Improving the Height of Independent Spanning Trees on Folded Hyper-Stars*

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Abstract

Hypercubes and star graphs are widespread topologies of interconnection networks. The class of hyper-stars was introduced as a new type of interconnection network to compete with both hypercubes and star graphs, and the class of folded hyper-stars is a strengthened variation of hyper-stars with additional links to connect nodes with complemented 0/1-strings. Constructing independent spanning trees (ISTs) has numerous applications in networks such as fault-tolerant broadcasting and secure message distribution. Recently, Yang and Chang [33] proposed an algorithm to construct $k+1$ ISTs on folded hyper-star $FHS(2k,k)$. For $k \geqslant 4$, their construction includes k ISTs with the height $2k-2$ and the other one with the height $k+1$. In this paper, we refine their constructing rules on $FHS(2k,k)$ for $k \geqslant 3$ and provide a set of construction including k ISTs with the height $k+2$ and the other one with the height $k+1$. As a by-product, we obtain an improvement on the upper bound of the fault diameter (respectively, the wide diameter) of $FHS(2k,k)$.

Keyword: folded hyper-stars; independent spanning trees; interconnection networks; fault-tolerant broadcasting; secure message distribution.

1 Introduction

Fault-tolerant broadcasting and secure message distribution are important issues for numerous applications in networks [1, 15, 26, 32]. It is a common idea to design multiple spanning trees in the underlying graph of a network to serve as a broadcasting scheme or a distribution protocol for receiving high levels of fault-tolerance and security. For achieving such an aim, it relies on the construction of *independent spanning trees* (ISTs). Two spanning trees in a graph G are said to be *independent* if they are rooted at the same node r and such that, for each node $v \neq r$ in G, the two different paths from v to r, one path in each tree, are internally node-disjoint. Moreover, a set of spanning trees of G is said to be independent if they are pairwise independent.

For a graph (or network) G, the *independent spanning trees* (IST) *problem* attempts to construct a maximal set of ISTs rooted at any node r of G and such that the cardinality of the set of ISTs matches the connectivity of G. Although the problem is hard for general graphs, several results are known for some special classes of graphs (especially, the graph classes related to interconnection networks), such as k-connected graphs with $k \leqslant 4$ (see [15], [9, 39] and [10] for $k = 2, 3, 4$, respectively), recursive circulant graphs [36, 37], deBruijn and Kautz graphs [12, 13], chordal ring [16, 35], graphs defined by Cartesian product [2, 25, 27, 28, 31, 34, 38], variations of hypercubes [3–6,23,24,29,30,32] and special classes of Cayley graphs [18, 19, 26, 33, 36, 37].

For $n \geqslant 3$ and $1 \leqslant k \leqslant n-1$, the *hyper-star* $HS(n,k)$ is a graph such that every node is associated with a distinct binary string of length n that contains exactly k 1's, and two nodes are adjacent if and only if one can be obtained from the other by exchanging the first symbol with a different symbol (1 with 0, or 0 with 1) in another position. Hyper-stars were introduced by Lee et al. [22] and Kim et al. [20] as a competitive model of interconnection network for both hypercubes and star graphs. Lee et al. [22] showed that $HS(n,k)$ is isomorphic to $HS(n,n-k)$, $HS(n,k)$ has the diameter $n-1$, $HS(2k,k)$ is maximally fault-tolerant (i.e., the connectivity equals the regularity), and $HS(2k,k)$ can be constructed from $HS(2k-1,k-1)$ and $HS(2k-1,k)$ by adding appropriate edges. In addition, they proposed a routing algorithm for $HS(2k,k)$. Kim et al. [20] particularly intensify properties of $HS(2k,k)$ as

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follows: $HS(2k,k)$ is node-symmetric, the wide-diameter of $HS(2k,k)$ is bounded by the shortest path length plus 4, the fault-diameter of $HS(2k,k)$ is bounded by its diameter plus 2 (i.e., $2k+1$). In addition, they proposed an efficient broadcasting scheme for $HS(2k,k)$ based on a spanning tree with the minimum height. Furthermore, stronger structural properties (such as edge-symmetry, super-connectivity and orientability) and some embedding schemes for hyper-stars were provided in [7,8] and [17], respectively.

Inspired by the idea of El-Amawy and Latifi [11] that proposed the so-called folded hypercubes to strengthen the structure of hypercubes, a variation of hyper-stars was introduced in [22] as follows. The *folded hyper-star* $FHS(2k,k)$ is constructed from $HS(2k,k)$ by adding edges to connect nodes whose binary strings are complements. A result in [22] also showed that hyper-stars and folded hyper-stars have a lower network cost (measured by the product of degree and diameter) than that of hypercubes, folded hypercubes, and other variants with the same number of nodes. In particular, $FHS(2k,k)$ has the diameter k. Recently, Yang and Chang [33] showed that $FHS(2k,k)$ is node-symmetric and proposed the following results of $FHS(2k,k)$: for $k \geqslant 4$ (respective, $k \leqslant 3$), there exist k ISTs with the height $2k-2$ (respectively, $2k-1$) and the other one with the height $k+1$. Consequently, $FHS(2k,k)$ has the connectivity $k+1$.

In this paper, we revisit the problem of constructing ISTs on $FHS(2k,k)$. For $k \geqslant 3$, we refine the constructing rules given in [33] to produce $k+1$ ISTs, where k ISTs have the height $k+2$ and the other one has the height $k+1$. The rest of this paper is organized as follows. Section 2 introduces the constructing rules of ISTs in [33], and then refine them to become a set of new ones. Section 3 analyzes the heights of ISTs. Section 4 shows the correctness of constructing rules. The final section contains our concluding remarks.

2 The rules of constructing ISTs

In this paper, we use the following notation. Let $V(G)$ and $E(G)$ be the node set and edge set, respectively, of a graph G. Two paths P and Q joining the same nodes x and y are *internally node-disjoint*, denoted by $P||Q$, if $V(P) \cap V(Q) = \{x,y\}$ and $E(P) \cap E(Q) = \emptyset$. A *spanning tree* T in a graph G is a subgraph containing all nodes of G and without forming a cycle. For $x,y \in V(T)$, the unique path from x to y in T is denoted by $T[x,y]$. Thus, two spanning trees T and T' with the same root r are ISTs if and only if $T[r,x] \,||\, T'[r,x]$ for every node $x \in V(T) \setminus \{r\}$.

For each node $x \in FHS(2k,k)$ with binary string $x = x_1x_2\cdots x_{2k}$, we define $\rho_i(x)$ to be the operation that exchanges x_1 with x_i provided that $2 \leqslant i \leqslant 2k$ and $x_i = \bar{x}_1$ (i.e., x_i is the complement of x_1). In this case, $(x, \rho_i(x))$ is called a *normal edge*. Also, we define $\rho_*(x)$ to be the operation that takes the complement of x_i for all $i = 1,2,\ldots,2k$. In this case, $(x,\rho_*(x))$ is called a *complement edge*. For example, Figure 1 shows the graph $FHS(6,3)$, where each node is labeled by its binary representation and octal representation. Solid lines indicate normal edges and dotted lines represent complement edges. For notational convenience, we use the notation $x \xrightarrow{i} y$ if $\rho_i(x) = y$ (respectively, $x \xrightarrow{*} y$ if $\rho_*(x) = y$) to mean that x and y are connected by a normal edge (respectively, complement edge). For example, $100110(46) \xrightarrow{*} 011001(31) \xrightarrow{2} 101001(51) \xrightarrow{4} 001101(15) \xrightarrow{3} 100101(45) \xrightarrow{5} 000111(07)$ represents a path connecting nodes 100110 and 000111 in $FHS(6,3)$.

For constructing ISTs of $FHS(2k,k)$, by node-symmetry, we consider $r = \overbrace{0\cdots0}^{k}\overbrace{1\cdots1}^{k}$ (0^k1^k for short) as the root of ISTs. Let $I = \{1,2,\ldots,2k\}$.

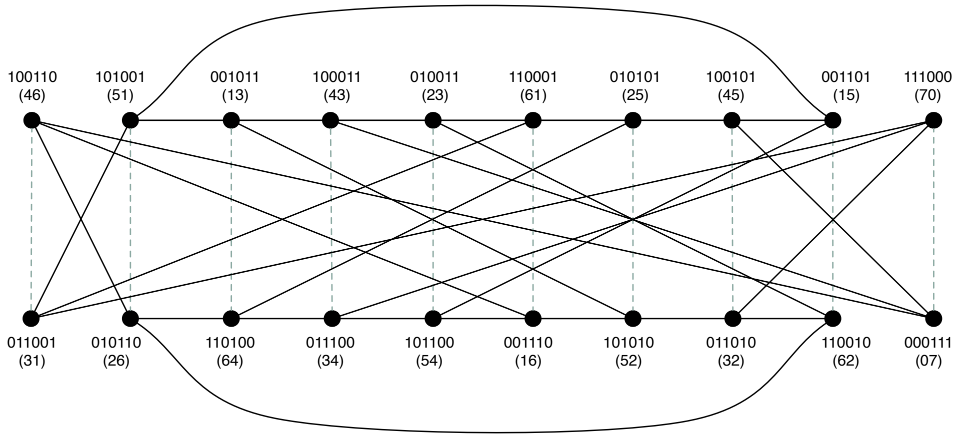

Figure 1: Folded hyper-star $FHS(6,3)$.

For each node $x(\neq r) \in FHS(2k,k)$ with binary representation $x = x_1x_2\cdots x_{2k}$ and $b \in \{0,1\}$, we define $H_x^b = \{i \in I\colon x_i = b\}$. Also, we write $H_x^b(i,i') = \{j \in H_x^b : i < j \leqslant i'\}$ to mean the restricted set of H_x^b. It is clear that $|H_x^0(0,k)| = |H_x^1(k,2k)|$ and $|H_x^1(0,k)| = |H_x^0(k,2k)|$. Let $F_0 : H_x^0(k,2k) \to H_x^1(0,k)$ and $F_1 : H_x^1(k,2k) \cup \{*\} \to H_x^0(0,k) \cup \{*\}$ be two increasing functions that preserves the related order of elements between their domain and codomain, where '$*$' is regarded as the largest element in $H_x^1(k,2k) \cup \{*\}$ (respectively, in $H_x^0(0,k) \cup \{*\}$). Obviously, $F_1(*) = *$. In [33], according to the parity of x_1, two crucial functions are defined as follows. For $x_1 = 0$, a bijection between $\{k+1, k+2, \ldots, 2k, *\}$ and $H_x^1 \cup \{*\}$ is given by

$$\textsc{next}_x^0(i) = \begin{cases} \min H_x^1(i,2k) & \text{if } i \neq *,\ x_i = 1,\ |H_x^1(k,2k)| \leqslant k/2, \text{ and } H_x^1(i,2k) \neq \emptyset; & (1.1)\\ * & \text{if } i \neq *,\ x_i = 1,\ |H_x^1(k,2k)| \leqslant k/2, \text{ and } H_x^1(i,2k) = \emptyset; & (1.2)\\ i & \text{if } i \neq *,\ x_i = 1, \text{ and } |H_x^1(k,2k)| > k/2; & (1.3)\\ F_0(\min H_x^0(i,2k)) & \text{if } i \neq *,\ x_i = 0, \text{ and } H_x^0(i,2k) \neq \emptyset; & (1.4)\\ F_0(\min H_x^0(k,2k)) & \text{if } i \neq *,\ x_i = 0, \text{ and } H_x^0(i,2k) = \emptyset; & (1.5)\\ \min H_x^1(k,2k) & \text{if } i = * \text{ and } |H_x^1(k,2k)| \leqslant k/2; & (1.6)\\ * & \text{if } i = * \text{ and } |H_x^1(k,2k)| > k/2. & (1.7) \end{cases}$$

For $x_1 = 1$, a bijection between $\{k+1, k+2, \ldots, 2k, *\}$ and $H_x^0 \cup \{*\}$ is given by

$$\textsc{next}_x^1(i) = \begin{cases} \min H_x^0(i,2k) & \text{if } i \neq *,\ x_i = 0,\ |H_x^0(k,2k)| \neq k, \text{ and } H_x^0(i,2k) \neq \emptyset; & (2.1)\\ \min H_x^0(k,2k) & \text{if } i \neq *,\ x_i = 0,\ |H_x^0(k,2k)| \neq k, \text{ and } H_x^0(i,2k) = \emptyset; & (2.2)\\ i & \text{if } i \neq *,\ x_i = 0, \text{ and } |H_x^0(k,2k)| = k; & (2.3)\\ F_1(\min H_x^1(i,2k)) & \text{if } i \neq *,\ x_i = 1,\ |H_x^1(k,2k)| \leqslant k/2, \text{ and } H_x^1(i,2k) \neq \emptyset; & (2.4)\\ F_1(*) & \text{if } i \neq *,\ x_i = 1,\ |H_x^1(k,2k)| \leqslant k/2, \text{ and } H_x^1(i,2k) = \emptyset; & (2.5)\\ F_1(i) & \text{if } i \neq *,\ x_i = 1, \text{ and } |H_x^1(k,2k)| > k/2; & (2.6)\\ F_1(\min H_x^1(k,2k)) & \text{if } i = * \text{ and } 0 \neq |H_x^1(k,2k)| \leqslant k/2; & (2.7)\\ F_1(*) & \text{if } i = * \text{ and } (|H_x^1(k,2k)| > k/2 \text{ or } |H_x^1(k,2k)| = 0). & (2.8) \end{cases}$$

We now refine Eqs.(1.4) and (1.5) by the following rules:

$$\textsc{next}_x^0(i) = \begin{cases} F_0(\min H_x^0(i,2k)) & \text{if } i \neq *,\ x_i = 0, |H_x^0(k,2k)| \leqslant k/2+1, \text{ and } H_x^0(i,2k) \neq \emptyset; & (1.8)\\ F_0(\min H_x^0(k,2k)) & \text{if } i \neq *,\ x_i = 0, |H_x^0(k,2k)| \leqslant k/2+1, \text{ and } H_x^0(i,2k) = \emptyset; & (1.9)\\ F_0(i) & \text{if } i \neq *,\ x_i = 0, \text{ and } |H_x^0(k,2k)| > k/2+1. & (1.10) \end{cases}$$

By contrast, we refine Eqs.(2.1), (2.2) and (2.3) by the following rules:

$$\textsc{next}_x^1(i) = \begin{cases} \min H_x^0(i,2k) & \text{if } i \neq *,\ x_i = 0,\ |H_x^0(k,2k)| \leqslant k/2+1, \text{ and } H_x^0(i,2k) \neq \emptyset; & (2.9)\\ \min H_x^0(k,2k) & \text{if } i \neq *,\ x_i = 0,\ |H_x^0(k,2k)| \leqslant k/2+1, \text{ and } H_x^0(i,2k) = \emptyset; & (2.10)\\ i & \text{if } i \neq *,\ x_i = 0, \text{ and } |H_x^0(k,2k)| > k/2+1. & (2.11) \end{cases}$$

The above functions mean that we consider $H_x^0(k,2k)$ and $H_x^1(k,2k) \cup \{*\}$ as two cyclic ordered set in increasing order and perform operation according to the following rules:

R1: For $x_1 = 0$ and $i \in H_x^1(k,2k) \cup \{*\}$, if the number of '1' in $H_x^1(k,2k)$ is no more than $k/2$, then it maps i to the next position of '1' or '$*$' in the cyclic order; otherwise, it maps i to itself. (cf. Eqs.(1.1), (1.2), (1.3), (1.6) and (1.7))

R2: For $x_1 = 0$ and $i \in H_x^0(k,2k)$, if the number of '0' in $H_x^0(k,2k)$ is no more than $k/2+1$, then it maps i to $F_0(j)$ where j is the next position of '0' in the cyclic order; otherwise, it maps i to $F_0(i)$. (cf. Eqs.(1.8), (1.9) and (1.10))

R3: For $x_1 = 1$ and $i \in H_x^1(k,2k) \cup \{*\}$, if the number of '1' in $H_x^1(k,2k)$ is no more than $k/2$, then it maps i to $F_1(\ell)$ where ℓ is the next position of '1' or '$*$' in the cyclic order; otherwise, it maps i to $F_1(i)$. (cf. Eqs.(2.4), (2.5), (2.6), (2.7) and (2.8))

R4: For $x_1 = 1$ and $i \in H_x^0(k,2k)$, if the number of '0' in $H_x^0(k,2k)$ is no more than $k/2+1$, then it maps i to the next position of '0' in the cyclic order; otherwise, it maps i to itself. (cf. Eqs.(2.9), (2.10) and (2.11))

For instance, if $x = 0101010011$, then $|H_x^0(5,10)| = |\{7,8\}| = 2 \leqslant 5/2+1$ and $|H_x^1(5,10)| = |\{6,9,10\}| = 3 > 5/2$. In this case, we have $\textsc{next}_x^0(i) = i$ for $i \in \{6,9,10\}$ (cf. Eq.(1.3)), $\textsc{next}_x^0(7) = F_0(8) = 4$ (cf. Eq.(1.8)),

$\text{NEXT}_x^0(8) = F_0(7) = 2$ (cf. Eq.(1.9)), and $\text{NEXT}_x^0(*) = *$ (cf. Eq.(1.7)). Also, if $x = 1011010010$, then $|H_x^0(5,10)| = |\{7,8,10\}| = 3 \leqslant 5/2+1$ and $|H_x^1(5,10)| = |\{6,9\}| = 2 \leqslant 5/2$. In this case, we have $\text{NEXT}_x^1(6) = F_1(9) = 5$ (cf. Eq.(2.4)), $\text{NEXT}_x^1(7) = 8$ and $\text{NEXT}_x^1(8) = 10$ (cf. Eq.(2.9)), $\text{NEXT}_x^1(9) = *$ (cf. Eq.(2.5)), $\text{NEXT}_x^1(10) = 7$ (cf. Eq.(2.10)), and $\text{NEXT}_x^1(*) = F_1(6) = 2$ (cf. Eq.(2.7)).

For $i \in \{k+1, k+2, \ldots, 2k, *\}$, let T_i denote the spanning tree such that the root and its unique child are connected by an edge with label i. The construction can be carried out by describing the parent of each node $x (\neq 0^k1^k) \in FHS(2k,k)$ in each spanning tree.

Algorithm CONSTRUCTING-ISTS

for each $i \in \{k+1, k+2, \ldots, 2k, *\}$ **do**
    for each node $x = x_1x_2\cdots x_{2k} (\neq 0^k1^k)$ in $FHS(2k,k)$ **do**
        let $j = \text{NEXT}_x^b(i)$, where $b = x_1 \in \{0,1\}$;
        $\text{PARENT}(T_i, x) = \rho_j(x)$;

Figure 2: Algorithm for constructing $k+1$ spanning trees in $FHS(2k,k)$.

Note that the set of $k+1$ ISTs can be constructed simultaneously in parallel. In fact, according to the algorithm, we provide a fully parallelized approach for the construction of each spanning tree. For $k = 3, 4$, the sets of spanning trees we constructed are the same as those in [33], and thus the reader can refer therein to view the case of $k = 3$.

3 Analysis of the heights of ISTs

For notational convenience, we write H_x^b instead of $H_x^b(k,2k)$ for $b \in \{0,1\}$ in this section. Note that $|H_x^0| + |H_x^1| = k$. Firstly, we prove the reachability between every node $x (\neq r)$ and the root $r = 0^k1^k$ in T_i, thereby proving the existence of a unique path from x to the root in the tree.

Lemma 1. *For each $i \in \{k+1, k+2, \ldots, 2k, *\}$, T_i is a spanning tree rooted at $r = 0^k1^k$ in $FHS(2k,k)$.*

Proof. Let $x (\neq r) \in FHS(k,2k)$ be any node with binary string $x = x_1x_2\cdots x_{2k}$. By the rule of our algorithm, x is adjacent to $\rho_j(x)$ in T_i, where $j = \text{NEXT}_x^b(i)$ and $b = x_1$. Suppose that $H_x^0 = \{j_1, j_2, \ldots, j_s\}$ with $j_1 < j_2 < \cdots < j_s$ and $H_x^1 = \{\ell_1, \ell_2, \ldots, \ell_t\}$ with $\ell_1 < \ell_2 < \cdots < \ell_t$, where $s + t = k$. We assume that the index arithmetic of j_p (respectively, ℓ_q) is taken modulo s (respectively, t). Consider the following four cases:

Case 1: $i \in H_x^0$ and $|H_x^0| \leqslant k/2+1$. There are two subcases as follows.

Case 1.1: $x_1 = 1$. By R4, we suppose that there exists $j_p \in H_x^0$ such that $\text{NEXT}_x^1(i) = j_p$ (i.e., $i = j_{p-1}$) and let $y = \rho_{j_p}(x) = y_1y_2\cdots y_{2k}$. Clearly, $y_1 = x_{j_p} = 0$, $y_{j_p} = x_1 = 1$, and $H_y^0 = \{j_1, j_2, \ldots, j_{p-1}, j_{p+1}, \ldots, j_s\}$. Since $y_1 = 0$, $i \in H_y^0$ and $|H_y^0| \leqslant k/2+1$, by R2, we let $j = F_0(j_{p+1}) = \text{NEXT}_y^0(i)$ and $y' = \rho_j(y) = y'_1y'_2\cdots y'_{2k}$. Clearly, $y'_1 = y_j = 1$, $y'_j = y_1 = 0$, and $H_{y'}^0 = H_y^0$. Again, since $y'_1 = 1$, $i \in H_{y'}^0$ and $|H_{y'}^0| \leqslant k/2+1$, by R4, we have $j_{p+1} = \text{NEXT}_{y'}^1(i)$ and let $z = \rho_{j_{p+1}}(y') = z_1z_2\cdots z_{2k}$. Clearly, $z_1 = y'_{j_{p+1}} = 0$, $z_{j_{p+1}} = y'_1 = 1$, and $H_z^0 = \{j_1, j_2, \ldots, j_{p-1}, j_{p+2}, \ldots, j_s\}$. By this way, we can find the unique path P connecting x and r in T_i as follows:

$$P : x \xrightarrow{j_p} y \xrightarrow{F_0(j_{p+1})} y' \xrightarrow{j_{p+1}} z \xrightarrow{F_0(j_{p+2})} \cdots \xrightarrow{F_0(j_{p-1})}$$

$$(10^{k-1}1^{i-k-1}01^{2k-i}) \xrightarrow{i=j_{p-1}} r (= 0^k1^k).$$

Case 1.2: $x_1 = 0$. By R2, we let $j = F_0(j_p) = \text{NEXT}_x^0(i)$ for some $1 \leqslant p \leqslant s$ and let $y = \rho_j(x) = y_1y_2\cdots y_{2k}$. Clearly, $y_1 = x_j = 1$, $y_j = x_1 = 0$, and $H_y^0 = H_x^0$. Since $y_1 = 1$, $i \in H_y^0$ and $|H_y^0| \leqslant k/2+1$, this shows that y is in the situation of Case 1.1. Let P be the path connecting y and r in such a case. Therefore, we can find the unique path $T_i[x,r]$ by concatenating $x \xrightarrow{j} y$ and P.

Case 2: $i \in H_x^1 \cup \{*\}$ and $|H_x^1| \leqslant k/2$. There are two subcases as follows.

Case 2.1: $x_1 = 0$. This case implies $H_x^1 \neq \emptyset$. By R1, either $\text{NEXT}_x^0(i) = *$ or there exists $\ell_q \in H_x^1$ such that $\text{NEXT}_x^0(i) = \ell_q$. Let $y = \rho_{\ell_q}(x) = y_1y_2\cdots y_{2k}$. If $\text{NEXT}_x^0(i) = *$, then $y_1 = \bar{x}_1 = 1$. Since $H_x^1 \neq \emptyset$, it follows that $i \in H_x^1$. In this case, since $y_1 = 1$, $i \in H_y^0$ and $|H_y^0| = |H_x^1| \leqslant k/2$, this shows that y is in the situation of Case 1.1. Let P be the path connecting y and r in such a case. Therefore, we can find the unique path $T_i[x,r]$ by concatenating $x \xrightarrow{*} y$ and P. On the other hand (i.e., $i = \ell_{q-1}$), we have $y_1 = x_{\ell_q} = 1$, $y_{\ell_q} = x_1 = 0$, and $H_y^1 = \{\ell_1, \ell_2, \ldots, \ell_{q-1}, \ell_{q+1}, \ldots, \ell_t\}$. Since $y_1 = 1$, $i \in H_y^1$ and $|H_y^1| \leqslant k/2$, by R3, we let $j = F_1(\ell_{q+1}) = \text{NEXT}_y^1(i)$ and $y' = \rho_j(y) = y'_1y'_2\cdots y'_{2k}$. Clearly, $y'_1 = y_j = 0$, $y'_j = y_1 = 1$, and $H_{y'}^1 = H_y^1$. Again, since $y'_1 = 0$, $i \in H_{y'}^1$ and $|H_{y'}^1| \leqslant k/2$, by R1, we have $\ell_{q+1} = \text{NEXT}_{y'}^0(i)$ and let $z = \rho_{\ell_{q+1}}(y') = z_1z_2\cdots z_{2k}$. Clearly, $z_1 = y'_{\ell_{q+1}} = 1$, $z_{\ell_{q+1}} = y'_1 = 0$, and $H_z^1 = \{\ell_1, \ell_2, \ldots, \ell_{q-1}, \ell_{q+2}, \ldots, \ell_t\}$. We can repeat a similar process until the path passes through an edge with label $*$ that connects nodes $\bar{w}$ and w, where $w = \rho_*(\bar{w}) = w_1w_2\cdots w_{2k}$. Let Q be the path described as above:

$$Q : x \xrightarrow{\ell_q} y \xrightarrow{F_1(\ell_{q+1})} y' \xrightarrow{\ell_{q+1}} z \xrightarrow{F_1(\ell_{q+2})} \cdots \xrightarrow{F_1(\ell_t)} v' \xrightarrow{\ell_t} \bar{w} \xrightarrow{*} w.$$

Clearly, we have $w_1 = 0$, $i \in H_w^0 = \{\ell_1, \ell_2, \ldots, \ell_{q-1}\}$ and $|H_w^0| \leqslant k/2$. Thus, w is in the situation of Case 1.2. Let P be the path connecting w and r in such a case. Therefore, we can find the unique path $T_i[x, r]$ by concatenating Q and P.

Case 2.2: $x_1 = 1$. By R3, we let $j = F_1(\ell_q) = \textsc{next}_x^1(i)$ for some $\ell_q \in H_x^1 \cup \{*\}$ and let $y = \rho_j(x) = y_1y_2 \cdots y_{2k}$. If $j = *$, then $y_1 = \bar{x}_1 = 0$. In this case, either $i = *$ (i.e., $H_x^1 = \emptyset$) or $i = \ell_t \in H_x^1$. If $i = *$, then $x(= 1^k0^k) \xrightarrow{*} r$ is the desired path. Otherwise, $i \in H_x^1$ implies $i \in H_y^0$. Since $y_1 = 0$, $i \in H_y^0$ and $|H_y^0| = |H_x^1| \leqslant k/2$, this shows that y is in the situation of Case 1.2. Let Q be the path connecting y and r in such a case. Therefore, we can find the unique path $T_i[x, r]$ by concatenating $x \xrightarrow{*} y$ and Q. On the other hand (i.e., $j \neq *$), we have $y_1 = x_j = 0$ and $y_j = x_1 = 1$. Since $y_1 = 0$, $i \in H_y^1 \cup \{*\}$ and $|H_y^1| = |H_x^1| \leqslant k/2$, this shows that y is in the situation of Case 2.1. Let Q be the path connecting y and r in such a case. Therefore, we can find the unique path $T_i[x, r]$ by concatenating $x \xrightarrow{j} y$ and Q.

Case 3: $i \in H_x^0$ and $|H_x^0| > k/2 + 1$. There are two subcases as follows.

Case 3.1: $x_1 = 1$. By R4, we have $\textsc{next}_x^1(i) = i$. Let $y = \rho_i(x) = y_1y_2 \cdots y_{2k}$. Clearly, $y_1 = x_i = 0$ and $y_i = x_1 = 1$. Since $y_1 = 0$, $i \in H_y^1$ and $|H_y^1| = k - (|H_x^0| - 1) < k/2$, this shows that y is in the situation of Case 2.1. Let Q be the path connecting y and r in such a case. Therefore, we can find the unique path $T_i[x, r]$ by concatenating $x \xrightarrow{i} y$ and Q.

Case 3.2: $x_1 = 0$. By R2, we let $j = F_0(i) = \textsc{next}_x^0(i)$ and $y = \rho_j(x) = y_1y_2 \cdots y_{2k}$. Clearly, $y_1 = x_j = 1$, $y_j = x_1 = 0$, and $H_y^0 = H_x^0$. Since $y_1 = 1$, $i \in H_y^0$ and $|H_y^0| > k/2 + 1$, this shows that y is in the situation of Case 3.1. Let P be the path connecting y and r in such a case. Therefore, we can find the unique path $T_i[x, r]$ by concatenating $x \xrightarrow{j} y$ and P.

Case 4: $i \in H_x^1 \cup \{*\}$ and $|H_x^1| > k/2$. There are two subcases as follows.

Case 4.1: $x_1 = 0$. By R1, we have $\textsc{next}_x^0(i) = i$. Let $y = \rho_i(x) = y_1y_2 \cdots y_{2k}$. If $i = *$, then $y_1 = \bar{x}_1 = 1$. In this case, since $y_1 = 1$, $i = *$ and $|H_y^1| = k - |H_x^1| < k/2$, this shows that y is in the situation of Case 2.2. Let P be the path connecting y and r in such a case. Therefore, we can find the unique path $T_i[x, r]$ by concatenating $x \xrightarrow{*} y$ and P. On the other hand (i.e., $i \neq *$), we have $y_1 = x_i = 1$ and $y_i = x_1 = 0$. Since $y_1 = 1$, $i \in H_y^0$ and $|H_y^0| = |H_x^0| + 1 < k/2 + 1$, this shows that y is in the situation of Case 1.1. Let P be the path connecting y and r in such a case. Therefore, we can find the unique path $T_i[x, r]$ by concatenating $x \xrightarrow{i} y$ and P.

Case 4.2: $x_1 = 1$. By R3, we let $j = F_1(i) = \textsc{next}_x^1(i)$ and $y = \rho_j(x) = y_1y_2 \cdots y_{2k}$. If $i = *$, then $j = *$, and thus $y_1 = \bar{x}_1 = 0$. Since $y_1 = 0$, $i = *$ and $|H_y^1| = k - |H_x^1| < k/2$, this shows that y is in the situation of Case 2.1. Let Q be the path connecting y and r in such a case. Therefore, we can find the unique path $T_i[x, r]$ by concatenating $x \xrightarrow{*} y$ and Q. On the other hand (i.e., $i \neq *$), we have $y_1 = x_j = 0$ and $y_j = x_1 = 1$. Since $y_1 = 0$, $i \in H_y^1$ and $|H_y^1| = |H_x^1| > k/2$, this shows that y is in the situation of Case 4.1. Let Q be the path connecting y and r in such a case. Therefore, we can find the unique path $T_i[x, r]$ by concatenating $x \xrightarrow{j} y$ and Q. □

According to Lemma 1, we can determine the length of the unique path from a node $x = x_1x_2 \cdots x_{2k-1}x_{2k}$ to r in each spanning tree. We summarize the length of $T_i[x, r]$ in Table 1. From this table, we can compute the longest paths in each tree. For T_i with $i \neq *$, the length of a longest path is $k + 2$, which occurs in Case 1.2 (i.e., when $x_i = x_1 = 0$ and $|H_x^0| = k/2+1$). For T_*, the length of a longest path is $k + 1$, which occurs in Case 2.2 (i.e., when $x_1 = 1$ and $|H_x^1| = k/2$). Let $\textsc{height}(T)$ denote the height of a tree T. Therefore, we have the following lemma.

Table 1: The length of the unique path from a node x to $r = 0^k1^k$ in T_i.

Conditions		cf. Lemma 1	Length of $T_i[x, r]$
$i \in H_x^0$ and	$x_1 = 1$	Case 1.1	$2\|H_x^0\| - 1$
$\|H_x^0\| \leqslant k/2 + 1$	$x_1 = 0$	Case 1.2	$2\|H_x^0\|$
$i \in H_x^1 \cup \{*\}$ and	$x_1 = 0$	Case 2.1	$2\|H_x^1\|$
$\|H_x^1\| \leqslant k/2$	$x_1 = 1$	Case 2.2	$2\|H_x^1\| + 1$
$i \in H_x^0$ and	$x_1 = 1$	Case 3.1	$2\|H_x^1\| + 3$
$\|H_x^0\| > k/2 + 1$	$x_1 = 0$	Case 3.2	$2\|H_x^1\| + 4$
$i \in H_x^1 \cup \{*\}$ and	$x_1 = 0$	Case 4.1	$2\|H_x^0\| + 2$
$\|H_x^1\| > k/2$	$x_1 = 1$	Case 4.2	$2\|H_x^0\| + 3$

Lemma 2. *For $k \geqslant 3$, the heights of the constructed spanning trees are as follows:*

$$\textsc{height}(T_i) = \begin{cases} k + 2 & \text{if } i \in \{k + 1, k + 2, \ldots, 2k\}; \\ k + 1 & \text{if } i = * \end{cases}$$

Comparing with the heights of ISTs given in [33], we can see that the results are the same for $k = 3, 4$. However, for $k \geqslant 5$, if $i \neq *$, then $\textsc{height}(T_i) = k + 2 < 2k - 2$. Thus, the heights of ISTs are significantly improved.

4 Proof of independency

In this section, we show the independency of ISTs. Before this, we need the following notation. Let $r = 0^k1^k$ and $x(\neq r) \in FHS(2k, k)$ be any node with binary string $x = x_1x_2 \ldots x_{2k-1}x_{2k}$. Suppose that $P = T_i[x, r]$ and $Q = T_j[x, r]$. If $x_1 = 1$, we let $i' = \textsc{next}^1_x(i)$ and $j' = \textsc{next}^1_x(j)$; otherwise, we let $i' = \textsc{next}^0_x(i)$ and $j' = \textsc{next}^0_x(j)$. In what follows, we always assume that $y = y_1y_2 \cdots y_{2k-1}y_{2k}$ is any node in $P \setminus \{x, r\}$ and $z = z_1z_2 \cdots z_{2k-1}z_{2k}$ is any node in $Q \setminus \{x, r\}$. In particular, let $\check{y} = \check{y}_1\check{y}_2 \cdots \check{y}_{2k-1}\check{y}_{2k}$ (respectively, $\check{z} = \check{z}_1\check{z}_2 \cdots \check{z}_{2k-1}\check{z}_{2k}$) be the node adjacent to x in P (respectively, in Q) and let $\hat{y} = \hat{y}_1\hat{y}_2 \cdots \hat{y}_{2k-1}\hat{y}_{2k}$ (respectively, $\hat{z} = \hat{z}_1\hat{z}_2 \cdots \hat{z}_{2k-1}\hat{z}_{2k}$) be the node adjacent to r in P (respectively, in Q). Also, we use $P(u, v)$ to denote the subpath of P between two nodes $u, v \in V(P)$. Moreover, for $b \in \{0, 1\}$ and $1 \leqslant \ell \leqslant 2k$, we write $y_\ell|P(u, v) = b$ to mean that the bit y_ℓ is assigned to b for every node $y \in P(u, v)$. Similarly, we can define $Q(u, v)$ and $z_\ell|Q(u, v) = b$ by the same way.

Lemma 3. *If $i, j \in H^0_x$ with $i \neq j$, then $P||Q$.*

Proof. We consider the following two cases:

Case 1: $|H^0_x| \leqslant k/2 + 1$. Since $|H^0_x| \geqslant 2$, by R2 or R4, we have $i' \neq i$, $j' \neq j$ and $i' \neq j'$. Note that it is possible $i' = j$ or $j' = i$. There are two subcases as follows.

Case 1.1: $x_1 = 1$. We observe paths P and Q constructed in Case 1.1 of Lemma 1. Since $x_1 = 1$ and P takes the first link with label i' to connect x, we have $\check{y}_{i'} = 1$. Moreover, since P never changes the bit $y_{i'}$ in the succedent path again, it follows that $y_{i'}|P(\check{y}, \hat{y}) = 1$. In addition, since it never changes the bit y_i in P until the last link connecting to r, we have $y_i|P(\check{y}, \hat{y}) = 0$. On the other hand, let $v, v' \in V(Q)$ be nodes such that $v' = \rho_i(v)$. Since it has not dealt with the bit $z_{i'}$ for every node in $Q(\check{z}, v)$, we have $z_{i'}|Q(\check{z}, v) = 0$. Also, since $Q(v', \hat{z})$ has dealt with the bit z_i, we have $z_i|Q(v', \hat{z}) = 1$. This shows that $P(\check{y}, \hat{y}) \cap (Q(\check{z}, v) \cup Q(v', \hat{z})) = \emptyset$. Thus, $P||Q$.

Case 1.2: $x_1 = 0$. We observe paths P and Q constructed in Case 1.2 of Lemma 1. Clearly, $\check{y} = \rho_{F_0(i')}(x)$ and $\check{z} = \rho_{F_0(j')}(x)$. Since $F_0(i') \neq F_0(j')$, it implies $\check{y} \neq \check{z}$. Also, since $\check{y}_{F_0(j')} = 1$ and $z_{F_0(j')}|Q(\check{z}, \hat{z}) = 0$, it follows $\check{y} \notin Q(\check{z}, \hat{z})$. Similarly, since $\check{z}_{F_0(i')} = 1$ and $y_{F_0(i')}|P(\check{y}, \hat{y}) = 0$, it follows $\check{z} \notin P(\check{y}, \hat{y})$. The remaining proof is similar to Case 1.1.

Case 2: $|H^0_x| > k/2 + 1$. By R2 or R4, we have $i' = i$ and $j' = j$. There are two subcases as follows.

Case 2.1: $x_1 = 1$. We observe paths P and Q constructed in Case 3.1 of Lemma 1. Let $u, u' \in V(P)$ be nodes such that $u' = \rho_*(u)$ and let $v, v' \in V(Q)$ be nodes such that $v' = \rho_*(v)$. Since $x_1 = 1$ and P takes the first link with label i to connect x, we have $y_i|P(\check{y}, u) = 1$ and it follows $y_i|P(u', \hat{y}) = 0$. On the other hand, since it has not dealt with the bit z_i for every node in $Q(\check{z}, v)$, we have $z_i|Q(\check{z}, v) = 0$ and it follows $z_i|Q(v', \hat{z}) = 1$. Also, since $k \geqslant 3$ and $|H^0_x| > k/2 + 1$, there is a position $m \in H^0_x \setminus \{i, j\}$ such that $x_m = 0$. Clearly, $y_m|P(\check{y}, u) = z_m|Q(\check{z}, v) = 0$ and $y_m|P(u', \hat{y}) = z_m|Q(v', \hat{z}) = 1$. This shows that $(P(\check{y}, u) \cup P(u, \hat{y})) \cap (Q(\check{z}, v) \cup Q(v', \hat{z})) = \emptyset$. Thus, $P||Q$.

Case 2.2: $x_1 = 0$. We observe paths P and Q constructed in Case 3.2 of Lemma 1. Clearly, $\check{y} = \rho_{F_0(i)}(x)$ and $\check{z} = \rho_{F_0(j)}(x)$. Since $F_0(i) \neq F_0(j)$, it implies $\check{y} \neq \check{z}$. Similar to Case 2.1, there is a position $m \in H^0_x \setminus \{i, j\}$ such that $\check{y}_m = \check{z}_m = 0$ and $y_m|P(u', \hat{y}) = z_m|Q(v', \hat{z}) = 1$. Also, since $\check{y}_{F_0(j)} = \check{z}_{F_0(i)} = 1$ and $y_{F_0(i)}|P(\check{y}, u) = z_{F_0(j)}|Q(\check{z}, v) = 0$, it follows $\check{y} \notin Q(\check{z}, \hat{z})$ and $\check{z} \notin P(\check{y}, \hat{y})$. The remaining proof is similar to Case 2.1. $\square$

Lemma 4. *If $i \in H^0_x$ and $j \in H^1_x$, then $P||Q$.*

Proof. We consider the following three cases:

Case 1: $|H^0_x| < k/2$ (i.e., $|H^1_x| > k/2$). By R2 or R4, if $|H^0_x| = 1$ then $i' = i$; otherwise, $i' \neq i$. By R1 or R3, we have $j' = j$. There are two subcases as follows.

Case 1.1: $x_1 = 1$. We observe the path P constructed in Case 1.1 of Lemma 1. Similar to Case 1.1 of Lemma 3, we can show that $y_{i'}|P(\check{y}, \hat{y}) = 1$. In addition, since it never changes the bit y_j in the path $P(\check{y}, \hat{y})$, we have $y_j|P(\check{y}, \hat{y}) = 1$. On the other hand, we observe the path Q constructed in Case 4.2 of Lemma 1. Clearly, $\check{z} = \rho_{F_1(j)}(x)$ and $\check{z}_{i'} = x_{i'} = 0$. Let $\tilde{z} = \rho_j(\check{z})$ be the node adjacent to $\check{z}$ on Q. Since Q takes the last link with label j to connect r, we have $z_j|Q(\tilde{z}, \hat{z}) = 0$. This shows that $P(\check{y}, \hat{y}) \cap (\{\check{z}\} \cup Q(\tilde{z}, \hat{z})) = \emptyset$. Thus, $P||Q$.

Case 1.2: $x_1 = 0$. We observe the path P constructed in Case 1.2 of Lemma 1. Clearly, $\check{y} = \rho_{F_0(i')}(x)$ and $\check{y}_j = x_j = 1$. Since it never changes the bit y_j in the path $P(\check{y}, \hat{y})$, we have $y_j|P(\check{y}, \hat{y}) = 1$. On the other hand, we observe the path Q constructed in Case 4.1 of Lemma 1. Since Q takes both the first link and the last link with label j to connect x and r, we have $z_j|Q(\check{z}, \hat{z}) = 0$. This shows that $P(\check{y}, \hat{y}) \cap Q(\check{z}, \hat{z}) = \emptyset$. Thus, $P||Q$.

Case 2: $k/2 \leqslant |H^0_x| \leqslant k/2+1$ (i.e., $k/2 \geqslant |H^1_x| \geqslant$

$k/2-1$). Since $|H_x^0| \geqslant 2$, by R2 or R4, we have $i' \neq i$. There are two subcases as follows.

Case 2.1: $x_1 = 1$. We observe the path P constructed in Case 1.1 of Lemma 1. Similar to Case 1.1 of Lemma 3, we can show that $y_{i'}|P(\check{y},\hat{y}) = 1$ and $y_i|P(\check{y},\hat{y}) = 0$. On the other hand, we observe the path Q constructed in Case 2.2 of Lemma 1. Clearly, $\check{z} = \rho_{F_1(j')}(x)$. Let $v, v' \in V(Q)$ be nodes such that $v' = \rho_*(v)$. Since it never changes the bits $z_{i'}$ and z_i in the path $Q(\check{z}, v)$, we have $z_{i'}|Q(\check{z},v) = z_i|Q(\check{z},v) = 0$. It follows that $z_i|Q(v',\hat{z}) = 1$. This shows that $P(\check{y},\hat{y}) \cap (Q(\check{z},v) \cup Q(v',\hat{z})) = \emptyset$. Thus, $P||Q$.

Case 2.2: $x_1 = 0$. We observe the path P constructed in Case 1.2 of Lemma 1. Clearly, $\check{y} = \rho_{F_0(i')}(x)$. Thus, $\check{y}_{F_0(i')} = x_1 = 0$ and $\check{y}_{i'} = x_{i'} = 0$. On the other hand, we observe the path Q constructed in Case 2.1 of Lemma 1. Let $v, v' \in V(Q)$ be nodes such that $v' = \rho_*(v)$. Since it never changes the bits $z_{F_0(i')}$ and $z_{i'}$ in the path $Q(\check{z}, v)$, we have $z_{F_0(i')}|Q(\check{z},v) = 1$ and $z_{i'}|Q(\check{z},v) = 0$. It further implies $z_{i'}|Q(v',\hat{z}) = 1$. Thus, $\check{y} \notin Q(\check{z},v) \cup Q(v',\hat{z})$. The remaining proof is similar to Case 2.1.

Case 3: $|H_x^0| > k/2+1$ (i.e., $|H_x^1| < k/2-1$). By R2 or R4, we have $i' = i$. There are two subcases as follows.

Case 3.1: $x_1 = 1$. We observe paths P and Q constructed in Case 3.1 and Case 2.2 of Lemma 1, respectively. Let $u, u' \in V(P)$ be nodes such that $u' = \rho_*(u)$ and let $v, v' \in V(Q)$ be nodes such that $v' = \rho_*(v)$. A proof similar to Case 2.1 of Lemma 3 shows that there is a position $m \in H_x^0 \setminus \{i,j\}$ such that $y_i|P(\check{y},u) = y_m|P(u',\hat{y}) = z_i|Q(v',\hat{z}) = z_m|Q(v',\hat{z}) = 1$ and $y_m|P(\check{y},u) = y_i|P(u',\hat{y}) = z_i|Q(\check{z},v) = z_m|Q(\check{z},v) = 0$. It follows that $(P(\check{y},u) \cup P(u,\hat{y})) \cap (Q(\check{z},v) \cup Q(v',\hat{z})) = \emptyset$. Thus, $P||Q$.

Case 3.2: $x_1 = 0$. We observe the path P constructed in Case 3.2 of Lemma 1. Clearly, $\check{y} = \rho_{F_0(i)}(x)$. Thus, $\check{y}_i = x_i = 0$ and $\check{y}_{j'} = x_{j'} = 1$. On the other hand, we observe the path Q constructed in Case 2.1 of Lemma 1. Let $v, v' \in V(Q)$ be nodes such that $v' = \rho_*(v)$. Since $x_1 = 0$ and Q takes the first link with label j' to connect x, we have $z_{j'}|Q(\check{z},v) = 0$. Also, since it never changes the bit z_i in the path $Q(\check{z},v)$, we have $z_i|Q(\check{z},v) = 0$ and it follows $z_i|Q(v',\hat{z}) = 1$. Thus, $\check{y} \notin Q(\check{z},v) \cup Q(v',\hat{z})$. The remaining proof is similar to Case 3.1. $\square$

Lemma 5. *If $i \in H_x^1$ and $j \in H_x^0$, then $P||Q$.*

Proof. By symmetry, the proof is similar to that in Lemma 4. $\square$

Lemma 6. *If $i, j \in H_x^1$ with $i \neq j$, then $P||Q$.*

Proof. We consider the following two cases:

Case 1: $|H_x^1| \leqslant k/2$. Since $|H_x^1| \geqslant 2$, by R1 or R3, we have $i' \neq i$, $j' \neq j$ and $i' \neq j'$. Note that it is possible $i' = j$ or $j' = *$. There are two subcases as follows.

Case 1.1: $x_1 = 0$. We observe paths P and Q constructed in Case 2.1 of Lemma 1. Since P deals with links with labels $i', j, j', *, i$ in sequence, for $\ell \in \{j, j'*\}$ we let $u_\ell, u'_\ell \in V(P)$ be nodes such that $u'_\ell = \rho_\ell(u_\ell)$. Clearly, we have the following setting for bits:

$$
\begin{array}{ll}
y_i|P(\check{y},u_j) = 1, & y_i|P(u'_j,u_{j'}) = 1, \\
y_{i'}|P(\check{y},u_j) = 0, & y_{i'}|P(u'_j,u_{j'}) = 0, \\
y_j|P(\check{y},u_j) = 1, & y_j|P(u'_j,u_{j'}) = 0, \\
y_{j'}|P(\check{y},u_j) = 1, & y_{j'}|P(u'_j,u_{j'}) = 1, \\
\\
y_i|P(u'_{j'},u_*) = 1, & y_i|P(u'_*,\hat{y}) = 0, \\
y_{i'}|P(u'_{j'},u_*) = 0, & y_{i'}|P(u'_*,\hat{y}) = 1, \\
y_j|P(u'_{j'},u_*) = 0, & y_j|P(u'_*,\hat{y}) = 1, \\
y_{j'}|P(u'_{j'},u_*) = 0, & y_{j'}|P(u'_*,\hat{y}) = 1.
\end{array}
$$

Similarly, since Q deals with links with labels $j', *, i, i', j$ in sequence, for $\ell \in \{*, i, i'\}$ we let $v_\ell, v'_\ell \in V(Q)$ be nodes such that $v'_\ell = \rho_\ell(v_\ell)$. Clearly, we have the following setting for bits:

$$
\begin{array}{ll}
z_i|Q(\check{z},v_*) = 1, & z_i|Q(v'_*,v_i) = 0, \\
z_{i'}|Q(\check{z},v_*) = 1, & z_{i'}|Q(v'_*,v_i) = 0, \\
z_j|Q(\check{z},v_*) = 1, & z_j|Q(v'_*,v_i) = 0, \\
z_{j'}|Q(\check{z},v_*) = 0, & z_{j'}|Q(v'_*,v_i) = 1, \\
\\
z_i|Q(v'_i,v_{i'}) = 1, & z_i|Q(v'_{i'},\hat{z}) = 1, \\
z_{i'}|Q(v'_i,v_{i'}) = 0, & z_{i'}|Q(v'_{i'},\hat{z}) = 1, \\
z_j|Q(v'_i,v_{i'}) = 0, & z_j|Q(v'_{i'},\hat{z}) = 0, \\
z_{j'}|Q(v'_i,v_{i'}) = 1, & z_{j'}|Q(v'_{i'},\hat{z}) = 1.
\end{array}
$$

Obviously, only $P(u'_j, u_{j'})$ and $Q(v'_i, v_{i'})$ have the same setting. Since $x \neq 0^k1^k$, there is a position $m \in H_x^0$ such that $x_m = 0$. Since P deals with links j and j' before $*$, we have $y_m|P(u'_j,u_{j'}) = 0$. By contrast, since Q deals with links i and i' after $*$, we have $z_m|Q(v'_i,v_{i'}) = 1$. This shows that $P(\check{y},\hat{y}) \cap Q(\check{z},\hat{z}) = \emptyset$. Thus, $P||Q$.

Case 1.2: $x_1 = 1$. We observe paths P and Q constructed in Case 2.2 of Lemma 1. Clearly, $\check{y} = \rho_{F_1(i')}(x)$ and $\check{z} = \rho_{F_1(j')}(x)$. Since $F_1(i') \neq F_1(j')$, it implies $\check{y} \neq \check{z}$. Let $u, u' \in V(P)$ be nodes such that $u' = \rho_*(u)$ and let $v, v' \in V(Q)$ be nodes such that $v' = \rho_*(v)$. Since $\check{y}_j = 1$ and $\check{y}_{F_1(j')} = 0$ while $z_{F_1(j')}|Q(\check{z},v) = 1$ and $z_j|Q(v',\hat{z}) = 0$, it follows $\check{y} \notin Q(\check{z},\hat{z})$. Similarly, since $\check{z}_i = 1$ and $\check{z}_{F_1(i')} = 0$ while $y_{F_1(i')}|P(\check{y},u) = 1$ and $y_i|P(u',\hat{y}) = 0$, it follows $\check{z} \notin P(\check{y},\hat{y})$. The remaining proof is similar to Case 1.1.

Case 2: $|H_x^1| > k/2$. By R1 or R3, we have $i' = i$ and $j' = j$. There are two subcases as follows.

Case 2.1: $x_1 = 0$. We observe paths P and Q constructed in Case 4.1 of Lemma 1. Since P takes both the first link and the last link with label i to connect x and r, we have $y_i|P(\check{y},\hat{y}) = 0$. On the other hand, since it never changes the bit z_i in the path $Q(\check{z},\hat{z})$, we have $z_i|Q(\check{z},\hat{z}) = 1$. This shows that $P(\check{y},\hat{y}) \cap Q(\check{z},\hat{z}) = \emptyset$. Thus, $P||Q$.

Case 2.2: $x_1 = 1$. We observe paths P and Q constructed in Case 4.2 of Lemma 1. Clearly, $\check{y} = \rho_{F_1(i)}(x)$ and $\check{z} = \rho_{F_1(j)}(x)$. Since $F_1(i) \neq F_1(j)$, it implies $\check{y} \neq \check{z}$. Clearly, $\check{y}_j = \check{z}_i = 1$. Let $\tilde{y} = \rho_i(\check{y})$ be the node adjacent to $\check{y}$ on P and let $\tilde{z} = \rho_j(\check{z})$ be the node adjacent to $\check{z}$ on Q. Since $\tilde{y}_i = \tilde{y}_1 = x_{F_1(i)} = 0$ and P takes the last link with label i to connect r, we have $y_i|Q(\tilde{y},\hat{y}) = 0$. Thus, $\check{z}_i = 1$ implies $\check{z} \notin P(\tilde{y},\hat{y})$. Similarly, since $\tilde{z}_j = \tilde{z}_1 = x_{F_1(j)} = 0$ and Q takes the last link with label j to connect r, we have $z_j|Q(\tilde{z},\hat{z}) = 0$. Thus, $\check{y}_j = 1$ implies $\check{y} \notin Q(\tilde{z},\hat{z})$. The remaining proof is similar to Case 2.1. □

Lemma 7. *If $i \in H_x^0$ and $j = *$, then $P||Q$.*

Proof. We consider the following three cases:

Case 1: $|H_x^0| < k/2$ (i.e., $|H_x^1| > k/2$). By R2 or R4, if $|H_x^0| = 1$ then $i' = i$; otherwise, $i' \neq i$. By R1 or R3, since $|H_x^1| > k/2$, we have $j' = *$ and there is a position $m \in H_x^1$ such that $x_m = 1$. We observe the path P constructed in Case 1.1 or Case 1.2 of Lemma 1. Since it never changes the bits y_m in P, we have $y_m|P(\check{y},\hat{y}) = 1$. On the other hand, we observe the path Q constructed in Case 4.1 or Case 4.2 of Lemma 1. Since Q takes the first link with label $*$ to connect x, we have $\check{z}_m = \bar{x}_m = 0$. Moreover, since it never changes the bit z_m in the succedent path of Q until the last link with label $*$ to connect r, we have $z_m|Q(\check{z},\hat{z}) = 0$. This shows that $P(\check{y},\hat{y}) \cap Q(\check{z},\hat{z}) = \emptyset$. Thus, $P||Q$.

Case 2: $k/2 \leqslant |H_x^0| \leqslant k/2+1$ (i.e., $k/2 \geqslant |H_x^1| \geqslant k/2-1$). Since $|H_x^0| \geqslant 2$, by R2 or R4, we have $i' \neq i$. Also, since $|H_x^1| \geqslant 1$, by R1 or R3, we have $j' \in H_x^1$. There are two subcases as follows.

Case 2.1: $x_1 = 0$. We observe the path P constructed in Case 1.2 of Lemma 1. Since it never changes the bit $y_{j'}$ in P, we have $y_{j'}|P(\check{y},\hat{y}) = 1$. On the other hand, we observe the path Q constructed in Case 2.1 of Lemma 1. Since Q takes the first link with label j' to connect x, we have $\check{z}_{j'} = \bar{x}_{j'} = 0$. Moreover, since it never changes the bit $z_{j'}$ in the succedent path of Q until the last link with label $*$ to connect r, we have $z_{j'}|Q(\check{z},\hat{z}) = 0$. This shows that $P(\check{y},\hat{y}) \cap Q(\check{z},\hat{z}) = \emptyset$. Thus, $P||Q$.

Case 2.2: $x_1 = 1$. We observe the path P constructed in Case 1.1 of Lemma 1. Since P takes the first link with label i' to connect x, we have $\check{y}_{i'} = x_1 = 1$. Moreover, since it never changes the bit $y_{i'}$ in the succedent path of P, it follows $y_{i'}|P(\check{y},\hat{y}) = 1$. On the other hand, we observe the path Q constructed in Case 2.2 of Lemma 1. Clearly, $\check{z} = \rho_{F_1(j')}(x)$ and $\check{z}_{i'} = x_{i'} = 0$. Thus, $\check{z} \notin P(\check{y},\hat{y})$. The remaining proof is similar to Case 2.1.

Case 3: $|H_x^0| > k/2+1$ (i.e., $|H_x^1| < k/2-1$). By R2 or R4, we have $i' = i$. Also, by R1 or R3, we have $j' = j = *$. Let $u, u' \in V(P)$ be nodes such that $u' = \rho_*(u)$. If $x_1 = 1$ (respectively, $x_1 = 0$), a proof similar to Case 2.1 (respectively, Case 2.2) of Lemma 3 shows that $(P(\check{y},u) \cup P(u',\hat{y})) \cap (Q(\check{z},\hat{z})) = \emptyset$. Thus, $P||Q$. □

Lemma 8. *If $i \in H_x^1$ and $j = *$, then $P||Q$.*

Proof. We consider the following two cases:

Case 1: $|H_x^1| > k/2$. By R1 or R3, we have $i' = i$ and $j' = *$. Since $|H_x^1| > k/2$, there is a position $m \in H_x^1 \setminus \{i\}$ such that $x_m = 1$. We observe paths P and Q constructed in Case 4.1 or Case 4.2 of Lemma 1. It is clear that the proof is similar to Case 1 of Lemma 7.

Case 2: $|H_x^1| \leqslant k/2$. By R1 or R3, we have $i' \neq i$, $j' \neq j$ and $i' \neq j'$. Note that it is possible $j' = i$, and if $H_x^1(i, 2k) = \emptyset$ then $i' = *$. In the following, if $i' = *$, we omit the setting of $y_{i'}$ in P and $z_{i'}$ in Q. There are two subcases as follows.

Case 2.1: $x_1 = 0$. We observe paths P and Q constructed in Case 2.1 of Lemma 1. Let $u, u' \in V(P)$ be nodes such that $u' = \rho_*(u)$. Since P takes the first link with label i' to connect x, we have $\check{y}_{i'} = x_1 = 0$ and $\check{y}_i = 1$. Moreover, since P takes the last link with label i to connect r, it is easy to check that $y_i|P(\check{y},u) = y_{i'}|P(u',\hat{y}) = 1$ and $y_{i'}|P(\check{y},u) = y_i|P(u',\hat{y}) = 0$. On the other hand, Q starts from the link with label j' and ends to the link with label $*$. Let $v, v', w, w' \in V(Q)$ be nodes such that $v' = \rho_i(v)$ and $w' = \rho_{i'}(w)$. Clearly, $\check{z}_i = \check{z}_{i'} = 1$, and thus $z_i|Q(\check{z},v) = z_{i'}|Q(\check{z},v) = 1$. This implies $z_i|Q(v',w) = 0$, $z_{i'}|Q(v'w) = 1$, and $z_i|Q(w',\hat{z}) = z_{i'}|Q(w',\hat{z}) = 0$. Obviously, the setting of bits y_i and $y_{i'}$ in $P(u',\hat{y})$ and the setting of bits z_i and $z_{i'}$ in $Q(v',w)$ are the same. We now distinguish nodes between $P(u',\hat{y})$ and $Q(v',w)$. Since $|H_x^1| \leqslant k/2$, there is a position $m \in H_x^0$ such that $x_m = 0$. Clearly, $y_m|P(u',\hat{y}) = 1$ and $z_m|Q(v',w) = 0$. This shows that $(P(\check{y},u) \cup P(u',\hat{y})) \cap (Q(\check{z},v) \cup Q(v',w) \cup Q(w',\hat{z})) = \emptyset$. Thus, $P||Q$.

Case 2.2: $x_1 = 1$. We observe paths P and Q constructed in Case 2.2 of Lemma 1. Let $u, u' \in V(P)$ be nodes such that $u' = \rho_*(u)$. Since $|H_x^1| \leqslant k/2$, there is a position $m \in H_x^0$ such that $x_m = 0$. Clearly, $\check{y} = \rho_{F_1(i')}(x)$ and $\check{z} = \rho_{F_1(j')}(x)$. Since $F_1(i') \neq F_1(j')$, it implies $\check{y} \neq \check{z}$. Clearly, $\check{y}_{j'} = \check{z}_{i'} = 1$ and $\check{y}_m = \check{z}_m = 0$. Let $\tilde{y} = \rho_{i'}(\check{y})$ be the node adjacent to $\check{y}$ on P and let $\tilde{z} = \rho_{j'}(\check{z})$ be the node adjacent to $\check{z}$ on Q. Since $\tilde{z}_{j'} = \tilde{z}_1 =$

$x_{F_1(j')} = 0$ and Q takes the last link with label $*$ to connect r, we have $z_{j'}|Q(\tilde{z},\hat{z}) = 0$. Thus, $\check{y}_{j'} = 1$ implies $\check{y} \notin Q(\tilde{z},\hat{z})$. In addition, since $\tilde{y}_{i'} = \check{y}_1 = x_{F_1(i')} = 0$ and $\check{y}_m = 0$, it implies $y_{i'}|P(\tilde{y},u) = y_m|P(\tilde{y},u) = 0$ and $y_m|P(u',\hat{y}) = 1$. Thus, $\check{z}_{i'} = 1$ implies $\check{z} \notin P(\tilde{y},u)$ and $\check{z}_m = 0$ implies $\check{z} \notin P(u',\hat{y})$. The remaining proof is similar to Case 2.1. □

From Lemma 3 to Lemma 8, we conclude that ISTs constructed in this paper are independent. According to Lemmas 1, 2 and the result of independency, we obtain the following main theorem.

Theorem 9. *Let $N = \binom{2k}{k}$. For $FHS(2k,k)$, Algorithm* Constructing-ISTs *can correctly construct $k+1$ ISTs with the heights at most $k+2$ in $\mathcal{O}(kN)$ time. In particular, the algorithm can be parallelized on $FHS(2k,k)$ by using N processors to run in $\mathcal{O}(k)$ time.*

5 Concluding remarks

The fault diameter [21] and the wide diameter [14] are important measurement for reliability and efficiency of interconnection networks. Note that, for a network G, the fault diameter and the wide diameter of G are bounded below by its diameter plus one, and are bounded above by the maximum height of a set of $\kappa(G)$ ISTs with an arbitrary root, where $\kappa(G)$ is the connectivity of G. Kim et al. [22] showed that $FHS(2k,k)$ has the diameter k, and Yang and Chang [33] showed that $FHS(2k,k)$ has the connectivity $k+1$. In this paper, we provide a set of $k+1$ ISTs with the heights at most $k+2$ in $FHS(2k,k)$. As a result, we conclude that the fault diameter (respectively, the wide diameter) of $FHS(2k,k)$ is either $k+1$ or $k+2$. An interesting question is to clarify the accurateness for these two parameters of $FHS(2k,k)$.

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