The Electric Vehicle Touring Problem

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Abstract

The increasing concern over global warming has led to the rapid development of the electric vehicle industry. Electric vehicles (EVs) have the potential to reduce the greenhouse effect and facilitate more efficient use of energy resources. In this paper, we study several EV route planning problems that take into consideration possible battery charging or swapping operations. Given a road network, the objective is to determine the shortest (travel time) route that a vehicle with a given battery capacity can take to travel between a pair of vertices or to visit a set of vertices with several stops, if necessary, at battery switch stations. We present polynomial time algorithms for the EV shortest travel time path problem *and the* fixed tour EV touring problem*, where the fixed tour problem requires visiting a set of vertices in a given order. Based on the result, we also propose constant factor approximation algorithms for the* EV touring problem*, which is a generalization of the traveling salesman problem.*

Keywords. *Approximation algorithm, electric vehicle, shortest path, traveling salesman problem, vehicle routing*

1 Introduction

Transportation is one of the fastest-growing sources of greenhouse gas emissions that contribute to climate change. In the United States, transportation accounts for approximately 25 percent of total greenhouse gas emissions [34]. Consequently, during the last decade, the automobile industry has developed an increasing number of electric (battery) vehicles or hybrid electric vehicles to deal with the rising cost of energy. Electric vehicles (EVs), which release almost no air pollutants, could make a significant contribution to maintaining the quality of the environment. The Electric Power Research Institute estimates that EVs will account for 6% - 30% of the vehicles in use by 2030 [11].

An efficient EV routing service would obviously encourage the transition to electric vehicle use. The U.S. Department of Energy has developed an online service [33] that provides a route map interface, as well as information about EV charging facilities for EV owners. However, it is very difficult to design an optimal EV route planning service because EVs have some serious limitations. The first is the low energy capacity of batteries. Currently, their range is only 150 to 200 kilometers; hence, EVs are used primarily in urban areas. The second problem is that EV batteries require a long charging time. At the moment, they can be fully recharged from empty in 2 to 6 hours, depending on the level of charging available at the station. These factors have delayed the growth of the EV market; however, EVs can now be refueled in a matter of minutes through a system called battery-swaps. Recently, Tesla Motors [32] provided the solution via a network of battery switch stations. The state-of-the-art technology leads to a new model of EV route planning.

In this paper, we explore some interesting models that incorporate the battery capacity constraint when an electric vehicle is driven. First, we begin with the *EV shortest travel time path problem*. In this problem, we determine a route from a source to a destination that an electric vehicle with a given battery capacity U can travel along so that the total time including traveling

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and battery-swaps is minimized. If necessary, the vehicle can stop at several battery switch stations on the route to maintain its movement. Note that we measure our objective in terms of time. That is, the weight of an edge represents the time required for the vehicle to travel through the edge, and the capacity represents the length of time the vehicle can travel with a full battery. Similarly, the *EV touring problem* involves organizing a tour of a set of cities so that the total time required is minimized. The vehicle visits each city and returns to the origin, stopping at battery switch stations whenever necessary. We consider two scenarios: the *on-site station* and the *off-site station* models in which each city has an *on-site* battery switch station and an *off-site* battery switch station within an acceptable distance, respectively.

Our contribution. The main results obtained in this paper are summarized as follows:

1. We consider the EV shortest travel time path problem and present a simple dynamic programming algorithm that runs in $O(kn^2)$ time where k is the upper bound of the number of battery-swaps and n is the order of the graph.

2. We develop polynomial time algorithms for the fixed tour EV touring problem, where the fixed tour constraint requires visiting a set of cities and returning to the origin in a given order. This result extends the previous studies of the *fixed path gas station problem* reported in [18, 22, 23] by using graph-theoretic techniques.

3. We propose two approximation algorithms within a $\frac{9}{4}$-factor and a $\frac{9}{2}$-factor, respectively, for the uniform and non-uniform cost on-site station EV touring problem. Moreover, if the battery capacity is sufficiently large, the approximation ratio is the same as that of the well-known Christofides algorithm for the TSP, i.e., $\frac{3}{2}$.

4. We also study the off-site station EV touring problem and propose a $\frac{3}{2}(\frac{3+2\alpha}{1-2\alpha})$-approximation algorithm to solve the problem, where α is a given acceptable distance between a city and its nearest battery switch station.

2 Preliminaries

A great deal of research has been devoted to the shortest route planning problem; and many variations and extensions of the problem have been proposed. One related problem is the well-known *capacitated vehicle routing problem*, which involves finding a set of routes that begin at a depot, visit multiple customers and deliver goods, and return to the depot such that the number of vehicles, each of which has a limited carrying capacity, is minimized or the total distance is minimized. Readers may refer to Laporte's survey [20] and Pillac *et al.*'s review [25] for further details on the constraints and conditions.

Another related work is the *orienteering problem* where the objective is to find a path of a fixed length from a single source that visits as many locations as possible [3, 4, 5]. The EV touring problem can be regarded as an extension of this problem because the goal is to visit as many cities as possible under a fixed (i.e., full) battery capacity.

Compared with the widely studied routing problems, there is a dearth of research on the optimal *refueling problem* [18, 22, 23, 30, 31] where the objective is to minimize the total cost of the fuel used. Lin *et al.* [22, 23] investigated the shortest path problem with optimal refueling policies. They proposed a linear time algorithm for the fixed route version and polynomial time algorithms for other variations. Suzuki [30, 31] developed a more comprehensive model that incorporates many operating costs, and conducted numerical studies. Recently, Khuller *et al.* [18] proposed the *gas station problem* where the price of gas may vary at every station, so the owner of a petrol-powered vehicle must decide the amount of gas he/she will purchase (i.e., a fraction of the tank's capacity) at a particular gas station in order to minimize the total cost of gas required. They also study the *tour gas station problem*, where the objective is to find the cheapest tour that can visit a set of vertices and return to the origin, so that the total cost of the gas required is minimized.

Subsequently, Erdoğan and Miller-Hooks presented the *green vehicle routing problem* [13] and Schneider *et al.* proposed the similar *electric vehicle routing problem* [29]. They combined the *vehicle routing problem* with the possibility of filling alternative fuels or charging a vehicle's battery at stations along the routes. Both works provided meta-heuristic algorithms and an analysis of numerical experiments. The setting of the problems is similar to that of the above optimal refueling problem; the only difference is that the objective is to minimize the total distance instead of the total fuel cost. In addition, a number of previous studies [2, 10, 19, 26, 28] discussed energy efficient routing of electric vehicles.

3 Electric Vehicle Shortest Travel Time Path

To define the EV shortest travel time path problem, we need some notation. First, each vertex $v_i \in V$ is associated with one variable u_i, which indicates the amount of power left in the battery. We also define a binary variable y_i to indicate if the vehicle swaps a battery at station v_i. The distance cost for each edge (v_i, v_j) in E is represented by W_{ij} in terms of the time required to traverse the edge; that is, W_{ij} is the time required when the vehicle drives along the edge. In addition, let parameter B_i be defined as the time required for a vehicle to replace a battery at v_i, and let U be defined as a battery's capacity, which is measured in terms of the amount of time a vehicle with a full battery can remain on the route. The integer programming model of the EV shortest (travel time) path problem (Problem EVSPP) can be described as follows.

$$\text{Minimize} \quad \sum_{(v_i,v_j)\in E} W_{ij}x_{ij} + \sum_{i=1}^{n} B_i y_i$$

$$\sum_{j:\, j\neq i} x_{ij} - \sum_{j:\, j\neq i} x_{ji} = \begin{cases} 1, & \text{if } i = s; \\ -1, & \text{if } i = t; \\ 0, & \text{otherwise.} \end{cases} \qquad i = 1, 2, \ldots, n$$

$$(((1 - y_i)u_i + y_i U) - W_{ij} - u_j)x_{ij} = 0, \qquad (v_i, v_j) \in E$$

$$0 \leq u_i \leq U, \qquad i = 1, 2, \ldots, n$$

$$y_i \in \{0, 1\}, \qquad i = 1, 2, \ldots, n$$

$$x_{ij} \in \{0, 1\}, \qquad (v_i, v_j) \in E$$

$$u_s = U_s$$

Note that the second set of constraints ensures battery power conservation. More precisely, the vehicle will stop at some stations to swap its battery if the power left is insufficient to reach the next stop; that is, the electric power left in the battery might be wasted. The total power needed to traverse the optimal path could be more than the total distance cost in terms of the time expended.

Here we assume that the vehicle starts at v_s with an empty battery, i.e., $U_s = 0$, where there is a battery switch station; otherwise, if $U_s \neq 0$, replace v_s by a new source vertex $v_{s'}$ with $W_{s'j} = W_{sj} + (U - U_s)$, for each $j \neq s$, $B_{s'} = 0$ and $U_{s'} = 0$. Then, the reduction shows that the problem of starting from v_s with a non-empty U_s is equivalent to that of starting from $v_{s'}$ with an empty battery when there is a battery switch station at $v_{s'}$ [18].

To gain more insights, we next discuss in detail the uniform cost case where the battery-swap time is identical at every battery switch station; that is, for each v in F, $b(v) = c$, for a constant c.

3.1 The uniform cost model

We use the dynamic programming (DP) technique for solving the uniform cost model. The reason we do so is that the EV shortest travel time path problem (Problem EVSPP) cannot be solved by straightforward greedy algorithms, even when the uniform battery-swap time cost model is considered. By contrast, the gas station problem (Problem GSP) can be simply solved by greedy approaches in the uniform cost model [18].

As mentioned earlier, we let k be the upper bound of the number of battery-swaps. Given a complete graph $G = (V \cup F, E)$, where $V = \{s, t\}$ and each $e \in E$ represents the shortest (travel time) distance between each pair of vertices, which can be derived by Dijkstra's algorithm, an optimal EV shortest travel time path from s to t with initial U_s units of power in G can be solved based on the following DP formulation:

$OPT[v, q]$ = The minimum time for the vehicle to travel from a vertex v to the destination t with exactly q battery-swaps, where the first battery-swap of these q operations is performed at v.

Suppose the vehicle starts with an empty battery $U_s = 0$ and there is a battery switch station at s. If there is no battery-switch station at s or $0 < U_s \leq U$, the problem can be reduced by a similar scheme above to the EV s', t-shortest travel time path problem in $G = (V' \cup F, E')$ with $V' = V \setminus \{s\} \cup \{s'\}$, $b(s') = c$, $U_{s'} = 0$, and $E' = E \setminus \{(s, u) \mid \forall u \in F\} \cup \{(s', u) \mid \forall (s, u) \in E)\}$, where $w(s', u) = w(s, u) + (U - U_s)$.

For every v in $F \cup \{s\}$, each entry of the form $OPT[v, q]$, $1 \leq q \leq k$, can be computed by the following recurrence:

$$OPT[v, q] = \min_{u\in F}\{OPT[u, q-1] + c + w(v, u) \mid w(v, u) \leq U\},$$

where the boundary condition of the recurrence is given as follows:

$$OPT[v, 1] = \begin{cases} c + w(v, t), & \text{if } w(v, t) \leq U; \\ \infty, & \text{otherwise.} \end{cases}$$

In the above DP recursion, let u be the first battery switch station after v when the vehicle travels from v through u to t and swaps a battery at u. We keep track of at most $|F|$ different values for all the stations to compute each entry of the table. The optimal solution is derived in the form of $\min_{1\leq \ell \leq k}\{OPT[s,\ell]\}$, which can be derived in $O(kn^2)$ time in a naive way by filling the table whose size is $O(kn)$, where n is the order of G.

Lemma 3.1. *The uniform cost model of the EV shortest travel time path problem with at most k battery-swaps can be optimally solved in $O(kn^2)$ time.*

Consider the properties behind the updates of the recursion. Note that the major difficulty is that each update of the above DP recursion does not have good properties to find a more efficient way for computing every entry of the table in sublinear time (rather than the $O(n)$-time operation) based on such a recurrence form, even using advanced data structures such as the Fibonacci heap [9, 14].

3.2 The non-uniform cost model

Next, we consider the case where each station v in F may have a different battery-swap time $b(v)$. This model of the EV shortest travel time path problem can be solved in a similar manner by using $b(v)$ instead of the constant c in Equations (1) and (2). The reduction scheme for the assumption that $U_s = 0$ can be done in a similar way: letting $b(s') = 0$ for the new source vertex s'. Thus, the time complexity analysis of the dynamic programming recursion for the non-uniform cost model remains the same, and the theorem is as follows:

Theorem 3.2. *There is an $O(kn^2)$-time algorithm for the EV shortest travel time path problem with at most k battery-swaps.*

We remark that the EV shortest travel time path problem (Problem EVSPP) is one case of the *Constrained Shortest Path problem with one resource* (Problem CSPPOR). The CSPPOR has been proved to be NP-complete [15], and the latest results show that the problem can be solved by using similar dynamic programming techniques that run in pseudo-polynomial time in terms of resource [12, 24, 27]. Readers can be referred to [17] for more details about the CSPPOR.

In the following sections, we introduce our main results and investigate the touring problem that incorporates the battery capacity constraint for an electric vehicle. Similar to the EV shortest travel time path problem, it ensures that the vehicle never runs out of power during the shortest (travel time) tour that visits every city and returns to the origin.

4 Fixed Tour EV Touring

Given a complete graph $G = (V \cup F, E)$ with a set of cities V, a set of battery switch stations F and the vehicle's battery capacity U, where each station $v \in F$ is associated with a battery-swap time $b(v)$, $b : F \to R^+$, and each edge $e \in E$ is associated with a distance weight $w(e)$, $w : E \to R^+$, the goal is to find a tour that enables the vehicle to visit each city in V and return to the origin, and stop at battery switch stations in F when needed, such that the total time cost including traveling and battery-swaps is minimized. We call this generalization of the TSP the *EV touring problem*.

Before discussing the EV touring problem, we consider the fixed tour model, in which the vehicle visits cities that have battery-switch stations (i.e., $V = F$) in a given order during the tour. This problem is an extension of the *fixed path gas station problem* proposed in [18, 22, 23].

We introduce some new notation: A path P from u to v, or shortly, a u,v-path, is denoted by $P : u \sim v$, and the path can also be represented by a sequence $P : u = v_1 - v_2 - \ldots - v_m - v_{m+1} = v$ if it is of length m. The u,v-path P is called a tour or a cycle of length m if $P : u = v_1 - v_2 - \ldots - v_m - v = u$. The distance weight of a path or a tour P is defined as $w(P) = \sum_{e \in P} w(e)$.

4.1 Uniform cost

Suppose the battery-swap time is identical at every station, i.e., $b(v_i) = c$ for each $v_i \in F$. First, consider a path in a given order $P : v_1 - v_2 - \ldots - v_n$, which consists exclusively of battery switch stations. Without loss of generality, assume the battery capacity U is larger than the distance weight of each edge in P in terms of time spent, i.e., $U > w(v_i, v_{i+1})$, $1 \leq i \leq n-1$. The vehicle starts at v_1 with an empty battery.

An intuitive strategy for the vehicle is to go as far as possible unless the battery power left is insufficient to reach the next vertex on the path. The next lemma shows that the greedy concept can be used to optimally select appropriate stations for battery-swaps for a given path in the uniform cost model.

Given a fixed path $P : v_1 - v_2 - \ldots - v_n$ consisting exclusively of battery switch stations, each of which has an identical battery-swap time, the optimal strategy for a vehicle to select the minimum number of stations to enable its movement and visit all the vertices sequentially can be described as the following lemma:

Lemma 4.1. *The EV would not stop at a station for a battery-swap unless the battery power left is insufficient to reach the next vertex in P.*

Proof. Let O be the optimal solution of battery switch stations whose total battery-swap time is the minimum. Assume $\sum_{v \in O} b(v) < \sum_{v \in F^*} b(v)$, where F^* is derived by the greedy strategy. It implies that $|O| < |F^*|$ for the uniform cost model.

Let v_j be the first vertex in $O \setminus F^*$, i.e., with the smallest index j, in P. Let the station v_i be selected immediately before v_j, that is, $v_i \in O \cap F^*$, where $i < j$. Suppose the station v_ℓ, $\ell \neq j$, is selected by the greedy strategy immediately after v_i. It implies that $\ell > j$; otherwise, the vehicle can reach v_j from v_i without any battery-swaps and it would drive through v_ℓ. However, this contradicts the greedy choice of v_ℓ. Thus, we replace v_j by v_ℓ, $\ell > j$ to obtain full battery power at v_ℓ and $O = O \setminus \{v_j\} \cup \{v_\ell\}$. Repeat this argument to show that $|F^*| > |O|$ is a contradiction. □

Subsequently, we extend the straightforward greedy concept to the fixed tour model. The major difference is that we have to determine the start vertex for the EV such that the number of battery-swaps is minimized in the fixed tour model. We refer to the result reported by Hsu and Tsai [16], who studied several optimization problems in *circular-arc graphs*. Based on Lemma 4.1, we present a linear time algorithm using graph-theoretic approaches for optimally selecting battery switch stations in a given tour in the uniform cost model. An intersection graph G is called a *circular-arc graph* if its vertices can be put into a one-to-one correspondence with a set of arcs on a circle, such that two vertices are adjacent in G if and only if their corresponding arcs have nonempty intersections. A circular ordering $v_1, v_2, \ldots, v_n$ of G is represented by $b_1 \preceq b_2 \preceq \ldots \preceq b_n \preceq b_1$, where b_i is the right endpoint of the arc v_i; and $b_i \preceq b_j$ means that b_j follows b_i in a clockwise direction.

Given a fixed tour of length n, $P : v_1 - v_2 - \ldots - v_n - v_1$; similarly, $v_i \prec v_j$ means that v_j follows v_i circularly in the ascending order of P. For any vertex v_i, $1 \leq i \leq n$, we denote a vertex v_ℓ as $\texttt{FAR}(v_i)$ if the vehicle can drive with a full battery from v_i to v_ℓ circularly such that $w(v_i \sim v_\ell)$ is maximized, i.e., $w(v_i \sim v_\ell) \leq U$ and $w(v_i \sim v_\ell - v_{\ell+1}) > U$. Moreover, it can be proved that the relationship between a vertex v_i and $\texttt{FAR}(v_i)$ has the following *interleaving property.*

Lemma 4.2. *For any two vertices v_i and v_j in a given tour P with $v_i \prec v_j$, we have $\texttt{FAR}(v_i) \preceq \texttt{FAR}(v_j)$.*

Proof. Assume there exist two vertices v_i and v_j with $v_i \prec v_j$, such that $\texttt{FAR}(v_j) \prec \texttt{FAR}(v_i)$. Because the vehicle can drive with a full battery from v_i through $\texttt{FAR}(v_j)$ to $\texttt{FAR}(v_i)$, it contradicts the fact that $w(v_j \sim \texttt{FAR}(v_j))$ is maximized. □

Based on the interleaving property, the computation of $\texttt{FAR}(v_i)$ for each vertex v_i can be solved in linear time [23]; that is, if $\texttt{FAR}(v_i) = v_\ell$, then we can compute $\texttt{FAR}(v_{i+1})$ by starting at v_ℓ. Repeating this argument from v_1 to v_{n-1} circularly to derive every $\texttt{FAR}(v_i)$ in linear time.

We use a similar idea to that of Hsu and Tsai [16] and construct a directed graph $D = (V, E_D)$, where $V = \{v_1, v_2, \ldots, v_n\}$ and a directed edge $\overrightarrow{(v_i, v_j)} \in E_D$ if and only if $v_j = \texttt{FAR}(v_i)$, $v_i \prec v_j$. First, we assume that every vertex $v_i \in V$ has its $\texttt{FAR}(v_i)$; otherwise, the vehicle can begin with a full battery and return to the origin. Consequently, there is at least one directed cycle in D because V is of finite cardinality. Besides, two directed cycles cannot share a common vertex since every vertex has out-degree exactly one in D.

Next, we define $F_{v_i} = \{v_i^{(0)}, v_i^{(1)}, \ldots, v_i^{(m-1)}\}$, where $v_i^{(j+1)} = \texttt{FAR}(v_i^{(j)})$, $v_i^{(0)} = v_i$, and $v_i \preceq \texttt{FAR}(v_i^{(m-1)})$. By Lemma 4.1, F_{v_i} is a feasible solution of stations containing v_i. Moreover, for any feasible solution F' containing v_i, we have $|F_{v_i}| \leq |F'|$. Let O be the optimal solution of stations and a vertex v_i be called *valid* if $|F_{v_i}| = |O|$. We have the following lemmas.

Lemma 4.3. *There is at least one directed cycle comprised exclusively of valid vertices in D.*

Lemma 4.4. *Every directed cycle in the directed graph D consists exclusively of valid vertices.*

Thus, for the uniform cost fixed tour model, in which the vehicle visits all the cities that have battery switch stations in a given order during the tour, we can optimally select appropriate stations for battery-swaps in linear time based on the above lemmas by incorporating the greedy concept of Lemma 4.1 with a similar idea to that of Hsu and Tsai [16] (see Algorithm 1).

Algorithm 1: Select battery switch stations for the fixed tour uniform cost model

Input: a fixed tour $P : v_1 - v_2 - \ldots - v_n - v_1$ consisting of battery switch stations, the distance of each edge in P and the battery capacity U;
Output: a set F^* consisting of battery switch stations selected;

Find $\texttt{FAR}(v_i)$ for each $v_i \in P$;
Let each vertex be marked 'unvisited' and let $v^* = v_1$ be the first vertex;
while v^* is 'unvisited' **do**
  Mark v^* as 'visited' and let $v^* = \texttt{FAR}(v^*)$;
end while
return $F^* = F_{v^*}$;

Theorem 4.5. *For the uniform cost fixed tour model, the minimum total battery-swap time as well as the corresponding battery switch stations can be optimally determined in linear time.*

Proof. Based on the above lemmas, when a vertex v^* is visited twice, there exists a directed cycle from v^* to v^*. Thus, v^* is valid and F_{v^*} is the optimal solution of stations.

Regarding the time complexity analysis, $\texttt{FAR}(v_i)$, for each v_i, can be totally derived in linear time in the initial step. The number of iterations in the while loop is at most $O(n)$; hence, the running time is linear in the order of G. □

4.2 Non-uniform cost

Next, we investigate the non-uniform cost model in which each vertex $v_i \in F$ has a different battery-swap time $b(v_i)$. Similarly, we consider this case in a fixed path, and then extend it to the fixed tour model. The assumptions of the previous subsection still hold; we define $\texttt{FAR}(v_i)$ for each vertex v_i in a similar way and have the interleaving property as well.

For the fixed path gas station problem, Lin *et al.* [23] proposed a linear time algorithm that solves the problem in a greedy manner. More precisely, when arriving at a vertex v_i, the petrol-powered vehicle refills its gas tank at v_i if there are no stations with cheaper gas price lying between v_i and $\texttt{FAR}(v_i)$; otherwise, the vehicle would partially refuel to be able to just reach the first station whose gas price is lower than that of v_i. Repeating the argument can derive the optimal solution in a given fixed path. However, this greedy manner cannot work in the *fixed path EV touring problem* which incorporates battery-swap operations, as discussed earlier. Thus, we refer to the result reported by Chang [6] who explored weighted optimization problems in circular-arc graphs, and use the dynamic programming technique to solve the non-uniform cost model as follows.

Given a fixed path $P : v_1 - v_2 - \ldots - v_n$ consisting of battery switch stations, each of which has a different battery-swap time, an optimal solution of stations required for traversing this path in a given order can be determined based on the DP formulation below:

$\texttt{MBS}(v_j)$ = The minimum time of total battery-swaps for the vehicle to travel from v_1 to $\texttt{FAR}(v_j)$, and the last battery-swap is performed at v_j.

Suppose the vehicle starts at v_1 with an empty battery. For every $v_i \in F$, $\texttt{FAR}(v_i)$ can be obtained in linear time as mentioned earlier; similarly, $v_i \prec v_j$ means that v_j follows v_i in the ascending order of P. Then, each entry of the form $\texttt{MBS}(v_j)$, $1 \le j \le n$, can be computed by the following recurrence:

$$\texttt{MBS}(v_j) = \begin{cases} b(v_1), & \text{if } j = 1; \\ b(v_j) + \\ \min\{\texttt{MBS}(v_i) \mid v_i \prec v_j \preceq \texttt{FAR}(v_i)\}, & \text{if } 2 \le j \le n. \end{cases}$$

In the above DP recursion, note that the vehicle performs a battery-swap at v_i immediately before v_j when traveling from v_1 to $\texttt{FAR}(v_j)$; therefore, station v_j lies between v_i and $\texttt{FAR}(v_i)$. The optimal solution is derived in the form of $\min\{\texttt{MBS}(v_i) \mid v_i \prec v_n \preceq \texttt{FAR}(v_i)\}$.

When computing each entry $\texttt{MBS}(v_{j+1})$, $1 \le j \le n-1$, we have to keep track of all the possibilities for every station v_i satisfying $v_i \prec v_{j+1} \preceq \texttt{FAR}(v_i)$. More precisely, at each recursion step $j+1$, we need to *insert* $\texttt{MBS}(v_j)$ into the current pool of possible solutions, and *delete* every station v_k whose $\texttt{FAR}(v_k)$ precedes v_{j+1}; besides, the minimum value in the pool has to be determined. By using the Fibonacci heap [14], which allows each of the *insert* and *return-minimum* operations to take $\Theta(1)$ time and the *delete* operation to take $O(\log n)$ amortized time, the recurrence can be solved in $O(n \log n)$ time.

Furthermore, for any v_i and v_j with $v_i \prec v_j$, if $\texttt{MBS}(v_i) \ge \texttt{MBS}(v_j)$, then $\texttt{MBS}(v_i)$ can be directly removed from the solution pool based on the interleaving property; that is, it is impossible to select $\texttt{MBS}(v_i)$ to comprise the optimal solution because $\texttt{FAR}(v_i) \preceq \texttt{FAR}(v_j)$. Therefore, when computing each entry of the recurrence, the pool of possible solutions can be maintained in a sorted list in increasing order. In other words, based on the interleaving property, each of the above *insert*, *delete* and *return-minimum* operations takes only $\Theta(1)$ time at every recursion step. The optimal solution $\min\{\texttt{MBS}(v_i) \mid v_i \prec v_n \preceq \texttt{FAR}(v_i)\}$ can thus be derived in linear time. We remark that the optimal solution of stations required can be obtained by backtracking the computation of the recurrence.

Lemma 4.6. *The non-uniform cost model of the fixed path EV touring problem can be optimally solved in linear time.*

We extend the DP technique to the fixed tour model where a tour $P : v_1 - v_2 - \ldots - v_n - v_1$ in a circular order is given. Recall that when computing each entry of the DP recursion for the fixed path model, a pool of possible solutions needs to be considered. For each v_j, $1 \leq j \leq n$, let the pool be defined by $S_j = \{v_i \mid v_i \prec v_j \preceq \texttt{FAR}(v_i)\}$. Based on the interleaving property, every vertex set S_j can be obtained in linear time. Note that each S_j is nonempty because of the natural assumption that the battery capacity U is larger than the distance weight of each edge in P. One observes that in any feasible solution, there is at least one vertex in each S_j. Thus, let $S_{\min}$ be the one of the minimum cardinality, denoted by δ, and an optimal solution of stations contains at least one vertex in $S_{\min}$. We use the DP technique in the fixed path model δ times; each starts at a different vertex in $S_{\min}$. After performing the DP procedure for the δ fixed paths, the one with the minimum time of total battery-swaps is the optimal solution for the non-uniform cost fixed tour model. Since the optimal DP procedure for each fixed path can be done in linear time by Lemma 4.6, the optimal solution for a fixed tour in a given order can be derived in $O(\delta n)$ time, where for every station v_j, δ is the minimum number of stations at which the vehicle can start with a full battery and reach v_j. It could be assumed that δ is sufficiently smaller than n in practice. The next theorem follows.

Theorem 4.7. *For the non-uniform cost fixed tour model, the minimum total battery-swap time as well as the corresponding battery-switch stations under a given fixed tour $P : v_1 - v_2 - \ldots - v_n - v_1$ can be optimally determined in $O(\delta n)$ time, where for every station v_j, δ is the minimum number of stations at which the vehicle can start with a full battery and reach v_j.*

5 Electric Vehicle Touring

In this section, we divide the EV touring problem into two models. In the first model, called the *on-site station model*, every city has an *on-site* battery switch station, i.e., $V = F$. The on-site EV touring problem (Problem OEVTP) involves determining a tour that visits all the vertices in $V = F = \{v_1, v_2, ..., v_n\}$. The objective is to minimize the total time expended on the tour. Similarly, suppose the vehicle starts with an empty battery.

The second model presents a more interesting scenario in that $V \nsubseteq F$, but there is at least one battery switch station within an acceptable distance of every city. This is called the *off-site station model.* According to the natural assumption in [18, 21], we let $\alpha = \max_{v \in V} \min_{u \in F}\{w(v, u)\}/U$ and let the acceptable distance be at most αU, $0 \leq \alpha < 1/2$. In practice, a vehicle cannot visit a city if the nearest station is more than the distance $U/2$ from the city. Note that the vehicle is allowed to visit a city multiple times, if necessary, in the off-site station model, because the city might have an on-site battery switch station. We also assume the distance weights satisfy the *triangle inequality* [18].

Khuller *et al.* [18] proposed the *tour gas station problem*, which involves finding the cheapest tour that visits all the cities in V and possibly some gas stations in F. They proved that the uniform cost on-site station model of this problem can be reduced to the original TSP, where the gas price is identical at each station. In contrast, the uniform cost on-site station model of the EV touring problem cannot be transformed into the TSP directly because of the 0-1 recharging operations, i.e., battery-swaps.

Table 1 summarizes the results of the EV touring problem, where $b_{\min} = \min_{v \in F}\{b(v)\}$ and $b_{\max} = \max_{v \in F}\{b(v)\}$. Note that comparisons between the results of the EV touring problem and those of the tour gas station problem [18] are presented in the last two columns. For the latter problem, although it was claimed that $\frac{c_{\max}}{c_{\min}}$ was around 1.2 in practice, where $c_{\min} = \min_{v \in F}\{c(v)\}$ and $c_{\max} = \max_{v \in F}\{c(v)\}$, the ratios derived in this study can be regarded as constants directly under such reasonable conditions in the on-site station model.

5.1 On-site station EV touring

The approximation algorithm is implemented in two phases. First, we exploit Christofides algorithm [7] to derive a route plan, i.e., a permutation of all the vertices in V. Christofides algorithm is a combination of the minimum spanning tree of a

Model	Cost	Ratio	Condition	Under condition	[18]
On-site	Uniform	$\frac{3U+6c}{2U+2c}$	$U \geq c \geq 0$	9/4	$\frac{3}{2}$
			$U \gg c \geq 0$	3/2	
	Non-uniform	$\frac{3U+6b_{\max}}{2U+2b_{\min}}$	$U \geq b_{\max}, b_{\min} \geq 0$	9/2	$\frac{3c_{\max}}{2c_{\min}}$
			$U \gg b_{\max}, b_{\min} \geq 0$	3/2	
Off-site	Uniform	$\frac{3(U+2\alpha U+2c)}{2(1-2\alpha)(U+c)}$	$U \geq c \geq 0$	$\frac{3}{4}(\frac{3+2\alpha}{1-2\alpha})$	$\frac{3}{2}(\frac{1+2\alpha}{1-2\alpha})$
			$U \gg c \geq 0$	$\frac{3}{2}(\frac{1+2\alpha}{1-2\alpha})$	
	Non-uniform	$\frac{3(U+2\alpha U+2b_{\max})}{2(1-2\alpha)(U+b_{\min})}$	$U \geq b_{\max}, b_{\min} \geq 0$	$\frac{3}{2}(\frac{3+2\alpha}{1-2\alpha})$	$\frac{3c_{\max}}{2c_{\min}}(\frac{1+2\alpha}{1-2\alpha})$
			$U \gg b_{\max}, b_{\min} \geq 0$	$\frac{3}{2}(\frac{1+2\alpha}{1-2\alpha})$	

Table 1: Comparison of the results of the EV touring problem with those reported in [18]

complete graph G with the minimum weight perfect matching on the vertices with odd degree in the tree. The result of this algorithm is a Hamiltonian tour with a 1.5-approximation ratio if the distance function satisfies the triangle inequality property. Then we use the algorithms derived in the previous section to optimally identify appropriate battery switch stations in both the uniform and non-uniform cost cases and ensure that a vehicle's movement can be maintained. The steps of the approach for solving the on-site EV touring problem are detailed in Algorithm 2.

Algorithm 2: Find an approximation to the on-site EV touring problem

Input: $G = (V, E)$ of order n; a distance function $w : E \to R^+$, a battery-swap function $b : V \to R^+$ and the vehicle's battery capacity U;

Output: A tour P that visits all the vertices in V with a set of stations $F^* \subseteq V$ for battery-swaps;

1: Use Christofides algorithm to determine the visiting order of all cities in V, denoted by $P : v_1 - v_2 - \ldots - v_n - v_1$;

2: Select a set of stations F^* for battery-swaps by Algorithm 1 in the uniform cost fixed tour model or by the DP recursion in the non-uniform cost fixed tour model;

Assume the vehicle in the on-site station model starts with an empty battery. Without loss of generality, the distance weight of every edge is not larger than the given battery capacity U. Note that once the permutation over all the vertices in a tour is determined, Algorithm 1 can optimally select stations for battery-swaps in linear time in the uniform cost on-site model. Similarly, the DP recursion can also be applied to the non-uniform cost on-site model and optimally solve the problem in $O(\delta n)$ time as mentioned earlier. Thus, the solution derived by Algorithm 2 is feasible, which represents the total time cost, denoted as SOL. Then, we analyze its approximation ratio. Note that the total time cost includes traveling and battery-swaps; therefore, let $SOL = SOL_{travel} + SOL_{swap}$ represent the respective time required. Let OPT be the minimum time needed to complete the EV tour, as shown by

$$OPT = OPT_{travel} + OPT_{swap}. \quad (1)$$

Consider the traveling time SOL_{travel}, i.e., $w(P)$. Let OPT_{TSP} be the minimum time required for the original TSP. It is trivial that $OPT_{TSP} \leq OPT_{travel}$. Because the vehicle is allowed to visit a city multiple times, Phase 1 combines the minimum spanning tree of G with the minimum weight perfect matching on the vertices with odd degrees in the tree to obtain an Euler tour instead of a Hamiltonian tour. Based on a similar analysis in [7], the $\frac{3}{2}$-approximation ratio can be derived, and the equation follows immediately.

$$SOL_{travel} \leq \frac{3}{2} OPT_{TSP} \leq \frac{3}{2} OPT_{travel} \quad (2)$$

Let $b_{\min}$ and $b_{\max}$ represent the smallest and largest battery-swap time respectively, i.e., $b_{\min} = \min_{v \in F}\{b(v)\}$ and $b_{\max} = \max_{v \in F}\{b(v)\}$. The vehicle starts with an empty battery and needs to stop at a minimum of $\lceil \frac{OPT_{travel}}{U} \rceil$ stations. Hence, the relationship between OPT_{travel} and OPT_{swap} can be formulated as follows:

$$b_{min}(\left\lceil \frac{OPT_{travel}}{U} \right\rceil) \leq OPT_{swap}$$
$$\Rightarrow \; OPT_{travel} \leq \frac{U}{b_{\min}} OPT_{swap}. \qquad (3)$$

Note that b_{min} can be replaced by a constant c in the uniform cost case.

Given the fixed tour P obtained in Phase 1 in Algorithm 2, let $F^* = \{v_{s_1}, v_{s_2}, \ldots, v_{s_k}\}$ be derived in Phase 2 in Algorithm 2, where k is the number of stations required. A path $p_i : v_{s_i} \sim v_{s_{i+1}}$ represents the subpath from v_{s_i} to $v_{s_{i+1}}$ along the tour P. The next lemma follows: (We omit the proof due to space limit.)

Lemma 5.1. *For both uniform and non-uniform cost models, we have $w(p_i) + w(p_{i+1}) > U$, $1 \leq i \leq k$, where we let $p_k : v_{s_k} \sim v_{s_1}$ and $p_{k+1} = p_1$.*

Then, by summing the equations for every i from Lemma 5.1, we have the following inequality:

$$2SOL_{travel} = \sum_{i=1}^{k} (w(p_i) + w(p_{i+1})) > kU$$
$$\Rightarrow k < \frac{2SOL_{travel}}{U}$$
$$\Rightarrow SOL_{swap} < \frac{2SOL_{travel}}{U} b_{max}. \qquad (4)$$

The last inequality holds because $SOL_{swap} \leq kb_{\max}$. Based on the above properties, the next theorem follows according to Equations (1)-(4).

Theorem 5.2. *The on-site EV touring problem can be approximated within a $\frac{3U+6b_{max}}{2U+2b_{min}}$-ratio for the non-uniform cost case.*

It is reasonable to assume that the battery capacity U is larger than b_{max} in terms of time expended, which leads to a constant approximation ratio $\frac{9}{2}$. For the uniform cost case, the ratio is within $\frac{3U+6c}{2U+2c}$, for a constant c; and it is $\frac{9}{4}$ based on the assumption that $U \geq c$. In addition, if U is significantly larger than b_{max} and c, the approximation factor is $\frac{3}{2}$, which is the same as that of Christofides algorithm for the TSP, as compared with the ratio $\frac{3c_{\max}}{2c_{\min}}$ derived in [18].

5.2 Off-site station EV touring

Consider the off-site station model in which the distance between every city in V and its nearest battery switch station is at most αU, $0 \leq \alpha < \frac{1}{2}$; when $\alpha = 0$, it is the on-site station model. A function $near : V \rightarrow F$ is defined as the nearest station to each city $v \in V$; thus, $\alpha = max_{v_i \in V}\{w(v_i, near(v_i))\}/U$. To ensure a vehicle's movement, assume the distance between any two vertices in $V \cup F$ is less than $U/2$. The assumption can be considered as route planning in urban areas. In practice, $U/2 \approx 100$ kilometers is sufficient for the diameter of a metropolitan region.

For the off-site EV touring problem, suppose the vehicle begins with a full battery because the vehicle might start from a vertex without any battery switch station. The rationale behind the approach for this problem is similar to the greedy concept of Lemma 4.1; however, this straightforward approach cannot produce an optimal solution, even for the uniform cost case in the off-site station model. Because the distance between a vertex v and $near(v)$ is not identical, the set of stations F^* selected by the greedy concept is only a feasible solution. Given a fixed tour $P : v_1 - v_2 - \ldots - v_n - v_1$ derived in Phase 1 of Algorithm 2, the vehicle will start from v_1 and stop at the nearest battery switch station of a vertex v_i, i.e., $near(v_i)$, unless it can reach $near(v_{i+1})$ of the vertex v_{i+1}. The steps of the approach for solving the off-site EV touring problem are detailed in the GreedyHeuristic procedure. In addition, the following analysis based on F^* is similar to that of the on-site station model.

procedure GreedyHeuristic(G, P)
  Let v_1 be the start vertex and let $F^* = \emptyset$;
  for $i = 1$ to n **do**
    if the vehicle cannot reach v_{i+1} and $near(v_{i+1})$ with remaining power at v_i, where $v_{n+1} = v_1$ **then**
      leave v_i for $near(v_i)$, and do a battery-swap there;
      $F^* = F^* \cup \{near(v_i)\}$;
    end if
  end for
  return a subset of battery switch stations F^* and a tour P^* that incorporates extra routes into P for battery-swaps;
end procedure

Lemma 5.3. *Given a fixed tour that traverses each vertex in V for the off-site EV touring problem, the* GreedyHeuristic procedure *can derive a feasible solution of stations F^* that satisfies the greedy concept of* Lemma 4.1*, and incorporate additional routes for battery-swaps in linear time.*

(We omit the proof due to space limit.)

In Algorithm 2, we replace Phase 2 by the GreedyHeuristic procedure. Consider the uniform cost case; that is, $b(v) = c$ for every vertex v, where c is a constant. Note that the GreedyHeuristic procedure not only selects stations, but also devises a tour P^* that incorporates routes between vertices and their nearest stations, if necessary. Similarly, let $SOL = SOL_{travel} + SOL_{swap}$ and $OPT = OPT_{travel} + OPT_{swap}$ by Equation (1). Given a fixed tour $P : v_1 - v_2 - \ldots - v_n - v_1$ derived in Phase 1 of Algorithm 2, the following also holds for the off-site station model.

$$w(P) \leq \frac{3}{2} OPT_{TSP} \leq \frac{3}{2} OPT_{travel} \qquad (5)$$

Assume $F^* = \{near(v_{s_1}), near(v_{s_2}), \ldots, near(v_{s_k})\}$ and the tour derived by the GreedyHeuristic procedure can be represented by $P^* : v_1 \sim v_{s_1} - near(v_{s_1}) \sim v_{s_2} - near(v_{s_2}) \sim \ldots \sim v_{s_k} - near(v_{s_k}) \sim v_1$, where k is the number of stations required. In addition, let $\triangle_i : v_{s_i} - near(v_{s_i}) - v_{s_i+1}$ represent the subpath from v_{s_i} to v_{s_i+1} along the tour P^*. We have $w(near(v_{s_i}), v_{s_i+1}) < w(v_{s_i}, near(v_{s_i})) + w(v_{s_i}, v_{s_i+1})$ because of the triangle inequality. Thus, for $1 \leq i \leq k$,

$$\begin{aligned} w(\triangle_i) &< 2w(v_{s_i}, near(v_{s_i})) + w(v_{s_i}, v_{s_i+1}) \\ &\leq 2\alpha U + w(v_{s_i}, v_{s_i+1}). \end{aligned}$$

Then, sum the equations for every i and the next inequality follows.

$$\sum_{i=1}^{k} w(\triangle_i) = SOL_{travel} - w(P) + \sum_{i=1}^{k} w(v_{s_i}, v_{s_i+1})$$
$$< 2k\alpha U + \sum_{i=1}^{k} w(v_{s_i}, v_{s_i+1})$$
$$\Rightarrow SOL_{travel} < w(P) + 2k\alpha U \leq \frac{3}{2} OPT_{travel} + 2k\alpha U$$

Moreover, for $1 \leq i \leq k-2$, let $p_i : v_{s_i+1} \sim v_{s_{i+1}} - v_{s_{i+1}+1} \sim v_{s_{i+2}}$ be the subpath from v_{s_i+1} to $v_{s_{i+2}}$ on the tour P; and let $p_0 : v_1 \sim v_{s_1} - v_{s_1+1} \sim v_{s_2}$ and $p_{k-1} : v_{s_{k-1}+1} \sim v_{s_k} - v_{s_k+1} \sim v_1$. We have the following lemma.

Lemma 5.4. *For every $1 \leq i \leq k-1$, $w(p_i) + w(v_{s_i}, v_{s_i+1}) > (l - 2\alpha)U$; in addition, $w(p_0) > (l - \alpha)U$.*

Similarly, we sum the equations for every i, $0 \leq i \leq k-1$, and derive the following formulation

$$\begin{aligned} 2w(P) &> w(p_0) + \Sigma_{i=1}^{k-1}(w(v_{s_i} - v_{s_i+1}) + w(p_i)) \\ &> k(1-2\alpha)U \\ \Rightarrow \quad k &< \frac{3OPT_{travel}}{(1-2\alpha)U} \\ \Rightarrow SOL_{swap} &< \frac{3cOPT_{travel}}{(1-2\alpha)U} \end{aligned}$$

The next theorem follows from the above discussion.

Theorem 5.5. *The uniform cost model of the off-site EV touring problem can be approximated within a $\frac{3(U+2\alpha U+2c)}{2(1-2\alpha)(U+c)}$-ratio, where $0 \leq \alpha < \frac{1}{2}$.*

(We omit the proof due to space limit.)

The scenario where U is larger than c leads to the approximation ratio $\frac{3}{4}(\frac{3+2\alpha}{1-2\alpha})$. In addition, if U is significantly larger than c, the approximation ratio is $\frac{3}{2}(\frac{1+2\alpha}{1-2\alpha})$, the same as that of the tour gas station problem [18]. When $\alpha = 0$, the ratio is exactly equal to the result of the on-site station model. For the non-uniform cost case, we refer to [18] and replace c with b_{max}/b_{min} in the approximation ratio based on a similar reduction scheme in the uniform cost case. The approximation ratio is $\frac{3}{2}(\frac{3+2\alpha}{1-2\alpha})$ when U is larger than $b_{\max}$.

6 Concluding Remarks

In this study, we have considered several EV route planning problems that incorporate 0-1 battery recharging operations. We have presented a simple dynamic programming algorithm that optimally solves the EV shortest travel time path problem in polynomial time. We have further studied the fixed tour EV touring problem and used graph-theoretic techniques to develop optimal algorithms, which extend the prior work of the fixed path gas station problem. We have also investigated the on-site station and off-site station EV touring problem, and proposed approximation algorithms with constant factors and a $\frac{3}{2}(\frac{3+2\alpha}{1-2\alpha})$-factor, respectively.

We remark that the latest result in the literature reported by An *et al.* [1] improved the approximation ratio of Christofides algorithm for the s,t-*path TSP*. Our approach can be combined directly with s,t-Hamiltonian path derived by An *et al.* (or any better approximation algorithms that may be proposed in the future) to obtain an improved performance ratio for variations of the EV touring problem.

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