Constructing Vertex-Disjoint Paths in Alternating Group Graphs*

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Abstract

This work derives a routing algorithm for constructing a container of width 2(n−2) between a pair of vertices in an alternating group graph with connectivity 2(n−2). Based on the provided algorithm, the wide diameter of an n-dimensional alternating group graph can be computed as its diameter plus 1 or 2.

1 Introduction*

There has been plenty of research on topological properties of interconnection networks by constructing vertex-disjoint paths, such as Hamiltonian laceability [1], performance [2], reliability [16]. The wide diameter [9, 15], fault tolerance [11, 12] and Rabin number [16]. Therefore, constructing vertex-disjoint paths [8, 9, 14, 15] becomes an increasingly important issue on fault-tolerant ability [10, 11, 12], reliability [16], maximum parallelism [4] and minimum transmission delay [3].

Jwo et al. [6] proposed alternating group graphs and showed that they have some favorable properties such as small diameter, rich connectivity, vertex symmetry, edge symmetry, embeddability, broadcastability, hierarchical structure, and hamiltonicity. In comparison with star graphs, an n-dimensional alternating group graph has half the number of vertices and approximately twice its degree. Alternating group graphs provide strong fault tolerance [10, 11, 12, 13], hamiltonicity [13]. Lai and Tsay [7] provided communication algorithms for all-to-all broadcast on an alternating group graph with all-port and store-and-forward routing. Lin and Chiu [8] derived a routing scheme for constructing vertex-disjoint paths, but paths may coincide on one vertex in the constructed paths.

Constructing vertex-disjoint paths in an interconnection network is very important issue for measuring reliability, fault tolerant ability, parallelism and transmission delay of an interconnection network. The wide-diameters of (n, k)-star graphs, and enhanced pyramid networks have been computed by Lin and Duh [9], and Hsieh and Duh [5], respectively. This work proposes a novel algorithm to construct vertex disjoint paths and determine wide diameters of alternating group graphs.

The remainder of this paper is organized as follows. Section 2 formally describes some background of graphs and the topological properties of an n-dimensional alternating group graph. Section 3 shows path routing rules, and discusses the lengths of the constructed paths. Section 4 first presents a routing algorithm for constructing $2(n-2)$ vertex-disjoint paths between every vertex pair in an alternating group graph. Then, the wide diameter of an alternating group graph is also computed as its diameter plus 1 or 2. Conclusion is finally drawn in Section 5.

2 Background & Notations

Let G denote a graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. Two vertices u and v are *adjacent* when they are joined by an edge e, where $u, v \in V(G)$ and $e \in E(G)$. All vertices adjacent to a vertex are its *neighbors*. The *distance* from vertex u to vertex v, represented by $d(u, v)$, refers to the length of a shortest path from u to v in G. The *diameter* of G, denoted by $d(G)$, is defined as the maximum distance for all pairs of distinct vertices u and v in G. A graph is *connected* when at least one path exists between any two vertices in it. The *vertex connectivity* (or *connectivity*) of a graph is

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defined as the minimum number of vertices whose removal renders it disconnected or trivial. Let $\kappa(G)$ (or κ) be the connectivity of G. According to Merger's theorem, at least κ vertex-disjoint paths exist between distinct vertices u and v in G. A set of κ vertex-disjoint paths between vertices u and v in G, denoted by $C_\kappa(u, v)$, is a *container* of width κ between u and v. The length of a $C_\kappa(u, v)$, denoted by $l(C_\kappa(u, v))$, is defined as the length of the longest path in $C_\kappa(u, v)$. A best-container between u and v, denoted by $C^*_\kappa(u, v)$, is the container with the shortest length among all $C_\kappa(u, v)$s. Let $d_\kappa(u, v)$ indicate the *κ-wide distance* (or *wide distance*) from u to v; thus, $d_\kappa(u, v)=l(C^*_\kappa(u, v))$. The *$\kappa$-wide diameter* (or *wide diameter*) of G, denoted by $d_\kappa(G)$, is defined as the maximum of $d_\kappa(u, v)$s for all pairs of distinct vertices u and v in G.

Let AG_n denote the n-dimensional alternating group graph. The vertex set $V(AG_n)$ is defined as $\{\rho_1\rho_2...\rho_n \mid \rho_1\rho_2...\rho_n$ is an even permutation of 1, 2, …, $n\}$, and the link set $E(AG_n) = \{(\rho_1\rho_2 \ldots \rho_n, \rho_2\rho_i\rho_3\rho_4\ldots\rho_{i-2}\rho_{i-1}\rho_1\rho_{i+1}\rho_{i+2} \ldots \rho_n), (\rho_1\rho_2 \ldots \rho_n, \rho_i\rho_1\rho_3\rho_4 \ldots \rho_{i-2}\rho_{i-1}\rho_2\rho_{i+1}\rho_{i+2}\ldots\rho_n) \mid \rho_1\rho_2\ldots\rho_n \in V(AG_n)$ and $3\leq i \leq n\}$. Operation i^+ (i^-) shifts ρ_1, ρ_2, ρ_i left (right) cyclically. *Symbol sequence* $\langle s_1, s_2, \ldots, s_m\rangle$ orderly fixes symbols $s_1, s_2 \ldots s_m$, where $s_1, s_2 \ldots s_m \in \{3, 4, \ldots, n\}$. *Correcting sequence* $\langle s^\alpha$: $s_1, s_2, \ldots s_x, \eta$^$w, s_{x+1}, s_{x+2}, \ldots s_y\rangle$ is a special symbol sequence that executes operation s^α, fixes symbols $s_1, s_2 \ldots s_x$, puts symbol η on w position, and then corrects $s_{x+1}, s_{x+2}, \ldots s_y$ orderly, where $x, y \geq 0$, $\alpha \in \{+, -\}$, $\eta \in \{1, 2\}$, $s, s_1, s_2, \ldots s_x, s_{x+1}, s_{x+2}, \ldots s_y$, and $w \in \{3, 4, \ldots, n\}$.

Moreover, symbol ρ_i in label $\rho_1\rho_2...\rho_n$ is as follows: (1) *fixed* if $\rho_i=i$ (notably, position i is the *desired position* for symbol i); (2) *misplaced* if $p_i\neq i$. For instance, with ρ=2315647 in AG_7, ρ_7=7 is a fixed symbol. Conversely, ρ_2=3, ρ_4=5 ρ_5=6, and ρ_6=4 are misplaced symbols. Significantly, symbols 1 and 2 are not misplaced symbols and not fixed symbols. Symbols 1 and 2 are automatically corrected after correcting 3, 4, $\cdots$, n because it should be an even permutation.

Vertex ε = 12...n is the *identity* vertex whose $\rho_i = i$ for all $1 \leq i \leq k$. Since AG_n is vertex symmetry, constructing paths between two distinct vertices can be regarded as constructing paths from a source vertex to ε [6]. This work attempts to build $2(n-2)$ vertex-disjoint paths from vertex ρ to ε by correcting each non-fixed symbol to a fixed symbol. A *cycle representation* represents all non-fixed symbols of a vertex identifier. Notably, non-fixed symbols may include symbols 1 and 2, and misplaced symbols. For example, when $\rho_x = y$, $\rho_y = z$, and $\rho_z = x$, these non-fixed symbols can be presented as a cycle $(x\ y\ z)$, where x denotes the cycle *head*, indicating that the desired position of a symbol is occupied by the next symbol in the cycle. Let vertex $\rho = C_1C_2\ldots C_ce_1e_2\ldots e_l$, where $C_i = (s_{i,1}\ s_{i,2} \ldots s_{i,ki})$ and the length of C_i be k_i, $1 \leq i \leq c$. For instance, ρ = 2315647 in AG_n can be represented as (1 2 3)(4 5 6)7. Let m and f be the numbers of misplaced and fixed symbols of vertex ρ, respectively. Also, let l be $n - |C_1\cup C_2\cup\ldots\cup C_c|$. The cycle representation then comprises m+f+2 symbols in total, implying that m symbols are not in their desired positions and f symbols are in their desired positions. Without loss of generality, if a cycle contains symbol 1, then the cycle is C_1 and symbol 1 is $s_{1,1}$ by rotating C_1. Again, if no cycles contain symbol 1, then the cycle containing symbol 2 is cycle 1 and symbol 2 is $s_{1,1}$. If symbols 1 and 2 belong to different cycles, then the cycle containing symbol 2 is cycle 2 and symbol 2 is $s_{2,1}$.

Interestingly, Jwo et al., showed in 1993 that $d(\rho, \varepsilon) = n+c-l$ if $\rho_1 = 1$ and $\rho_2 = 2$; $d(\rho, \varepsilon) = n+c-l-3$ if $\rho_1 = 2$ and $\rho_2 = 1$; $d(\rho, \varepsilon) = n+c-l-2$ if $\rho_1 \neq 1$ and $\rho_2 = 2$; $n+c-l-2$ if $\rho_1 = 1$ and $\rho_2 \neq 2$; $d(\rho, \varepsilon) = n+c-l-3$ if $1, 2 \in C_i$, $1 \leq i \leq k$ and $|C_i| \geq 3$, and $d(\rho, \varepsilon) = n+c-l-4$ if $1 \in C_i$ and $2 \in C_j$, $1 \leq i \neq j \leq k$. Thus, $d(AG_n) = \lfloor 3(n-2)\rfloor/2$ [6]. In other words, $d(AG_n) = (3n-7)/2$ ($(3n-6)/2$) if n is odd (even).

3 Path Routing Rules

For simplicity, let $\Pi(u, v)$ denote the path from vertex u to vertex v in an AG_n. To construct a path $\Pi(\rho, \varepsilon)$ in an AG_n, all misplaced symbols of ρ should be corrected one by one to transform the label of ρ into the label of ε. Significantly, after correcting all misplaced symbols, symbols 1 and 2 are automatically fixed in their desired position. The following routing rules are applied to fix symbol s. Each rule can join 1 or 2 edges (vertices) in a path.

R1:

- If symbol s is at position 1, do s^+ operation to fix s.
- If symbol s is at position 2, do s^- operation to fix s.

R2:

- If symbol s is not at position 1 or position 2 and at position p, then do p^+ (or p^-) to put s at position 2 (or position 1), and then apply R1.

Naturally, R1 contributes 1 edge and R2 contributes 2 edges in each constructed routing path. For example, let vertex ρ = 15432. Symbols 3, 4 and 5 are misplaced. To correct symbol 5, do 5^- operation according to R1 since 5 is at position 2. Thus, the first intermediate vertex, denoted by u, of the routing path is 21435. That is, symbol 5 is fixed. Notably, symbol 4 of u is at position 3 and rule R2 should be apply to correct symbol 4. By applying R2, execute 3^+ (or 3^-) and 4^- (or 4^+)

operations orderly and then the second and third intermediate vertices v = 14235 (or 42135) and w = 31245 (or 23145) are obtained. Restated, R1 takes 1 step and R2 takes 2 steps.

Lemma 1. Given two symbol sequences, the first (last) symbol of symbol sequences are a and c (b and d), respectively, and the order of b in the second symbol sequence is prior to a. If vertex ρ corrected by <a, …, c, …, d, …, b> and <c, …, b, …, a, …, d>, then the two paths constructed according to these two symbol sequences are vertex-disjoint to each other.

Proof. As shown in Fig. 1, ρ have two distinct neighbors ρ_α and ρ_β, ε have two distinct neighbors ε_α and ε_β. Paths $\Pi(u, v)$ and $\Pi(\rho_\beta, \varepsilon_\beta)$ are vertex-disjoint since every vertex in $\Pi(u, v)$ has $\rho_a = a$ and $\rho_b \neq b$ and every vertex in $\Pi(\rho_\beta, \varepsilon_\beta)$ has $\rho_a \neq a$ or $\rho_b = b$. Vertex ρ_α is different from every vertex in path $\Pi(w, \varepsilon_\beta)$ because symbols a, b, c and d of ρ_α are not yet fixed and each vertex in $\Pi(w, \varepsilon_\beta)$ has at least one of a, b, c and d is fixed. Vertex ε_α is different from every vertex in path $\Pi(\rho_\beta, x)$ because symbols a, b, c and d of ε_α are fixed and each vertex in $\Pi(\rho_\beta, x)$ has at least one of a, b, c and d is not fixed. Therefore, $\Pi(\rho_\alpha, \varepsilon_\alpha)$ and $\Pi(\rho_\beta, \varepsilon_\beta)$ are vertex-disjoint to each other. ■

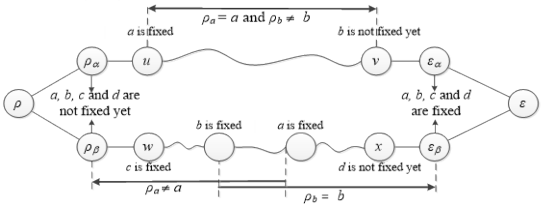

Fig. 1. The two paths constructed according to two symbol sequences <a, …, c, …, d, …, b> and <c, …, b, …, a, …, d> are vertex-disjoint.

Lemma 2. For any vertex $\rho = C_1C_2 \ldots C_ce_1e_2 \ldots e_l$ in AG_n, let $C_i = (s_{i,1}\ s_{i,2} \ldots s_{i,k_i})$ and $s_{i,1}$, $s_{i,2}$, …, s_{i,k_i} are all misplaced symbols, where $1 \le i \le c$. If two paths constructed according to $\langle s_{i,k_i}{}^+: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i}\rangle$ and $\langle s_{i,k_i}{}^-: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i}\rangle$, these two paths are vertex-disjoint except the beginning and ending vertices.

Proof. We prove this lemma constructively. Two $\Pi(\rho, \rho')$s are constructed by $\langle s_{i,k_i}{}^+: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i}\rangle$ and $\langle s_{i,k_i}{}^-: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i}\rangle$, respectively. Moreover, they apply operations $s_{i,k_i}{}^+$, $s_{i,1}{}^-$, $s_{i,2}{}^+$, … and $s_{i,k_i}{}^-$: $s_{i,1}{}^+$, $s_{i,2}{}^-$, … and they are therefore vertex-disjoint except the beginning and ending vertices. ■

Lemma 2 indicates that if two paths are constructed by $\langle s_{i,k_i}{}^+: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i}\rangle$ and $\langle s_{i,k_i}{}^-: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i}\rangle$, which are composed of all misplaced symbols in cycle $C_i = (s_{i,1}\ s_{i,2} \ldots s_{i,k_i})$, these two paths must has the same ending vertex. In other words, if the cycle representation of vertex ρ has two or more cycles, all misplaced symbols in the first selected cycle C_i cannot be completely fixed prior than any other cycle for building vertex-disjoint paths. According to Lemmas 1 and 2, we have the following corollary.

Corollary 3. Let $C_i = (s_{i,1}\ s_{i,2} \ldots s_{i,k_i})$, where each of $s_{i,1}$, $s_{i,2}$, …, and s_{i,k_i} is not symbols 1 and 2 for $1 \le i \le c$. The paths constructed by the following sequences are vertex-disjoint.

$\langle s_{i,1}{}^+: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, C_1, C_2, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,1}{}^-: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, C_1, C_2, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,2}{}^+: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, C_1, C_2, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,
$\langle s_{i,2}{}^-: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, C_1, C_2, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,
…
$\langle s_{i,k_i}{}^+: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, C_1, C_2, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$,
$\langle s_{i,k_i}{}^-: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, C_1, C_2, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$.

4 Construction of Disjoint Paths

Recall that our goal is to construct a $C_{2(n-2)}(\rho, \varepsilon)$. Based on the structure and adjacency rules of AG_n, symbols in ρ_1 and ρ_2 directly determine each constructed path from ρ to ε. Hence, we divide all vertices in AG_n into three groups with some subgroups according to symbols in ρ_1 and ρ_2 and the content of C_1. Naturally, we thus need to build a $C_{2(n-2)}(\rho, \varepsilon)$ from a vertex ρ in each subgroup to ε. Three cases and some subcases of them should be considered in the following subsections.

4.1 Case 1: $\rho 1, \rho 2 \in \{1, 2\}$

Let $\sigma_\alpha(C_i)$ denote that performs a left circular shift on C_i α times, where $0 \le \alpha \le k_i - 1$ and $1 \le i \le c$. Hence, $\sigma_\alpha(C_i)$ is a variant cycle of C_i and $\sigma_0(C_i) = C_i$. Assume C_i is the first selected cycle among all c cycles. According to Lemma 2, all misplaced symbols in C_i cannot be completely fixed prior than any other cycle for building vertex-disjoint paths. In other words, at least one misplaced symbol in C_i should be fixed after each misplaced symbol in C_h, where $1 \le h \le c$ and $h \neq i$, for building vertex-disjoint paths. In order to construct vertex-disjoint paths, we first divide $\sigma_\alpha(C_i)$ into two parts and then combine them with the reminding cycles to form a symbol sequence as $\langle S_{\alpha,il}, C_1, C_2, \ldots, C_{i-1}, C_{i+1}, \ldots, C_c, S_{\alpha,ir}\rangle$. Thus, $\sigma_\alpha(C_i) = (S_{\alpha,il}\ S_{\alpha,ir})$, where $|S_{\alpha,il}| \ge 1$, $|S_{\alpha,ir}| \ge 1$. Two subcases are discussed in the following.

Case 1.1: $\rho_1 = 1, \rho_2 = 2$

In this case, $\{1, 2\} \not\subset C_1 \cup C_2 \cup \ldots \cup C_c$. As mentioned above, $(|C_1| + |C_2| + \ldots + |C_c|) = m$ symbol sequences can be built. Each symbol sequence $\langle S_{\alpha,il}, C_1, C_2, \ldots, C_{i-1}, C_{i+1}, \ldots, C_c, S_{\alpha,ir}\rangle$ can construct 2 paths which are an *m-pair* because symbol s in one path apply s^+ (s^-) operation and s in the other path must apply s^- (s^+) operation. Thus, $2m$ paths are constructed. Additionally, each fixed symbol can also be used to construct 2 paths. Unlike the misplaced symbols, each fixed symbol e_j, where $1 \le j \le l$ and $e_j \notin \{1, 2\}$, form 2 correcting sequences $\langle e_j{}^+: C_1, C_2, \ldots, C_c, e_j\rangle$ and

$\langle e_j^-$: C_1, C_2, …, C_c, $e_j\rangle$ for building two paths. These two paths also form an *f-pair* because if every vertex in one path has symbol 1 at position e_j, then every vertex in the other path has symbol 2 at position e_j. Thus, $2f$ paths are constructed since $f = l - 2$. Totally, $2m + 2f = 2(n - 2)$ paths are established.

Case 1.2: $\rho_1 = 2$, $\rho_2 = 1$

Without loss of generality, assume $\{1, 2\} \subseteq C_1$. Hence, there is no misplaced symbol in C_1. Thus, only $(|C_2| + \ldots + |C_c|) = m$ symbol sequences can be established. Very similar to Case 1.1, each symbol sequence $\langle S_{\alpha,il}$, C_2, …, C_{i-1}, C_{i+1}, …, C_c, $S_{\alpha,ir}\rangle$ can construct 2 paths. Therefore, all misplaced symbols can also construct $2m$ paths. Moreover, each fixed symbol can also be used to construct 2 paths. The method of constructing paths by fixed symbols is the same as Case 1.1. Therefore, all fixed symbols can construct $2f$ paths. Totally, $2m + 2f = 2(n - 2)$ paths are established.

Consequently, in Case 1, $2m + 2f = 2(n - 2)$ paths are built. In addition, Lemma 4 shows that these $2(n - 2)$ paths are vertex-disjoint.

Lemma 4. The $2(n - 2)$ paths constructed in Case 1 are vertex-disjoint.

Proof. As mention above, $2(n-2)$ paths are built in Case 1. By Lemma 1, each path of one *m*-pair is vertex-disjoint to each path of any other *m*-pair. Naturally, a symbol sequence can be regarded as a correcting sequence by definitions of them. With the aid of Lemma 2, two paths of an *m*-pair are vertex-disjoint. Therefore, the $2m$ paths constructed in Case 1 are vertex-disjoint.

The first step to construct $2f$ paths is to perform e_j^+ (e_j^-) operation, so symbol 1 (2) of each *f*-pair is placed at e_j position. Thus, two paths of an *f*-pair are vertex-disjoint and every two *f*-pairs are vertex-disjoint. Since $2m$ paths e_j position are always e_j, every *m*-pair is vertex-disjoint to every *f*-pair. Therefore, the built $2(n-2)$ paths are vertex-disjoint and form a $C_{2(n-2)}(\rho, \varepsilon)$. ■

Lemma 5 shows that the upper bound of $l(C_{2(n-2)}(\rho, \varepsilon))$ is $d(\mathrm{AG}_n)+1$.

Lemma 5. In Case 1, $l(C_{2(n-2)}(\rho, \varepsilon)) \leq d(\mathrm{AG}_n)+1$.

Proof. In Case 1, the upper bound of $l(C_{2(n-2)}(\rho, \varepsilon))$ is calculated as follows:

In Case 1.1, paths constructed by $\langle S_{\alpha,il}$, C_2, …, C_{i-1}, C_{i+1}, …, C_c, $S_{\alpha,ir}\rangle$ have length at most $m+c = (n-2)+\lfloor(n-2)/2\rfloor = (3n-7)/2$ ($(3n-6)/2) = d(\mathrm{AG}_n)$ if n is odd (even). Paths constructed by $\langle e_j^+$: C_1, C_2, …, C_c, $e_j\rangle$ and $\langle e_j^-$: C_1, C_2, …, C_c, $e_j\rangle$ have length at most $2+m+c = 2+(n-2-f)+\lfloor(n-2-f)/2\rfloor = 2+(n-3)+\lfloor(n-3)/2\rfloor = (3n-5)/2$ ($(3n-6)/2) = d(\mathrm{AG}_n)+1$ ($d(\mathrm{AG}_n)$) if n is odd (even). The length of the longest path built in Case 1.2 are the same as Case 1.1. Therefore, $l(C_{2(n-2)}(\rho, \varepsilon)) \leq d(\mathrm{AG}_n)+1$.■

4.2 Case 2: $\{\rho_1, \rho_2\} \cap \{1, 2\} = \{1\}$ or $\{2\}$

Let *head* (*tail*) represent the first (last) operation of constructing a path pair according to a symbol sequence. Hence, one path of the path pair takes $head^+$ ($tail^+$) and the other takes $head^-$ ($tail^-$). For ease of description, a path pair with head s is named *s-pair* which is composed of s^+*-path* and s^-*-path*. Every pair of paths in Case 2 is distributed a unique tail, occupied symbol 2 (or symbol 1), where tail $\in \{3, 4, \ldots, n\}$. In general, an s-pair is also assigned s as its tail. Notably, in the rest of this work, ρ and ε are excluded when considering vertex-disjoint path.

Let $\Pi_s(\Pi;\, s_f, s_m)$ denote a subpath of Π in which every vertex has fixed s_f and unfixed s_m. Specially, s_f (s_m) $= \eta$^w indicates the vertex has (has not) 1 or 2 at position w. Moreover, s_f (or s_m) = 0 reveals that no symbol is specified. Hence, s^+-path (s^--path) is composed of $\Pi_s(s^+\text{-path};\, 0, s_{i,\alpha})$ and $\Pi_s(s^+\text{-path};\, s_{i,\alpha}, tail)$ ($\Pi_s(s^-\text{-path};\, 0, s_{i,\alpha})$ and $\Pi_s(s^-\text{-path};\, s_{i,\alpha}, tail)$), where $1 \leq i \leq c$ and $1 \leq \alpha \leq k_i$.

However, the head and tail of ρ_2-pair is not the same because ρ_2 is already fixed after applying ρ_2^-. Thus, the tail of ρ_2-pair is t and should not be ρ_2. Undoubtedly, the tail of t-pair should be ρ_2. In other words, the head and tail of t-pair is also different.

Although container is not unique, this section provides a routing scheme for constructing a $C_{2(n-2)}(\rho, \varepsilon)$ for each ρ in Case 2. Referring to Table 3, Sections 2.1−2.4 discuss four cases in the following.

Case 2.1: ρ_1=1 and $\rho_2 \neq 2$

A set of correcting sequences is provided for building a $C_{2(n-2)}(\rho, \varepsilon)$ for each of k_1=2, k_1=3, and $k_1 \geq 4$.

Case 2.1.1 k_1=2

Since $V(\mathrm{AG}_n)$ is defined as an even permutation of 1, 2, …, n, if $k_1 = |C_1| = 2$, then $C_1 = (2\ s_{1,2})$ and $c \geq 2$. As mentioned above, the tail position of $s_{1,2}$-pair should not be $s_{1,2}$ and is distributed as $s_{2,1}$. The correcting sequences of $s_{1,2}$-pair are listed as follows:

$\langle s_{1,2}^+$: 2^$s_{2,1}$, $s_{2,2}$, $s_{2,3}$, …, s_{2,k_2}, C_3, C_4, …, C_c, $s_{1,2}$, $s_{2,1}\rangle$,

$\langle s_{1,2}^-$: 1^$s_{2,1}$, $s_{2,2}$, $s_{2,3}$, …, s_{2,k_2}, C_3, C_4, …, C_c, $s_{2,1}\rangle$.

The first vertex of $s_{1,2}^+$-path and the first vertex of $s_{1,2}^-$-path are disjoint, because they are two distinct neighbors of ρ. Symbol 1 (2) of the first vertex of $s_{1,2}^+$-path ($s_{1,2}^-$-path) is not at position $s_{2,1}$. Additionally, every vertex of $s_{1,2}^+$-path ($s_{1,2}^-$-path) excluding the first vertex has symbol 2 (1) at position $s_{2,1}$. Thus $s_{1,2}^+$-path and $s_{1,2}^-$-path are vertex-disjoint.

Since the tail of $s_{1,2}$-pair is $s_{2,1}$, the tail position of $s_{2,1}$-pair should not be $s_{2,1}$ and is distributed as $s_{1,2}$. The correcting sequences of $s_{2,1}$-pair are listed as follows:

$\langle s_{2,1}^+$: $s_{2,2}$, $s_{2,3}$, …, s_{2,k_2}, C_3, C_4, …, C_c, $s_{2,1}$, $s_{1,2}\rangle$,

$\langle s_{2,1}^-$: $s_{2,2}$, $s_{2,3}$, …, s_{2,k_2}, 1^$s_{1,2}$, C_3, C_4, …, C_c, $s_{2,1}$, $s_{1,2}\rangle$.

Actually, every vertex in $\Pi_s(s_{2,1}^+\text{-path};\, 0, s_{1,2})$

has symbol 2 at position $s_{1,2}$. That is, $\Pi_s(s_{2,1}^{+}$-path; 0, $s_{1,2})$ is disjoint to $\Pi_s(s_{2,1}^{-}$-path; $1{\wedge}s_{1,2}$, $s_{1,2})$. Let μ be the first symbol in $\langle C_3, C_4, \ldots, C_c, s_{2,1}\rangle$. Notably, C_3, C_4, …, and C_c may not exist. Symbol μ of every vertex in $\Pi_s(s_{2,1}^{-}$-path; 0, $1{\wedge}s_{1,2})$ is not fixed. Thus, $\Pi_s(s_{2,1}^{-}$-path; 0, $1{\wedge}s_{1,2})$ is disjoint to $\Pi_s(s_{2,1}^{+}$-path; μ, $s_{1,2})$. According to Lemma 2, $\Pi_s(s_{2,1}^{+}$-path; 0, $\mu)$ is disjoint to $\Pi_s(s_{2,1}^{-}$-path; 0, $1{\wedge}s_{1,2})$. Therefore, $s_{2,1}^{+}$-path and $s_{2,1}^{-}$-path are vertex-disjoint.

Based on misplaced symbols in $C_i = (s_{i,1}\ s_{i,2}\ldots s_{i,k_i})$ and the built $s_{2,1}$-pair, $(k_2–1 + k_3 + k_4 + \ldots + k_c)$ pairs or $2(k_2–1 + k_3 + k_4 +\ldots + k_c)$ paths are constructed as follows:

$\langle s_{i,1}^{+}: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, s_{1,2}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,

$\langle s_{i,1}^{-}: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, 2{\wedge}s_{i,1}, s_{1,2}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,

$\langle s_{i,2}^{+}: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, s_{1,2}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,

$\langle s_{i,2}^{-}: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, 2{\wedge}s_{i,2}, s_{1,2}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,

……

$\langle s_{i,k_i}^{+}: s_{i,1}, s_{i,2} \ldots, s_{i,k_i-1}, s_{1,2}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$,

$\langle s_{i,k_i}^{-}: s_{i,1}, s_{i,2} \ldots, s_{i,k_i-1}, 2{\wedge}s_{i,k_i}, s_{1,2}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$.

Because every vertex in $\Pi_s(s_{i,1}^{+}$-path; 0, $s_{i,1})$ has symbol 1 at position $s_{i,1}$, $\Pi_s(s_{i,1}^{+}$-path; 0, $s_{i,1})$ is disjoint to $\Pi_s(s_{i,1}^{-}$-path; $2{\wedge}s_{i,1}$, $s_{i,1})$. Naturally, symbol $s_{1,2}$ of every vertex in $\Pi_s(s_{i,1}^{-}$-path; 0, $s_{1,2})$ is not fixed. Thus, $\Pi_s(s_{i,1}^{+}$-path; $s_{1,2}$, $s_{i,1})$ is disjoint to $\Pi_s(s_{i,1}^{-}$-path; 0, $s_{1,2})$. Since $\Pi_s(s_{i,1}^{-}$-path; 0, $2{\wedge}s_{i,1})$ has not symbol 1 at position $s_{i,1}$, $\Pi_s(s_{i,1}^{+}$-path; 0, $s_{1,2})$ and $\Pi_s(s_{i,1}^{-}$-path; 0, $2{\wedge}s_{i,1})$ are vertex-disjoint. Therefore, $s_{i,1}^{+}$-path and $s_{i,1}^{-}$-path are vertex-disjoint. Similarly, $s_{i,2}$-pair, $s_{i,3}$-pair, …, and s_{i,k_i}-pair are also vertex-disjoint.

Let $s_{i,\alpha}$-pair and $s_{i',\beta}$-pair be any two pairs of the constructed $(k_2–1 + k_3 + k_4 +\ldots + k_c)$ pairs, where $2\le i$, $i'\le c$, $1\le\alpha$, $\beta\le k_i$, $s_{i,\alpha} \ne s_{2,1}$ and $s_{i',\beta} \ne s_{2,1}$. Since symbol 1 of every vertex in $s_{i,\alpha}^{+}$-path ($s_{i',\beta}^{+}$-path) is placed at position $s_{i,\alpha}$ ($s_{i',\beta}$), $s_{i,\alpha}^{+}$-path and $s_{i',\beta}^{+}$-path are vertex-disjoint. Undoubtedly, $\Pi_s(s_{i,\alpha}^{-}$-path; 0, $s_{i,\alpha})$ is vertex-disjoint to $\Pi_s(s_{i',\beta}^{+}$-path; $s_{i,\alpha}$, $s_{i',\beta})$. Because symbol 2 of every vertex in $\Pi_s(s_{i',\beta}^{+}$-path; 0, $s_{i,\alpha})$ is not placed at position $s_{i,\alpha}$, $\Pi_s(s_{i,\alpha}^{-}$-path; $2{\wedge}s_{i,\alpha}$, $s_{i,\alpha})$ is vertex-disjoint to $\Pi_s(s_{i',\beta}^{+}$-path; 0, $s_{i,\alpha})$. Let $s_{i,\alpha\oplus1}$ ($s_{i',\beta\oplus1}$) represent the next symbol of $s_{i,\alpha}$ ($s_{i',\beta}$) in c_i ($c_{i'}$). Every vertex in $\Pi_s(s_{i,\alpha}^{-}$-path; $s_{i,\alpha\oplus1}$, $\underline{2{\wedge}s_{i,\alpha}})$ has fixed $s_{i,\alpha\oplus1}$ and every vertex in $\Pi_s(s_{i',\beta}^{+}$-path; 0, $s_{i,\alpha})$ has not; every vertex in $\Pi_s(s_{i',\beta}^{+}$-path; $s_{i',\beta\oplus1}$, $s_{i,\alpha})$ has fixed $s_{i',\beta\oplus1}$ and every vertex in $\Pi_s(s_{i,\alpha}^{-}$-path; 0, $s_{i,\alpha\oplus1})$ has not, and $\Pi_s(s_{i,\alpha}^{-}$-path; 0, $s_{i,\alpha\oplus1})$ and $\Pi_s(s_{i',\beta}^{+}$-path; 0, $s_{i',\beta\oplus1})$ are two distinct neighbors of ρ. Thus, $\Pi_s(s_{i,\alpha}^{-}$-path; 0, $\underline{2{\wedge}s_{i,\alpha}})$ and $\Pi_s(s_{i',\beta}^{+}$-path; 0, $s_{i,\alpha})$ are vertex-disjoint. Consequently, $s_{i,\alpha}^{-}$-path and $s_{i',\beta}^{+}$-path are vertex-disjoint. Similarly, $s_{i,\alpha}^{+}$-path and $s_{i',\beta}^{-}$-path are vertex-disjoint.

For every fixed symbol e_j, the correcting sequences of the e_j-pair are shown below:

$\langle e_j^{+}: s_{1,2}, C_2, C_3, \ldots, C_c, e_j\rangle$,

$\langle e_j^{-}: 2{\wedge}e_j, s_{1,2}, C_2, C_3,\ldots, C_c, e_j\rangle$.

Every vertex in $\Pi_s(e_j^{+}$-path; 0, $e_j)$ has symbol 1 at position e_j. Therefore, $\Pi_s(e_j^{+}$-path; 0, $e_j)$ is vertex-disjoint to $\Pi_s(e_j^{-}$-path; $2{\wedge}e_j$, $e_j)$. Because symbol 1 in $\Pi_s(e_j^{-}$-path; 0, $2{\wedge}e_j)$ is not at position e_j, $\Pi_s(e_j^{-}$-path; 0, $2{\wedge}e_j)$ and $\Pi_s(e_j^{+}$-path; 0, $e_j)$ are vertex-disjoint. Thus, $\Pi_s(e_j^{+}$-path; 0, $e_j)$ and $\Pi_s(e_j^{-}$-path; 0, $e_j)$ are vertex-disjoint.

Let e_α-pair and e_β-pair be any two distinct f-pairs, where $1\le\alpha$, $\beta\le f$. Since symbol 1 of every vertex in e_α^{+}-path (e_β^{+}-path) is placed at position e_α (e_β), e_α^{+}-path and e_β^{+}-path are vertex-disjoint. e_α^{+}-path is vertex-disjoint to e_β^{-}, because every vertex in $\Pi_s(e_\alpha^{+}$-path; 0, $e_\alpha)$ has symbol 1 at position e_α but symbol 1 in $\Pi_s(e_\beta^{-}$-path; 0, $e_\beta)$ is not at position e_β. Certainly, e_α^{-}-path is vertex-disjoint to e_β^{+}-path.

Let η represent vertex $\Pi_s(s_{1,2}^{+}$-path; 0, $2{\wedge}s_{2,1})$. Uniquely, η_1 and η_2 are $s_{1,2}$ and $s_{1,1}$, respectively. Thus, η is vertex-disjoint to $s_{2,1}$-pair. $\Pi_s(s_{1,2}^{+}$-path; $2{\wedge}s_{2,1}$, $s_{2,1})$ is vertex-disjoint to $s_{2,1}$-pair because symbol 2 is not at position $s_{2,1}$ in every vertex of $s_{2,1}$-pair. $s_{1,2}^{-}$-path is vertex-disjoint to $s_{2,1}$-pair by Lemma 1, since $s_{2,1}$ is prior to $s_{1,2}$ in the correcting sequences for $s_{2,1}$-pair and the first (last) symbol of the correcting sequence for $s_{1,2}^{-}$-path is $s_{1,2}$ ($s_{2,1}$). Therefore, $s_{1,2}$-pair and $s_{2,1}$-pair are vertex-disjoint.

Restated, η is unique and vertex-disjoint to $s_{i,\alpha}$-pair, where $2\le i\le c$, $1\le\alpha\le k_i$, $s_{i,\alpha} \ne s_{2,1}$. $\Pi_s(s_{1,2}^{+}$-path; $2{\wedge}s_{2,1}$, $s_{2,1})$ is vertex-disjoint to $s_{i,\alpha}$-pair because every vertex in $s_{i,\alpha}$-pair has symbol 2 not at position $s_{2,1}$. Thus, $s_{1,2}^{+}$-path is vertex-disjoint to $s_{i,\alpha}$-pair. Let δ represent vertex $\Pi_s(s_{1,2}^{-}$-path; 0, $1{\wedge}s_{2,1})$. Uniquely, δ_1 and δ_2 are $s_{1,1}$ and symbol 1, respectively. Hence, δ is vertex-disjoint to $s_{i,\alpha}$-pair. $\Pi_s(s_{1,2}^{-}$-path; $1{\wedge}s_{2,1}$, $s_{2,1})$ is vertex-disjoint to $s_{i,\alpha}$-pair because symbol 1 is not at position $s_{2,1}$ in every vertex of $s_{i,\alpha}$-pair. Thus, $s_{1,2}^{-}$-path is vertex-disjoint to $s_{i,\alpha}$-pair. Therefore, $s_{1,2}$-pair and $s_{i,\alpha}$-pair are vertex-disjoint.

Every vertex in $s_{2,1}^{+}$-path has symbol 2 at position $s_{1,2}$ and every vertex in $s_{i,\alpha}$-pair has symbol 2 not at position $s_{1,2}$. Thus, $s_{2,1}^{+}$-path are vertex-disjoint to $s_{i,\alpha}$-pair.

Let θ represent vertex $\Pi_s(s_{2,1}^{-}$-path; 0, $s_{2,2})$. Uniquely, θ_1 and θ_2 are $s_{2,2}$ and symbol 1, respectively. Hence, θ is vertex-disjoint to $s_{i,\alpha}$-pair. $\Pi_s(s_{2,1}^{-}$-path; $s_{2,2}$, $s_{1,2})$ has first fixed $s_{2,2}$ and finally fixed $s_{1,2}$, and $s_{i,\alpha}$-pair has fixed $s_{1,2}$ prior to $s_{2,2}$. By Lemma 1, $s_{2,1}^{-}$-path is vertex-disjoint to $s_{i,\alpha}$-pair.

f-pair and $s_{i,\alpha}$-pair are vertex-disjoint because every e_j-pair has modified e_j symbol but m-pair never modifies any e_j symbol. Similarly, f-pair is disjoint to $s_{1,2}$-pair and $s_{2,1}$-pair.

From the above discussion, it has become clear that every two distinct pairs are vertex-disjoint. Referring to Case 2.1.1 described above, there are 4 groups of paths constructed which are 1 $s_{1,2}$-pair, 1 $s_{2,1}$-pair, $(k_2–1 + k_3 + k_4 + \ldots + k_c)$ m-pairs, and f

f-pairs. In total, $2(n-2)$ vertex-disjoint paths are built. Significantly, every e_j^--path is the longest path. Thus, the length of every constructed path is bound above by $d(\rho, \varepsilon)+4$ because at most 4 extra edges, where 1 (3) is for e_j^- operation (unfixing and refixing symbol 2), should be added to the corresponding shortest path.

Case 2.1.2 k_1=3

Namely, $C_1 = (2\ s_{1,2}\ s_{1,3})$. The correcting sequences are listed as follows:

$\langle s_{1,2}^+ : s_{1,3}, C_2, C_3, \ldots, C_c, s_{1,2}\rangle$,
$\langle s_{1,2}^- : C_2, C_3, \ldots, C_c, s_{1,3}\rangle$.
$\langle s_{1,k_1}^+ : s_{1,2}, C_2, C_3, \ldots, C_c, s_{1,k_1}\rangle$,
$\langle s_{1,k_1}^- : 2\wedge s_{1,2}, C_2, C_3, \ldots, C_c, s_{1,k_1}, s_{1,2}\rangle$.
$\langle s_{i,1}^+ : s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, s_{1,2}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,1}^- : s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, 2\wedge s_{i,1}, s_{1,2}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,2}^+ : s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, s_{1,2}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,
$\langle s_{i,2}^- : s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, 2\wedge s_{i,2}, s_{1,2}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,
……
$\langle s_{i,k_i}^+ : s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, s_{1,2}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$,
$\langle s_{i,k_i}^- : s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, 2\wedge s_{i,k_i}, s_{1,2}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$.
$\langle e_j^+ : s_{1,2}, s_{1,k_1}, C_2, C_3, \ldots, C_c, e_j\rangle$,
$\langle e_j^- : 2\wedge e_j, s_{1,2}, s_{1,k_1}, C_2, C_3, \ldots, C_c, e_j\rangle$.

Similar to Case 2.1.1, any two distinct built paths in Case 2.1.2 are vertex-disjoint. This case provides rules for constructing $s_{1,2}$-pair, $s_{1,3}$-pair, $(k_2 + k_3 + \ldots + k_c)$ m-pairs, and f f-pairs. Totally, $2(n-2)$ vertex-disjoint paths are built. Referring to Case 2.1.1, the length of every constructed path in Case 2.1.2 is bound above by $d(\rho, \varepsilon)+4$ because at most 4 extra edges should be added to the corresponding shortest path.

Case 2.1.3 $k_1 \geq 4$

Under this condition, $|C_1| \geq 4$ and $C_1 = (2\ s_{1,2}\ s_{1,3}\ s_{1,4} \ldots s_{1,k_1})$. The correcting sequences are listed as follows:

$\langle s_{1,2}^+ : s_{1,3}, s_{1,4}, \ldots, s_{1,k_1-1}, C_2, C_3, \ldots, C_c, s_{1,2}, s_{1,k_1}\rangle$,
$\langle s_{1,2}^- : 1\wedge s_{1,k_1}, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1-1}, C_2, C_3, \ldots, C_c, s_{1,k_1}\rangle$.
$\langle s_{1,k_1}^+ : 2\wedge s_{1,2}, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_c, s_{1,2}\rangle$,
$\langle s_{1,k_1}^- : 1\wedge s_{1,2}, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_c, s_{1,2}\rangle$.
$\langle s_{1,3}^+ : s_{1,4}, s_{1,5}, \ldots, s_{1,k_1}, s_{1,2}, C_2, C_3, \ldots, C_c, s_{1,3}\rangle$,
$\langle s_{1,3}^- : s_{1,4}, s_{1,5}, \ldots, s_{1,k_1}, 2\wedge s_{1,3}, s_{1,2}, C_2, C_3, \ldots, C_c, s_{1,3}\rangle$,
$\langle s_{1,4}^+ : s_{1,5}, s_{1,6}, \ldots, s_{1,k_1}, s_{1,2}, s_{1,3}, C_2, C_3, \ldots, C_c, s_{1,4}\rangle$,
$\langle s_{1,4}^- : s_{1,5}, s_{1,6}, \ldots, s_{1,k_1}, 2\wedge s_{1,4}, s_{1,2}, s_{1,3}, C_2, C_3, \ldots, C_c, s_{1,4}\rangle$,
……
$\langle s_{1,k_1-1}^+ : s_{1,k_1}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1-2}, C_2, C_3, \ldots, C_c, s_{1,k_1-1}\rangle$,
$\langle s_{1,k_1-1}^- : s_{1,k_1}, 2\wedge s_{1,k_1-1}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1-2}, C_2, C_3, \ldots, C_c, s_{1,k_1-1}\rangle$.
$\langle s_{i,1}^+ : s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,1}^- : s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, 2\wedge s_{i,1}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,2}^+ : s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,
$\langle s_{i,2}^- : s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, 2\wedge s_{i,2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,
……
$\langle s_{i,k_i}^+ : s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$,
$\langle s_{i,k_i}^- : s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, 2\wedge s_{i,k_i}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$.
$\langle e_j^+ : s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_c, e_j\rangle$,
$\langle e_j^- : 2\wedge e_j, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_c, e_j\rangle$.

Similar to Case 2.1.1, any two distinct built paths in Case 2.1.3 are vertex-disjoint. This case provides rules for constructing $s_{1,2}$-pair, s_{1,k_1}-pair, $(k_1 - 2 + k_2 + k_3 + \ldots + k_c)$ m-pairs, and f f-pairs. Totally, $2(n-2)$ vertex-disjoint paths are built. Referring to Cases 2.1.1, the length of every constructed path in Case 2.1.3 is bound above by $d(\rho, \varepsilon)+4$ because at most 4 extra edges should be added to the corresponding shortest path.

Case 2.2: ρ_1=2 and $\rho_2 \neq 1$

A set of correcting sequences is proposed for building a $C_{2(n-2)}(\rho, \varepsilon)$ for each of k_1=3 and $k_1 \geq 4$.

Case 2.2.1 k_1 = 3

When c = 1, $C_1 = (1\ 2\ s_{1,3})$ and the correcting sequences of $s_{1,3}$-pair are $\langle s_{1,3}^+ : s_{1,3}\rangle$ and $\langle s_{1,3}^- :\rangle$. Notably, the sequence $\langle s_{1,3}^- :\rangle$ indicates that no more symbols should be fixed after doing $s_{1,3}^-$ operation.

For every fixed symbol e_j, the correcting sequences of e_j-pair are $\langle e_j^+ : s_{1,k_1}, e_j\rangle$ and $\langle e_j^- : 1\wedge e_j, s_{1,k_1}, e_j\rangle$.

When $c \geq 2$, $\rho = (1\ 2\ s_{1,3})C_2C_3 \ldots C_c$. The correcting sequences of $s_{1,3}$-pair are listed below:

$\langle s_{1,3}^+ : 1\wedge s_{2,1}, s_{1,3}, s_{2,2}, s_{2,3}, \ldots s_{2,k_2}, C_3\ C_4 \ldots C_c, s_{2,1}\rangle$,
$\langle s_{1,3}^- : 2\wedge s_{2,1}, s_{2,2}, s_{2,3}, \ldots s_{2,k_2}, C_3\ C_4 \ldots C_c, s_{2,1}\rangle$.
$\langle s_{2,1}^+ : s_{2,2}, s_{2,3}, \ldots s_{2,k_2}, C_3\ C_4 \ldots C_c, s_{2,1}, s_{1,3}\rangle$,
$\langle s_{2,1}^- : s_{2,2}, 2\wedge s_{1,3}, s_{2,3}, s_{2,4}, \ldots s_{2,k_2}, s_{2,1}, C_3\ C_4 \ldots C_c, s_{1,3}\rangle$.
$\langle s_{2,j}^+ : s_{2,(j \bmod k_2)+1}, s_{2,((j+1) \bmod k_2)+1}, \ldots s_{2,((j+k_2-3) \bmod k_2)+1}, s_{2,((j+k_2-2) \bmod k_2)+1}, s_{1,3}, C_3\ C_4 \ldots C_c, s_{2,j}\rangle$,
$\langle s_{2,j}^- : s_{2,(j \bmod k_2)+1}, s_{2,((j+1) \bmod k_2)+1}, \ldots s_{2,((j+k_2-3) \bmod k_2)+1}, s_{2,((j+k_2-2)) \bmod k_2)+1}, 1\wedge s_{2,j}, s_{1,3}, C_3\ C_4 \ldots C_c, s_{2,j}\rangle$.
$\langle s_{i,1}^+ : s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,1}^- : s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, 1\wedge s_{i,1}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,2}^+ : s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,
$\langle s_{i,2}^- : s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, 1\wedge s_{i,2}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,
……
$\langle s_{i,k_i}^+ : s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$,
$\langle s_{i,k_i}^- : s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, 1\wedge s_{i,k_i}, s_{1,3}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$.
$\langle e_j^+ : s_{1,k_1}, C_2, C_3, \ldots, C_c, e_j\rangle$,
$\langle e_j^- : 1\wedge e_j, s_{1,k_1}, C_2, C_3, \ldots, C_c, e_j\rangle$.

Similar to the construction of e_j-pair in Case 2.1.1, e_j^+-path and e_j^--path are vertex-disjoint.

Any two distinct built paths in Case 2.2.1 are vertex-disjoint because their constructions are very similar to Case 2.1.1. This case provides rules for constructing $s_{1,3}$-pair, $s_{2,1}$-pair, (k_2-1) $s_{2,j}$-pair, and $(k_3 + \ldots + k_c)$ m-pairs, and f f-pairs. Totally, $2(n-2)$ vertex-disjoint paths are built.

Case 2.2.2 $k_1 \geq 4$

Under this condition, $|C_1| \geq 4$ and $C_1 = (1\ 2\ s_{1,3}\ s_{1,4} \ldots s_{1,k_1})$. The correcting sequences of $s_{1,3}$-pair are shown below:

$\langle s_{1,3}^+ : s_{1,4}, \ldots, s_{1,k_1-1}, C_2, C_3, \ldots, C_c, s_{1,3}, s_{1,k_1}\rangle$,
$\langle s_{1,3}^- : s_{1,4}, \ldots, s_{1,k_1-1}, 2\wedge s_{1,k_1}, C_2, C_3, \ldots, C_c, s_{1,k_1}\rangle$.
$\langle s_{1,k_1}^+ : 1\wedge s_{1,3}, C_2, C_3, \ldots, C_c, s_{1,4}, s_{1,5}, \ldots, s_{1,k_1}, s_{1,3}\rangle$,
$\langle s_{1,k_1}^- : 2\wedge s_{1,3}, C_2, C_3, \ldots, C_c, s_{1,4}, s_{1,5}, \ldots, s_{1,k_1}, s_{1,3}\rangle$.
$\langle s_{1,4}^+ : s_{1,5}, s_{1,6}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_c, s_{1,3}, s_{1,4}\rangle$,
$\langle s_{1,4}^- : s_{1,5}, s_{1,6}, \ldots, s_{1,k_1}, 1\wedge s_{1,4}, C_2, C_3, \ldots, C_c, s_{1,3}, s_{1,4}\rangle$,
$\langle s_{1,5}^+ : s_{1,6}, s_{1,7}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_c, s_{1,3}, s_{1,4}, s_{1,5}\rangle$,
$\langle s_{1,5}^- : s_{1,6}, s_{1,7}, \ldots, s_{1,k_1}, 1\wedge s_{1,5}, C_2, C_3, \ldots, C_c, s_{1,3}, s_{1,4}, s_{1,5}\rangle$,

……

$\langle s_{1,k_1-1}{}^{+}: s_{1,k_1}, C_2, C_3, \ldots, C_c, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1-1}\rangle$,

$\langle s_{1,k_1-1}{}^{-}: s_{1,k_1}, 1\wedge s_{1,k_1-1}, C_2, C_3, \ldots, C_c, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1-1}\rangle$.

$\langle s_{i,1}{}^{+}: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, s_{i,1}\rangle$,

$\langle s_{i,1}{}^{-}: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, 1\wedge s_{i,1}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, s_{i,1}\rangle$,

$\langle s_{i,2}{}^{+}: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, s_{i,2}\rangle$,

$\langle s_{i,2}{}^{-}: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, 1\wedge s_{i,2}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, s_{i,2}\rangle$,

……

$\langle s_{i,k_i}{}^{+}: s_{i,1}, s_{i,2}, \ldots s_{i,k_i-1}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, s_{i,k_i}\rangle$,

$\langle s_{i,k_i}{}^{-}: s_{i,1}, s_{i,2}, \ldots s_{i,k_i-1}, 1\wedge s_{i,k_i}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, s_{i,k_i}\rangle$.

$\langle e_j{}^{+}: s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_c, e_j\rangle$,

$\langle e_j{}^{-}: 1\wedge e_j, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, C_2, C_3, \ldots, C_c, e_j\rangle$.

Any two distinct built paths are vertex-disjoint because their constructions are very similar to Case 2.2.1. This case provides rules for construction $s_{1,3}$-pair, s_{1,k_1}-pair, $(k_1-2+k_2+k_3+\ldots+k_c)$ m-pairs, and f f-pairs. Totally, $2(n-2)$ vertex-disjoint paths are built. Significantly, every e_j^{-}-path is the longest path. Thus, the length of every constructed path is bound above by $d(\rho, \varepsilon)+4$ because at most 4 extra edges, where 1 (3) is for e_j^{-} operation (unfixing and refixing symbol 2), should be added to the corresponding shortest path.

Case 2.3: $\rho_1 \neq 1$ and $\rho_2=2$

The rules of correcting sequences of Case 2.3 are divided into three cases according to k_1=2, k_1=3, and k_1≥4. Those constructed rules in Case 2.1 can be applied on Case 2.3 since every cycle structure in Case 2.3 are similar to Case 2.1 and only symbol 2 in C_1 is replaced by symbol 1. Nevertheless, replacing symbol 1 or symbol 2 in C_1 delivers a new vertex, which has even permutation, so that is also a vertex in AG$_n$ absolutely.

The paths construction of Case 2.3 is similar to Case 2.1. The length of the longest $C_{2(n-2)}(\rho, \varepsilon)$ built in Case 2.3 is the same as that of Case 2.1.

Case 2.4: $\rho_1 \neq 2$ and $\rho_2=1$

The paths construction of Case 2.4 is similar to Case 2.2 likewise and only symbol 1 (respectively, 2) in C_1 is replaced by symbol 2 (respectively, 1). Thus, the length of the longest $C_{2(n-2)}(\rho, \varepsilon)$ built in Case 2.4 is the same as that of Case 2.2.

According to the statements described in Case 2, the following lemma holds.

Lemma 6. The $2(n-2)$ paths constructed in Case 2 are vertex-disjoint.

Lemma 7. In Case 2, $l(C_{2(n-2)}(\rho, \varepsilon)) \leq d(\mathrm{AG}_n)+2$.

Proof. Referring to the constructions of all paths in Cases 2 the length of every longest path is at most $d(\rho, \varepsilon)+2$. Two extra edges are added for path disjoint.

Recall that $d(\rho, \varepsilon)=n+c-l-2$ if $\rho_1 \neq 1$ and $\rho_2 = 2$ or $\rho_1 = 1$ and $\rho_2 \neq 2$ [6]. After maximizing $(n+c-l-2)+2$ and substituting c with $\lfloor n-1/2\rfloor$, l with 1, the length of the longest path is at most $n+\lfloor n-1/2\rfloor-1$. Then, $n+\lfloor n-1/2\rfloor-1 = n+(n-1)/2-1 = (3n-3)/2 = d(\mathrm{AG}_n)+2$ if n is odd, and $n+\lfloor n-1/2\rfloor-1 = n+(n-2)/2-1 = (3n-4)/2 = d(\mathrm{AG}_n)+2$ if n is even.

Recall that $d(\rho, \varepsilon)$ is $n+c-l-3$ if $1, 2 \in C_i$, $1\leq i\leq k$ and $|C_i|\geq 3$ [6] After maximizing $(n+c-l-3)+2$ and substituting c with $1+\lfloor n-3/2\rfloor$ and l with 0, the length of the longest path is at most $n+(1+\lfloor n-3/2\rfloor)-3+2$. Hence, $n+(1+(n-3)/2)-3+2 = (3n-3)/2 = d(\mathrm{AG}_n)+2$ if n is odd, and $n+(1+(n-4)/2)-3+2 = (3n-4)/2 = d(\mathrm{AG}_n)+1$ if n is even. In Case 2, $l(C_{2(n-2)}(\rho, \varepsilon)) \leq d(\mathrm{AG}_n)+2$.

Therefore, the length of the containers constructed in Case 2 is bounded above by $d(\mathrm{AG}_n)+2$. ■

4.3 Case 3: $\{\rho_1, \rho_2\}\cap\{1, 2\} = \phi$

This case is divided into two subcases Cases 3.1 and 3.2 according to symbols 1 and 2 belong to C_1 or not. Let $C_i^{\#} = (s_{i,1}\ s_{i,2}\ldots s_{i,k_i})^{\#}$, where $2\leq i\leq c$, denote a cycle in a correcting sequence and indicate that symbol 1 or 2 should be kept at position 1 or 2 when fixing each symbol of $C_i^{\#}$.

Case 3.1: 1, 2 ∈ C_1

Assume $\rho=(1\ t_1\ t_2\ \ldots t_w\ 2\ u_1\ u_2\ \ldots u_{w'})C_2C_3\ldots C_c e_1 e_2\ldots e_l$, where w (w') represents the number of symbols from symbol 1 (2) to symbol 2 (1) in C_1. Hence, Case 3.1 is divided into Case 3.1.1 w=1, w'=1, Case 3.1.2 w=1, w'≥2, Case 3.1.3 w≥2, w'≥2, and Case 3.1.4 w≥2, w'=1.

In Case 3.1, some paths have similar structures between subcases. Each of those paths having similar structures has the same first and last symbols of its correcting sequence. Generally, the paths built by first doing α^{+} or α^{-} operation and $\alpha \in \{C_2, C_3,\ldots, C_c, e_1, e_2,\ldots, e_l\}$ have similar structures. Moreover, in Cases 3.1.2 and 3.1.3, the paths built by first doing α^{+} or α^{-} operation and $\alpha \in \{u_2, u_3, \ldots, u_{w'-1}\}$ have similar structures. Additionally, in Cases 3.1.3 and 3.1.4, paths built by first doing α^{+} or α^{-} operation and $\alpha \in \{t_2, t_3, \ldots, t_{w-1}\}$ having similar structures. Thus, the proofs of vertex-disjoint (respectively, length computation) of those paths with similar structures are the same.

Case 3.1.1 w = 1, w' = 1

Assume that symbol 1 (2) of ρ is at position α and let t denote the tail symbol of a correcting sequence. For vertex-disjoint, build every path with $\alpha \neq t$ by first doing α^{+} (α^{-}) operation and then t^{-} (t^{+}) operation.

The correcting sequences are listed as follows:

$\langle t_1{}^{+}: 2\wedge u_1, C_2, C_3, \ldots, C_c, u_1\rangle$,

$\langle t_1{}^{-}: C_2{}^{\#}, C_3{}^{\#}, \ldots, C_c{}^{\#}, t_1, u_1\rangle$.

$\langle u_1{}^{+}: C_2{}^{\#}, C_3{}^{\#}, \ldots, C_c{}^{\#}, u_1, t_1\rangle$,

$\langle u_1{}^{-}: 1\wedge t_1, C_2, C_3, \ldots, C_c, t_1\rangle$.

$\langle s_{i,1}{}^{+}: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, u_1, 1\wedge s_{i,1}, t_1, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c,$

$s_{i,1}\rangle$,
$\langle s_{i,1}^-: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, t_1, 2\wedge s_{i,1}, u_1, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,2}^+: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, u_1, 1\wedge s_{i,2}, t_1, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,
$\langle s_{i,2}^-: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, t_1, 2\wedge s_{i,2}, u_1, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,2}\rangle$,
……
$\langle s_{i,k_i}^+: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, u_1, 1\wedge s_{i,k_i}, t_1, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$,
$\langle s_{i,k_i}^-: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, t_1, 2\wedge s_{i,k_i}, u_1, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$.
$\langle e_j^+: u_1, 1\wedge e_j, t_1, , C_2, C_3, \ldots, C_c, e_j\rangle$,
$\langle e_j^-: t_1, 2\wedge e_j, u_1, C_2, C_3, \ldots, C_c, e_j\rangle$.

With the aid of Corollary 3 and carefully checking each vertex of each constructed path, every two built paths are vertex-disjoint. There are 4 groups of paths constructed which are 1 t_1-pair, 1 u_1-pair, $(k_2 + k_3 + k_4 + \ldots + k_c)$ m-pairs, and f f-pairs. In total, $2(2+k_2 + k_3 + k_4 + \ldots + k_c) = 2(n-2)$ vertex-disjoint paths are built. Significantly, every e_j^--path is the longest path. Thus, the length of every constructed path is bound above by $d(\rho, \varepsilon)+4$ because at most 4 extra edges, where 1 (3) is for e_j^- operation (unfixing and refixing symbol 2), should be added to the corresponding shortest path.

Case 3.1.2 $w = 1, w' \geq 2$

In the following, we construct each pair one by one according to the provided correcting sequences.

$\langle t_1^+: 2\wedge u_{w'}, u_1, u_2, \ldots, u_{w'-1}, C_2, C_3, \ldots, C_c, u_{w'}\rangle$,
$\langle t_1^-: 2\wedge u_1, u_2, u_3, \ldots, u_{w'}, C_2, C_3, \ldots, C_c, t_1, u_1\rangle$.
$\langle u_1^+: u_2, u_3, \ldots, u_{w'-1}, C_2, C_3, \ldots, C_c, u_{w'}, u_1, t_1\rangle$,
$\langle u_1^-: u_2, u_3, \ldots, u_{w'-1}, C_2, C_3, \ldots, C_c, t_1, u_{w'}\rangle$.
$\langle u_{w'}^+: 1\wedge t_1, u_1, u_2, \ldots, u_{w'}, C_2, C_3, \ldots, C_c, t_1\rangle$,
$\langle u_{w'}^-: 1\wedge u_1, u_2, u_3, \ldots, u_{w'}, t_1, C_2, C_3, \ldots, C_c, u_1\rangle$.
$\langle u_2^+: u_3, u_4, \ldots, u_{w'}, u_1, 1\wedge u_2, t_1, C_2, C_3, \ldots, C_c, u_2\rangle$,
$\langle u_2^-: u_3, u_4, \ldots, u_{w'}, t_1, 2\wedge u_2, u_1, C_2, C_3, \ldots, C_c, u_2\rangle$,
$\langle u_3^+: u_4, u_5, \ldots, u_{w'}, u_1, u_2, 1\wedge u_3, t_1, C_2, C_3, \ldots, C_c, u_3\rangle$,
$\langle u_3^-: u_4, u_5, \ldots, u_{w'}, t_1, 2\wedge u_3, u_1, u_2, C_2, C_3, \ldots, C_c, u_3\rangle$,
……
$\langle u_{w'-1}^+: u_{w'}, u_1, u_2, \ldots, u_{w'-2}, 1\wedge u_{w'-1}, t_1, C_2, C_3, \ldots, C_c, u_{w'-1}\rangle$,
$\langle u_{w'-1}^-: u_{w'}, u_1, u_2, \ldots, u_{w'-2}, t_1, 2\wedge u_{w'-1}, C_2, C_3, \ldots, C_c, u_{w'-1}\rangle$.
$\langle s_{i,1}^+: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, u_1, u_2, \ldots, u_{w'}, 1\wedge s_{i,1}, t_1, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,1}^-: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, t_1, 2\wedge s_{i,1}, u_1, u_2 \ldots, u_{w'}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,2}^+: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, u_1, u_2, \ldots, u_{w'}, 1\wedge s_{i,2}, t_1, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,2}^-: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, t_1, 2\wedge s_{i,2}, u_1, u_2, \ldots, u_{w'}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
……
$\langle s_{i,k_i}^+: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, u_1, u_2, \ldots, u_{w'}, 1\wedge s_{i,k_i}, t_1, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$,
$\langle s_{i,k_i}^-: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, t_1, 2\wedge s_{i,k_i}, u_1, u_2 \ldots, u_{w'}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$.
$\langle e_j^+: u_1, u_2, \ldots, u_{w'}, 1\wedge e_j, t_1, C_2, C_3, \ldots, C_c, e_j\rangle$,
$\langle e_j^-: t_1, 2\wedge e_j, u_1, u_2, \ldots, u_{w'}, C_2, C_3, \ldots, C_c, e_j\rangle$.

According to Lemma 1 and Corollary 3 and carefully checking each vertex of each constructed path, the built $2(n-2)$ paths are vertex-disjoint.

Case 3.1.3 $w \geq 2, w' \geq 2$

The correcting sequences are listed below:

$\langle t_1^+: t_2, t_3, \ldots, t_w, 2\wedge u_{w'}, u_1, u_2, \ldots, u_{w'-1}, C_2, C_3, \ldots, C_c, u_w\rangle$,
$\langle t_1^-: t_2, t_3, \ldots, t_w, t_1, u_1, u_2, \ldots, u_{w'-1}, C_2, C_3, \ldots, C_c, u_w\rangle$.
$\langle t_w^+: 2\wedge t_1, t_2, t_3, \ldots, t_w, u_1, u_2, \ldots, u_{w'}, C_2, C_3, \ldots, C_c, t_1\rangle$,
$\langle t_w^-: 2\wedge u_1, u_2, u_3, \ldots, u_{w'}, t_1, t_2, \ldots, t_w, C_2, C_3, \ldots, C_c, u_1\rangle$.
$\langle u_1^+: u_2, u_3, \ldots, u_{w'}, 1\wedge t_w, u_1, t_1, t_2, \ldots, t_{w-1}, C_2, C_3, \ldots, C_c, t_w\rangle$,
$\langle u_1^-: u_2, u_3, \ldots, u_{w'-1}, t_1, t_2, \ldots, t_{w-1}, C_2, C_3, \ldots, C_c, u_{w'}, t_w\rangle$.
$\langle u_{w'}^+: 1\wedge t_1, t_2, t_3, \ldots, t_w, u_1, u_2, \ldots, u_{w'}, C_2, C_3, \ldots, C_c, t_1\rangle$,
$\langle u_{w'}^-: 1\wedge u_1, u_2, u_3, \ldots, u_{w'}, t_1, t_2, \ldots, t_w, C_2, C_3, \ldots, C_c, u_1\rangle$.
$\langle t_2^+: t_3, t_4, \ldots, t_w, u_1, u_2, \ldots, u_{w'}, 1\wedge t_2, t_1, C_2, C_3, \ldots, C_c, t_2\rangle$,
$\langle t_2^-: t_3, t_4, \ldots, t_w, 2\wedge t_2, u_1, u_2, \ldots, u_{w'}, t_1, C_2, C_3, \ldots, C_c, t_2\rangle$,
$\langle t_3^+: t_4, t_5, \ldots, t_w, u_1, u_2, \ldots, u_{w'}, 1\wedge t_3, t_1, t_2, C_2, C_3, \ldots, C_c, t_3\rangle$,
$\langle t_3^-: t_4, t_5, \ldots, t_w, 2\wedge t_3, u_1, u_2, \ldots, u_{w'}, t_1, t_2, C_2, C_3, \ldots, C_c, t_3\rangle$,
……
$\langle t_{w-1}^+: t_w, u_1, u_2, \ldots, u_{w'}, 1\wedge t_{w-1}, t_1, t_2, \ldots, t_{w-2}, C_2, C_3, \ldots, C_c, t_{w-1}\rangle$,
$\langle t_{w-1}^-: t_w, 2\wedge t_{w-1}, u_1, u_2, \ldots, u_{w'}, t_1, t_2, \ldots, t_{w-2}, C_2, C_3, \ldots, C_c, t_{w-1}\rangle$.
$\langle u_2^+: u_3, u_4, \ldots, u_{w'}, u_1, 1\wedge u_2, t_1, t_2, \ldots, t_w, C_2, C_3, \ldots, C_c, u_2\rangle$,
$\langle u_2^-: u_3, u_4, \ldots, u_{w'}, t_1, t_2, \ldots, t_w, 2\wedge u_2, u_1, C_2, C_3, \ldots, C_c, u_2\rangle$,
$\langle u_3^+: u_4, u_5, \ldots, u_{w'}, u_1, u_2, 1\wedge u_3, t_1, t_2, \ldots, t_w, C_2, C_3, \ldots, C_c, u_3\rangle$,
$\langle u_3^-: u_4, u_5, \ldots, u_{w'}, t_1, t_2, \ldots, t_w, 2\wedge u_3, u_1, u_2, C_2, C_3, \ldots, C_c, u_3\rangle$,
……
$\langle u_{w'-1}^+: u_{w'}, u_1, u_2, \ldots, u_{w'-2}, 1\wedge u_{w'-1}, t_1, t_2, \ldots, t_w, C_2, C_3, \ldots, C_c, u_{w'-1}\rangle$,
$\langle u_{w'-1}^-: u_{w'}, t_1, t_2, \ldots, t_w, 2\wedge u_{w'-1}, u_1, u_2, \ldots, u_{w'-2}, C_2, C_3, \ldots, C_c, u_{w'-1}\rangle$.
$\langle s_{i,1}^+: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, u_1, u_2 \ldots, u_{w'}, 1\wedge s_{i,1}, t_1, t_2, \ldots, t_w, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,1}^-: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, t_1, t_2, \ldots, t_w, 2\wedge s_{i,1}, u_1, u_2 \ldots, u_{w'}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,2}^+: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, u_1, u_2 \ldots, u_{w'}, 1\wedge s_{i,2}, t_1, t_2, \ldots, t_w, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
$\langle s_{i,2}^-: s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, t_1, t_2, \ldots, t_w, 2\wedge s_{i,2}, u_1, u_2 \ldots, u_{w'}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,1}\rangle$,
……
$\langle s_{i,k_i}^+: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, u_1, u_2, \ldots, u_{w'}, 1\wedge s_{i,k_i}, t_1, t_2, \ldots, t_w, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$,
$\langle s_{i,k_i}^-: s_{i,1}, s_{i,2}, \ldots, s_{i,k_i-1}, t_1, t_2, \ldots, t_w, 2\wedge s_{i,k_i}, u_1, u_2 \ldots, u_{w'}, C_2, C_3, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{i,k_i}\rangle$.
$\langle e_j^+: u_1, u_2 \ldots, u_{w'}, 1\wedge e_j, t_1, t_2, \ldots, t_w, C_2, C_3, \ldots, C_c, e_j\rangle$,
$\langle e_j^-: t_1, t_2, \ldots, t_w, 2\wedge e_j, u_1, u_2 \ldots, u_{w'}, C_2, C_3, \ldots, C_c, e_j\rangle$.

Similar to Case 3.1.2, the built $2(n-2)$ paths are vertex-disjoint.

Case 3.1.4 $w \geq 2, w' = 1$

The correcting sequences are listed below:

$\langle t_1^+: t_2, t_3, \ldots, t_{w-1}, C_2, C_3, \ldots, C_c, u_1, t_w\rangle$,
$\langle t_1^-: t_2, t_3, \ldots, t_{w-1}, C_2, C_3, \ldots, C_c, t_w, t_1, u_1\rangle$.
$\langle t_w^+: 2\wedge t_1, t_2, t_3, \ldots, t_w, u_1, C_2, C_3, \ldots, C_c, t_1\rangle$,
$\langle t_w^-: 2\wedge u_1, t_1, t_2, \ldots, t_{w}, C_2, C_3, \ldots, C_c, u_1\rangle$.
$\langle u_1^+: 1\wedge t_1, t_2, t_3, \ldots, t_w, C_2, C_3, \ldots, C_c, u_1, t_1\rangle$,
$\langle u_1^-: 1\wedge t_w, t_1, t_2, \ldots, t_{w-1}, C_2, C_3, \ldots, C_c, t_w\rangle$.

Based on misplaced symbols in $C_i = (s_{i,1}\ s_{i,2} \ldots s_{i,k_i})$, $(k_2 + k_3 + \ldots + k_c)$ pairs or $2(k_2 + k_3 + \ldots + k_c)$ paths are constructed the same to Case 3.1.3. e_j-path are constructed the same to Case 3.1.3.

According to Lemma 1 and Corollary 3 and carefully checking each vertex of each constructed path, the built $2(n-2)$ paths are vertex-disjoint.

Significantly, every e_j^--path is the longest path. Thus, the length of every constructed path is bound above by $d(\rho, \varepsilon)+4$ because at most 4 extra edges, where 1 (3) is for e_j^- operation (unfixing and refixing symbol 2), should be added to the corresponding shortest path.

Case 3.2: $1 \in C_1$, $2 \in C_2$

In Case 3.2, assume $\rho=(1\ s_{1,2}\ s_{1,3} \ldots s_{1,k_1})(2\ s_{2,2}\ s_{2,3} \ldots s_{2,k_2})C_3C_4 \ldots C_c e_1 e_2 \ldots e_l$. This case are divided into Case 3.2.1 k_1=2, k_2=2, Case 3.2.2 k_1=2, k_2≥3, Case 3.2.3 k_1≥3, k_2≥3, and Case 3.2.4 k_1=2, k_2≥3.

Case 3.2.1 $k_1 = 2, k_2 = 2$

Assume $\rho=(1\ s_{1,2})(2\ s_{2,2})C_3C_4 \ldots C_c e_1 e_2 \ldots e_l$. The correcting sequences are listed as follows:

$\langle s_{1,2}^+: C_3^\#, C_4^\#, \ldots, C_c^\#, s_{2,2}\rangle$,
$\langle s_{1,2}^-: 1\wedge s_{2,2}, s_{1,2}, C_3, C_4, \ldots, C_c, s_{2,2}\rangle$.
$\langle s_{2,2}^+: 2\wedge s_{1,2}, s_{2,2}, C_3, C_4, \ldots, C_c, s_{1,2}\rangle$,
$\langle s_{2,2}^-: C_3^\#, C_4^\#, \ldots, C_c^\#, s_{1,2}\rangle$.
$\langle s_{i,1}^+: s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, 2\wedge s_{i,1}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, s_{i,1}\rangle$,

$\langle s_{i,1}^{-}$: $s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, 1\wedge s_{i,1}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, s_{i,1}\rangle$,

$\langle s_{i,2}^{+}$: $s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, 2\wedge s_{i,2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, s_{i,2}\rangle$,

$\langle s_{i,2}^{-}$: $s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, 1\wedge s_{i,2}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, s_{i,2}\rangle$,

……

$\langle s_{i,k_i}^{+}$: $s_{i,1}, s_{i,2} \ldots, s_{i,k_i-1}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, 2\wedge s_{i,k_i}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, s_{i,k_i}\rangle$,

$\langle s_{i,k_i}^{-}$: $s_{i,1}, s_{i,2} \ldots, s_{i,k_i-1}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, 1\wedge s_{i,k_i}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, s_{i,k_i}\rangle$.

$\langle e_j^{+}$: $s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, 2\wedge e_j, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, C_3, C_4, \ldots, C_c, e_j \rangle$,

$\langle e_j^{-}$: $s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, 1\wedge e_j, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, C_3, C_4, \ldots, C_c, e_j \rangle$.

According to Lemma 1 and Corollary 3 and carefully checking each vertex of each constructed path, all built paths are vertex-disjoint. This case provides rules for construction $s_{1,2}$-pair, $s_{2,1}$-pair, $(k_3 + k_4 + \ldots + k_c)$ m-pairs and f f-pairs. Totally, $2(n-2)$ vertex-disjoint paths are built.

Case 3.2.2 $k_1 = 2, k_2 \geq 3$

The correcting sequences are listed as follows:

$\langle s_{1,2}^{+}$: $1\wedge s_{2,k_2}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2-1}, C_3, C_4, \ldots, C_c, s_{2,k_2}\rangle$,

$\langle s_{1,2}^{-}$: $1\wedge s_{2,2}, s_{2,3}, s_{2,4}, \ldots, s_{2k_2-1}, s_{1,2}, C_3, C_4, \ldots, C_c, s_{2,2}\rangle$.

$\langle s_{2,2}^{+}$: $s_{2,3}, s_{2,4}, \ldots, s_{2,k_2-1}, C_3, C_4, \ldots, C_c, s_{2,k_2}, s_{2,2}, s_{1,2} \rangle$,

$\langle s_{2,2}^{-}$: $s_{2,3}, s_{2,4}, \ldots, s_{2,k_2-1}, C_3, C_4, \ldots, C_c, s_{1,2}, s_{2,k_2}\rangle$.

$\langle s_{2,k_2}^{+}$: $2\wedge s_{1,2}, s_{2,2}, s_{2,3} \ldots, s_{2,k_2-1}, C_3, C_4, \ldots, C_c, s_{2,k_2}, s_{1,2} \rangle$,

$\langle s_{2,k_2}^{-}$: $2\wedge s_{2,2}, s_{2,3}, s_{2,4} \ldots, s_{2,k_2}, s_{1,2}, C_3, C_4, \ldots, C_c, s_{2,2}\rangle$.

$\langle s_{2,3}^{+}$: $s_{2,4}, s_{2,5}, \ldots, s_{2,k_2}, 2\wedge s_{2,3}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, s_{2,2}, C_3, C_4, \ldots, C_c, s_{2,3} \rangle$,

$\langle s_{2,3}^{-}$: $s_{2,4}, s_{2,5}, \ldots, s_{2,k_2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, 1\wedge s_{2,3}, s_{2,2}, C_3, C_4, \ldots, C_c, s_{2,3} \rangle$,

$\langle s_{2,4}^{+}$: $s_{2,5}, s_{2,6}, \ldots, s_{2,k_2}, 2\wedge s_{2,4}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, s_{2,2}, s_{2,3}, C_3, C_4, \ldots, C_c, s_{2,4} \rangle$,

$\langle s_{2,4}^{-}$: $s_{2,5}, s_{2,6}, \ldots, s_{2,k_2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, 1\wedge s_{2,4}, s_{2,2}, s_{2,3}, C_3, C_4, \ldots, C_c, s_{2,4} \rangle$,

……

$\langle s_{2,k_2-1}^{+}$: $s_{2,k_2}, 2\wedge s_{2,k_2-1}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2-2}, C_3, C_4, \ldots, C_c, s_{2,k_2-1} \rangle$,

$\langle s_{2,k_2-1}^{-}$: $s_{2,k_2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, 1\wedge s_{2,k_2-1}, s_{2,2}\ s_{2,3}, \ldots, s_{2,k_2-2}, C_3, C_4, \ldots, C_c, s_{2,k_2-1} \rangle$,

$\langle s_{i,1}^{+}$: $s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, 2\wedge s_{i,1}, s_{1,2}, s_{i,1}\rangle$,

$\langle s_{i,1}^{-}$: $s_{i,2}, s_{i,3}, \ldots, s_{i,k_i}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,2}, 1\wedge s_{i,1}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, s_{i,1}\rangle$,

$\langle s_{i,2}^{+}$: $s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, 2\wedge s_{i,2}, s_{1,2}\ s_{i,2}\rangle$,

$\langle s_{i,2}^{-}$: $s_{i,3}, s_{i,4}, \ldots, s_{i,k_i}, s_{i,1}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,2}, 1\wedge s_{i,2}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, s_{i,2}\rangle$,

……

$\langle s_{i,k_i}^{+}$: $s_{i,1}, s_{i,2} \ldots, s_{i,k_i-1}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, 2\wedge s_{i,k_i}, s_{1,2}, s_{i,k_i}\rangle$,

$\langle s_{i,k_i}^{-}$: $s_{i,1}, s_{i,2} \ldots, s_{i,k_i-1}, C_3, C_4, \ldots, C_{i-1}, C_{i+1}, C_{i+2}, \ldots, C_c, s_{1,2}, 1\wedge s_{i,k_i}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, s_{i,k_i}\rangle$.

The correcting sequences of e_j-pair are the same as Case 3.2.1. Similar to Case 3.2.1, the constructed $2(n-2)$ paths are vertex-disjoint.

Case 3.2.3 $k_1 \geq 3, k_2 \geq 3$

The correcting sequences are listed as follows:

$\langle s_{1,2}^{+}$: $s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2-1}, C_3, C_4, \ldots, C_c, s_{2,k_2}\rangle$,

$\langle s_{1,2}^{-}$: $s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, 1\wedge s_{2,k_2}, s_{1,2}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2-1}, C_3, C_4, \ldots, C_c, s_{2,k_2}\rangle$.

$\langle s_{1,k_1}^{+}$: $1\wedge s_{1,2}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, C_3, C_4, \ldots, C_c, s_{1,2}\rangle$,

$\langle s_{1,k_1}^{-}$: $1\wedge s_{2,2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, s_{2,3}, \ldots, s_{2,k_2}, C_3, C_4, \ldots, C_c, s_{2,2}\rangle$.

$\langle s_{2,2}^{+}$: $s_{2,3}, s_{2,4}, \ldots, s_{2,k_2}, 2\wedge s_{1,k_1}, s_{2,2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1-1}, C_3, C_4, \ldots, C_c, s_{1,k_1}\rangle$,

$\langle s_{2,2}^{-}$: $s_{2,3}, s_{2,4}, \ldots, s_{2,k_2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1-1}, C_3, C_4, \ldots, C_c, s_{1,k_1}\rangle$.

$\langle s_{2,k_2}^{+}$: $2\wedge s_{1,2}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, C_3, C_4, \ldots, C_c, s_{1,2}\rangle$,

$\langle s_{2,k_2}^{-}$: $2\wedge s_{2,2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, s_{2,3}, \ldots, s_{2,k_2}, C_3, C_4, \ldots, C_c, s_{2,2}\rangle$.

$\langle s_{1,3}^{+}$: $s_{1,4}, s_{1,5}, \ldots, s_{1,k_1}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, 2\wedge s_{1,3}, s_{1,2}, C_3, C_4, \ldots, C_c, s_{1,3}\rangle$,

$\langle s_{1,3}^{-}$: $s_{1,4}, s_{1,5}, \ldots, s_{1,k_1}, 1\wedge s_{1,3}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, s_{1,2}, C_3, C_4, \ldots, C_c, s_{1,3}\rangle$,

$\langle s_{1,4}^{+}$: $_{1,5}, s_{1,6}, \ldots, s_{1,k_1}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, 2\wedge s_{1,3}, s_{1,2}, s_{1,3}, C_3, C_4, \ldots, C_c, s_{1,4}\rangle$,

$\langle s_{1,4}^{-}$: $_{1,5}, s_{1,6}, \ldots, s_{1,k_1}, 1\wedge s_{1,4}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, s_{1,2}, s_{1,3}, C_3, C_4, \ldots, C_c, s_{1,4}\rangle$,

……

$\langle s_{1,k_1-1}^{+}$: $s_{1,k_1}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, 2\wedge s_{1,k_1-1}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1-2}, C_3, C_4, \ldots, C_c, s_{1,k_1-1}\rangle$,

$\langle s_{1,k_1-1}^{-}$: $s_{1,k_1}, 1\wedge s_{1,k_1-1}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1-2}, C_3, C_4, \ldots, C_c, s_{1,k_1-1}\rangle$.

$\langle s_{2,3}^{+}$: $s_{2,4}, s_{2,5}, \ldots, s_{2,k_2}, 2\wedge s_{2,3}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, s_{2,2}, C_3, C_4, \ldots, C_c, s_{2,3} \rangle$,

$\langle s_{2,3}^{-}$: $s_{2,4}, s_{2,5}, \ldots, s_{2,k_2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, 1\wedge s_{2,3}, s_{2,2}, C_3, C_4, \ldots, C_c, s_{2,3} \rangle$,

$\langle s_{2,4}^{+}$: $s_{2,5}, s_{2,6}, \ldots, s_{2,k_2}, 2\wedge s_{2,4}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, s_{2,2}, s_{2,3}, C_3, C_4, \ldots, C_c, s_{2,4} \rangle$,

$\langle s_{2,4}^{-}$: $s_{2,5}, s_{2,6}, \ldots, s_{2,k_2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, 1\wedge s_{2,4}, s_{2,2}, s_{2,3}, C_3, C_4, \ldots, C_c, s_{2,4} \rangle$,

……

$\langle s_{2,k_2-1}^{+}$: $s_{2,k_2}, 2\wedge s_{2,k_2-1}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, s_{2,2}, s_{2,3}, \ldots, s_{2,k_2-2}, C_3, C_4, \ldots, C_c, s_{2,k_2-1} \rangle$,

$\langle s_{2,k_2-1}^{-}$: $s_{2,k_2}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1}, 1\wedge s_{2,k_2-1}, s_{2,2}\ s_{2,3}, \ldots, s_{2,k_2-2}, C_3, C_4, \ldots, C_c, s_{2,k_2-1} \rangle$.

Based on misplaced symbols in $C_i = (s_{i,1}\ s_{i,2} \ldots s_{i,k_i})$ where $3 \leq i \leq c$, $(k_3 + k_4 + \ldots + k_c)$ pairs or $2(k_3 + k_4 + \ldots + k_c)$ paths are constructed as that of Case 3.2.1.

The correcting sequences of e_j-pair are the same as Case 3.2.1.

This case provides rules for construction $s_{1,2}$-pair, s_{1,k_1}-pair, $s_{2,2}$-pair, s_{2,k_2}-pair, $((k_1-3) + (k_2-3) + (k_3 + k_4 + \ldots + k_c))$ m-pairs and f f-pairs. Similar to Case 3.2.1, the constructed $2(n-2)$ paths are vertex-disjoint.

Case 3.2.4 $k_1 \geq 3, k_2 = 2$

The correcting sequences are listed as follows:

$\langle s_{1,2}^{+}$: $s_{1,3}, s_{1,4}, \ldots, s_{1,k_1-1}, C_3, C_4, \ldots, C_c, s_{2,2}, s_{1,k_1}\rangle$,

$\langle s_{1,2}^{-}$: $s_{1,3}, s_{1,4}, \ldots, s_{1,k_1-1}, C_3, C_4, \ldots, C_c, s_{1,k_1}, s_{1,2}, s_{2,2}\rangle$.

$\langle s_{1,k_1}^{+}$: $1\wedge s_{1,2}, s_{1,3}, s_{1,4} \ldots, s_{1,k_1}, s_{2,1}, C_3, C_4, \ldots, C_c, s_{1,2}\rangle$,

$\langle s_{1,k_1}^{-}$: $1\wedge s_{2,2}, s_{1,2}, s_{1,3} \ldots, s_{1,k_1-1}, C_3, C_4, \ldots, C_c, s_{1,k_1}, s_{2,2} \rangle$.

$\langle s_{2,2}^{+}$: $2\wedge s_{1,2}, s_{1,3}, s_{1,4}, \ldots, s_{1,k_1}, s_{2,2}, C_3, C_4, \ldots, C_c, s_{1,2}\rangle$,

$\langle s_{2,2}^{-}$: $2\wedge s_{1,k_1}, s_{1,2}, s_{1,3}, \ldots, s_{1,k_1-1}, C_3, C_4, \ldots, C_c, s_{1,k_1}\rangle$.

Based on misplaced symbols in $C_i = (s_{i,1}\ s_{i,2} \ldots s_{i,k_i})$ where $3 \leq i \leq c$, $(k_3 + k_4 + \ldots + k_c)$ pairs or $2(k_3 + k_4 + \ldots + k_c)$ paths are constructed as that of Case 3.2.1. The correcting sequences of e_j-pair are listed as the same to Case 3.2.1.

This case provides rules for construction $s_{1,2}$-pair, s_{1,k_1}-pair, $s_{2,2}$-pair, $((k_1-2) + (k_2-1) + (k_3 + k_4 + \ldots + k_c))$ m-pairs and f f-pairs. Similar to Case 3.2.1, $2(n-2)$ vertex-disjoint paths are totally built.

According to the statements described in Case 3, the following lemma holds.

Lemma 8. The $2(n-2)$ paths constructed in Case 3 are vertex-disjoint.

Lemma 9. In Case 3, $l(C_{2(n-2)}(\rho, \varepsilon)) \leq d(\mathrm{AG}_n)+2$.

Proof. Referring to the constructions of all paths in Case 3, the length of every longest path is at most $d(\rho, \varepsilon)+2$. Two extra edges are added for path disjoint. One is the first edge of the path and the other one is a bold edge, because adding these two edges do not correct any symbol.

Recall that $d(\rho, \varepsilon)$ is $n+c-l-3$ if $1, 2 \in C_i$, $1 \leq i \leq k$

and $|C_i|\geq 3$ [6]. After maximizing $(n+c-l-3)+2$ and substituting c with $1+\lfloor n-4/2\rfloor$, the length of the longest path is at most $n+(1+\lfloor n-4/2\rfloor)-1$. Additionally, $n+(1+\lfloor n-4/2\rfloor)-1 = n+(n-5)/2 = (3n-5)/2 = d(\mathrm{AG}_n)+2$ if n is odd, and $n+(1+\lfloor n-4/2\rfloor)-1 = n+(n-4)/2 = (3n-4)/2 = d(\mathrm{AG}_n)+2$ if n is even.

Similarly, recall that $d(\rho, \varepsilon)$ is $n+c-l-4$ if $1\in C_i$, $2\in C_j$, $1\leq i, j \leq k$, and $i\neq j$. [6]. After maximizing $(n+c-l-4)+2$ and substituting c with $\lfloor n/2\rfloor$, the length of the longest path is at most $n+\lfloor n/2\rfloor-2$. Moreover, $n+\lfloor n/2\rfloor-2 = n+(n-1)/2-2 = (3n-5)/2 = d(\mathrm{AG}_n)+2$ if n is odd, and $n+\lfloor n/2\rfloor-2 = n+(n/2)-2 = (3n-4)/2 = d(\mathrm{AG}_n)+2$ if n is even.

Therefore, the length of the containers constructed in Case 3 is at most $d(\mathrm{AG}_n)+2$. ■

Lemma 10 [9]. If G is a regular graph with connectivity $\kappa\geq 2$, then $d_\kappa(G)\geq d(G)+1$.

Combining Lemma 4−10, we then have Theorem 11 as described below.

Theorem 11. $d(\mathrm{AG}_n)+1 \leq d_{2(n-2)}(\mathrm{AG}_n) \leq d(\mathrm{AG}_n)+2$.

5 Concluding Remarks

In designing a massively parallel computer, we try to maximize parallelism and minimize transmission delay for fast parallel communication in the interconnection network of the computer. Moreover, in order to evaluate reliability, fault tolerant ability, parallelism and transmission delay of an interconnection network, constructing node-disjoint paths in an interconnection network (or a graph) is very important issue. This work presents a novel routing algorithm to construct a $C_{2(n-2)}(\rho, \varepsilon)$ in an AG_n for each vertex ρ. Analysis results indicate that the length of each constructed $C_{2(n-2)}(\rho, \varepsilon)$ is at most $d(\mathrm{AG}_n)+2$, and $d_{2(n-2)}(\mathrm{AG}_n)$ is at least $d(\mathrm{AG}_n)+1$. These measurement results demonstrate that the proposed algorithm can built a $C_{2(n-2)}(\rho, \varepsilon)$ with length at most $d(\mathrm{AG}_n)+1$ or $d(\mathrm{AG}_n)+2$ in an AG_n. Therefore, we assert that the wide diameter of any AG_n is bounded below (above) by $d(\mathrm{AG}_n)+1$ ($d(\mathrm{AG}_n)+2$).

References

[1] T. Araki, Y. Kikuchi, Hamiltonian laceability of bubble-sort graphs with edge faults. *Information Sciences*, vol. 177, no. 13 pp. 2679–2691, 2007.

[2] B.J. Chang and R.H. Hwang, "Modeling and analyzing the performance of adaptive hierarchical networks," *Information Sciences*, vol. 176, no. 5, pp. 522–549, 2006.

[3] Y. Cho and I. Chung, "A parallel routing algorithm on circulant networks employing the Hamiltonian," *Information Sciences*, vol. 176, no. 21, pp. 3132–3142, 2006.

[4] K. Day and A. Tripathi, "A comparative study of topological properties of hypercubes and star graphs," *IEEE Transactions on Parallel and Distributed Systems*, vol. 5, no. 1, pp. 31–38, 1994.

[5] H.J Hsieh, D.R. Duh, "ω-wide diameters of enhanced pyramid networks," *Theor. Comput. Sci*, vol. 412, no.29, pp. 3658–3675, 2011.

[6] J.S. Jwo, S. Lakshmivarahan, and S.K. Dhall, "A new class of interconnection networks based on the alternating group," *Networks*, vol. 23, no. 4, pp. 315–326, 1993.

[7] C.M. Lai, J.J. Tsay, "Communication algorithms on alternating group graph," *In Proceedings of the Second Aizu International Symposium, Parallel Algorithms/Architecture Synthesis*, pp. 104-110, 1997.

[8] C.H. Lin, W.C. Chiu, "A routing scheme for constructing node-to-node disjoint paths in alternating group graphs," *Proc. 19th Workshop on Combinatorial Math. and Computat. Theory*, pp. 89–99, 2002.

[9] T.C. Lin, D.-R. Duh, "Constructing vertex-disjoint paths in (*n*, *k*)-star graphs," *Information Sciences*, vol. 178, no.3, pp. 788–801, 2008.

[10] A. Szepietowski, "Fault tolerance of vertex pancyclicity in alternating group graph," *App. Math. Comput.*, vol. 217, no. 16, pp. 6785–6791, 2011.

[11] P.Y. Tsai, J.S. Fu, and G.H. Chen, "Edge-fault-tolerant pancyclicity of alternating group graph," *Networks*, vol. 53, no. 3, pp. 307–313, 2009.

[12] P.Y. Tsai, J.S. Fu, and G.H. Chen, "Embedding hamiltonian cycles in alternating group graphs under conditional fault model", *Information Sciences*, vol. 179, no. 6, pp. 951–957, 2009.

[13] Y.H. Teng, Jimmy J.M. Tan, L.H. Hsu, "Panpositionable hamiltonicity of the alternating group graphs," *Networks*, vol. 50, no. 2, pp.146–156, 2007.

[14] R.Y. Wu, G. J. Chang, G.H. Chen, "Node-disjoint paths in hierarchical hypercube Networks," *Information Sciences*, vol. 177, no. 19, pp. 4200–4207, 2007.

[15] S. Zhou, W. Xiao, B. Parhami, "Construction of vertex-disjoint paths in alternating group networks," *J Supercomput*, vol. 54, no.2, pp. 206–228, 2010.

[16] Q. Zhu, J.M. Xu, X. Hou, and M. Xu, "On reliability of the folded hypercubes," *Information Sciences*, vol. 177, no. 8, pp. 1782–1788, 2007.