - f_0 - 1$. By Lemma 1, CQ^1_{n-1} has a path S joining $(\boldsymbol{c})^n$ and $\boldsymbol{y}$ with $\lceil n / 2 \rceil + 1 \leq l(S) \leq 2^{n-1} - 1$. We set $P = \langle \boldsymbol{x}, R_1, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, (\boldsymbol{c})^n, S, \boldsymbol{y} \rangle$. Since $(2n - 8) + (\lceil n / 2 \rceil + 1) + 1 < 2^{n-1} + 2$ for $n \geq 6$, P can be a path of $CQ_n - F$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2^{n-1} + 2 \leq l \leq 2^n - f - 1$. See Figure 3(b) for illustration.

Condition 2.2.2. $(\boldsymbol{c})^n = \boldsymbol{y}$. In this condition, we can find the neighbors $(\boldsymbol{a})^n$ and $(\boldsymbol{b})^n$ of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, in CQ^1_{n-1}. By induction, there exists a path S in $CQ^1_{n-1} - \{\boldsymbol{y}\}$ joining $(\boldsymbol{a})^n$ and $(\boldsymbol{b})^n$ with $2n - 7 \leq l(S) \leq 2^{n-1} - 2$. Moreover, path R_1 is with $2n - 10 \leq l(R_1) \leq 2^{n-1} - f_0 - 3$. Then, $P = \langle \boldsymbol{x}, R_1, \boldsymbol{a}, (\boldsymbol{a})^n, S, (\boldsymbol{b})^n, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{y} \rangle$ can be a path of $CQ_n - F$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2^{n-1} + 2 \leq l \leq 2^n - f - 1$. See Figure 3(c) for illustration.

Case 3. $\{\boldsymbol{x}, \boldsymbol{y}\} \subset V(CQ^1_{n-1})$ and $f_0 = n - 3$ and $f_1 = 0$. In this case, we need to consider the following two subcases.

Subcase 3.1. $2n - 5 \leq l \leq 2^{n-1} - 1$. By Lemma 1, we can obtain a path S of CQ^1_{n-1} joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $\lceil n / 2 \rceil + 1 \leq l(S) \leq 2^{n-1} - 1$. Note that $\lceil n / 2 \rceil + 1 < 2n - 5$ for $n \geq 6$. Then, $P = \langle \boldsymbol{x}, S, \boldsymbol{y} \rangle$ can be a path of CQ^1_{n-1} with $2n - 5 \leq l \leq 2^{n-1} - 1$. See Figure 4(a) for illustration.

Subcase 3.2. $2^{n-1} \leq l \leq 2^n - f - 1$. Consider the following conditions.

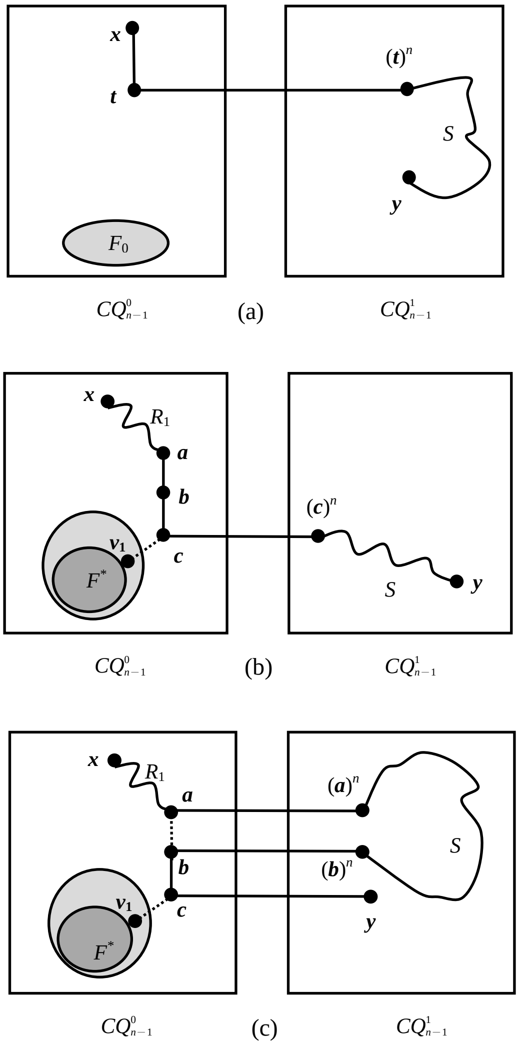

Figure 3. Illustration of Case 2

Condition 3.2.1. $|\{(\boldsymbol{x})^n, (\boldsymbol{y})^n\} \cap F_0| \leq 1$. Assume $(\boldsymbol{x})^n \notin F_0$. By induction, we have a path R of $CQ^0_{n-1} - F^*$ joining $(\boldsymbol{x})^n$ and $\boldsymbol{v_1}$ with $2n - 7 \leq l(R) \leq 2^{n-1} - f^* - 1 = 2^{n-1} - f_0$. We can write path R as $\langle(\boldsymbol{x})^n, R_1, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{v_1}\rangle$. If $(\boldsymbol{c})^n \neq \boldsymbol{y}$, by induction, $CQ^1_{n-1} - \{\boldsymbol{x}\}$ has a path S joining $(\boldsymbol{c})^n$ and $\boldsymbol{y}$ with $2n - 7 \leq l(S) \leq 2^{n-1} - 2$. We set $P = \langle\boldsymbol{x}, (\boldsymbol{x})^n, R_1, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, (\boldsymbol{c})^n, S, \boldsymbol{y}\rangle$, and P can be a path of $CQ_n - F$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2^{n-1} \leq l \leq 2^n - f - 1$. See Figure 4(b) for illustration. If $(\boldsymbol{c})^n = \boldsymbol{y}$, by induction, $CQ^1_{n-1} - \{\boldsymbol{x}, \boldsymbol{y}\}$ has a path S joining $(\boldsymbol{a})^n$ and $(\boldsymbol{b})^n$ with $2n - 7 \leq l(S) \leq 2^{n-1} - 3$. We set $P = \langle\boldsymbol{x}, (\boldsymbol{x})^n, R_1, \boldsymbol{a}, (\boldsymbol{a})^n, S, (\boldsymbol{b})^n, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{y}\rangle$, and P can be a path of $CQ_n - F$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2^{n-1} \leq l \leq 2^n - f - 1$. See Figure 4(c) for illustration.

Condition 3.2.2. $|\{(\boldsymbol{x})^n, (\boldsymbol{y})^n\} \cap F_0| = 2$. Since $\boldsymbol{x}$ has $n - 1$ neighbors in CQ^1_{n-1}, we can find a neighbor $\boldsymbol{a}$ of $\boldsymbol{x}$ with $(\boldsymbol{a})^n \notin F_0$. By induction, there exists a path R of $CQ^0_{n-1} - F^*$ joining $(\boldsymbol{a})^n$ and $\boldsymbol{v_1}$ with $2n - 7 \leq l(R) \leq 2^{n-1} - f_0$. We can write path R as $\langle(\boldsymbol{a})^n, R_1, \boldsymbol{b}, \boldsymbol{v_1}\rangle$. Since $(\boldsymbol{y})^n \in F_0$, we have $(\boldsymbol{b})^n \neq \mathbf{y}$. By induction, there exists a path S of $CQ^1_{n-1} - \{\boldsymbol{x}, \boldsymbol{a}\}$ joining $(\boldsymbol{b})^n$ and $\mathbf{y}$ with $2n - 7 \leq l(S) \leq 2^{n-1} - 3$. We set $P = \langle\boldsymbol{x}, \boldsymbol{a}, (\boldsymbol{a})^n, R_1, \boldsymbol{b}, (\boldsymbol{b})^n, S, \boldsymbol{y}\rangle$, and P can be a path of $CQ_n - F$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2^{n-1} \leq l \leq 2^n - f - 1$. See Figure 4(d) for illustration.

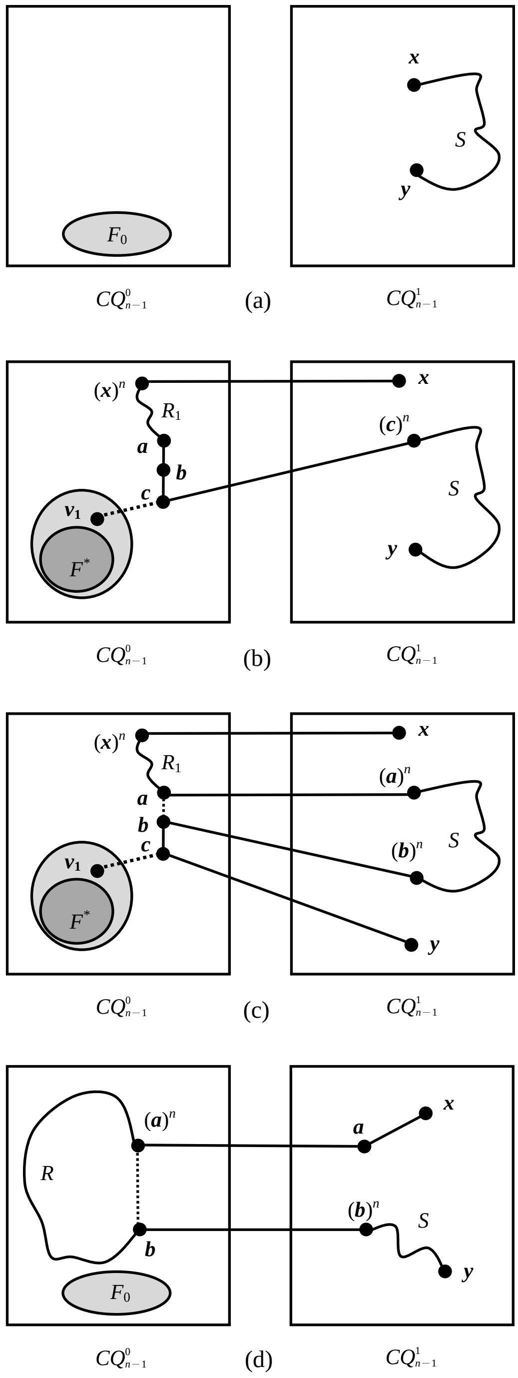

Figure 4. Illustration of Case 3

Case 4. $\{\boldsymbol{x}, \boldsymbol{y}\} \subset V(CQ^i_{n-1})$, $i \in \{0, 1\}$, and $1 \leq f_1 \leq f_0 \leq n - 4$. By induction, we can obtain a path R of $CQ^i_{n-1} - F_i$, $i \in \{0, 1\}$, joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2n - 7 \leq l(R) \leq 2^{n-1} - f_i - 1$. Certainly, R is a path of $CQ_n - F$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2n - 5 \leq l(R) \leq 2^{n-1} - f_i - 1$. We can write path R as $\langle\boldsymbol{x}, R_1, \boldsymbol{a}, \boldsymbol{b}, R_2, \boldsymbol{y}\rangle$ for some vertices $\boldsymbol{a}$ and $\boldsymbol{b}$ with $\{(\boldsymbol{a})^n, (\boldsymbol{b})^n\} \cap F_1 = \varnothing$. Then, by induction, there exists a path S of CQ_{n-1}^{1-i} joining $(\boldsymbol{a})^n$ and $(\boldsymbol{b})^n$ with $2n - 7 \le l(S) \le 2^{n-1} - f_{1-i} - 1$. Since $(2n - 7) + (2n - 7) + 1 < 2^{n-1} - f_i$ for $n \ge 6$, $P = \langle \boldsymbol{x}, R_1, \boldsymbol{a}, (\boldsymbol{a})^n, S, (\boldsymbol{b})^n, \boldsymbol{b}, R_2, \boldsymbol{y}\rangle$ can be a path of $CQ_n - F$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2^{n-1} - f_i \le l \le 2^n - f - 1$. See Figure 5 for illustration.

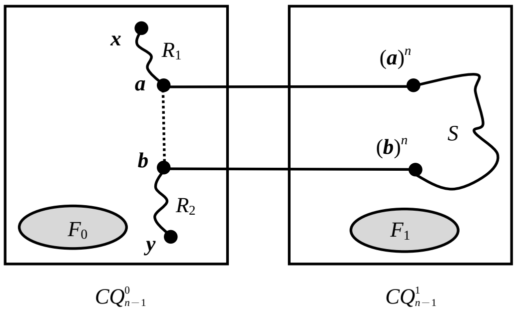

Figure 5. Illustration of Case 4

Case 5. $\boldsymbol{x} \in V(CQ_{n-1}^0)$ and $\boldsymbol{y} \in V(CQ_{n-1}^1)$ and $1 \le f_1 \le f_0 \le n - 4$. The following two subcases have to be considered.

Subcase 5.1. $2n - 5 \le l \le 2^{n-2}$. Since $\boldsymbol{x}$ has $n - 1$ neighbors in CQ_{n-1}^0, we can find a neighbor $\boldsymbol{t}$ of $\boldsymbol{x}$ with $\boldsymbol{t} \notin F_0$, $(\boldsymbol{t})^n \notin F_1$, and $(\boldsymbol{t})^n \ne \boldsymbol{y}$. By induction, we can obtain a path S of $CQ_{n-1}^1 - F_1$ joining $(\boldsymbol{t})^n$ and $\boldsymbol{y}$ with $2n - 7 \le l(S) \le 2^{n-2} - 2$. Then, $P = \langle \boldsymbol{x}, \boldsymbol{t}, (\boldsymbol{t})^n, S, \boldsymbol{y}\rangle$ can be a path of $CQ_n - F$ with $2n - 5 \le l \le 2^{n-2}$. Figure 6(a) illustrates this subcase.

Subcase 5.2. $2^{n-2} + 1 \le l \le 2^n - f - 1$. Let $\boldsymbol{a}$ be a vertex of $CQ_{n-1}^0 - F_0$ with $(\boldsymbol{a})^n \notin F_1$ and $(\boldsymbol{a})^n \ne \boldsymbol{y}$. By induction, we can obtain a path R of $CQ_{n-1}^0 - F_0$ joining $\boldsymbol{x}$ and $\boldsymbol{a}$ with $2n - 7 \le l(R) \le 2^{n-1} - f_0 - 1$. Also by induction, there exists a path S of $CQ_{n-1}^1 - F_1$ joining $(\boldsymbol{a})^n$ and $\boldsymbol{y}$ with $2n - 7 \le l(S) \le 2^{n-1} - f_0 - 1$. We set $P = \langle \boldsymbol{x}, R, \boldsymbol{a}, (\boldsymbol{a})^n, S, \boldsymbol{y}\rangle$. Since $(2n - 7) + (2n - 7) + 1 < 2^{n-2} + 1$ for $n \ge 6$, P can be a path of $CQ_n - F$ joining $\boldsymbol{x}$ and $\boldsymbol{y}$ with $2^{n-2} + 1 \le l \le 2^n - f - 1$. See Figure 6(b) for illustration.

The above argument of all cases completes the proof.

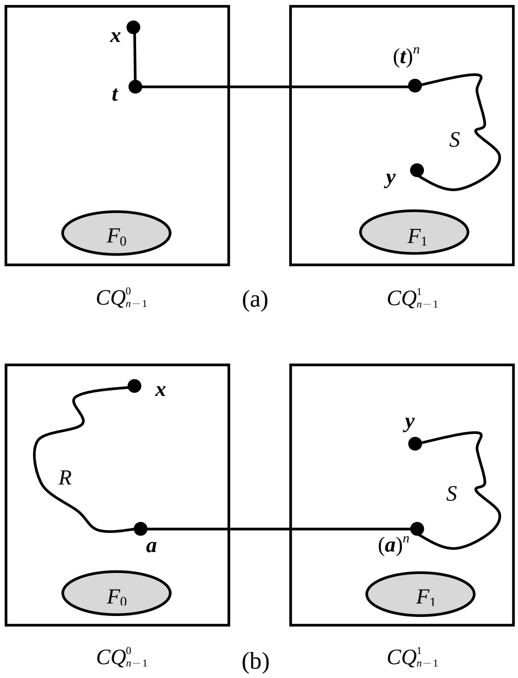

Figure 6. Illustration of Case 5

4. Concluding remarks

In this paper, we investigate the path embedding of various lengths in crossed cubes with faulty vertices. For an n-dimensional crossed cube CQ_n, if the number of faulty vertices is f with $f \le n - 3$, then there exists a fault-free path of length l between any two distinct fault-free vertices in CQ_n for each integer l satisfying $2n - 5 \le l \le 2^n - f - 1$. Motivated by this result, our future work will be devoted to find paths of shorter lengths with the same number of faults in the crossed cube.

References

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