Remarks on the complexity of the total domatic partition problem in graphs

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Abstract

In this paper, we study the total domatic partition problem for bipartite graphs, circular-arc graphs, and graphs with balanced adjacency matrices. We show that the total domatic partition problem is NP-complete for bipartite graphs and circular-arc graphs, and show that the total domatic partition problem is polynomial-time solvable for graphs with balanced adjacency matrices. Furthermore, we show that the total domatic partition problem is linear-time solvable for chordal bipartite graphs.

1 Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph. Unless stated otherwise, it is understood that $|V| = n$ and $|E| = m$. We also use $V(G)$ and $E(G)$ to denote vertex set and edge set of G, respectively. For any vertex $v \in V$, the *open neighborhood* of v in G is $N_G(v) = \{u \in V | (u, v) \in E\}$ and the *closed neighborhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in G is $deg_G(v) = |N_G(v)|$. The *minimum degree* of a vertex of G is denoted by $\delta(G)$.

A vertex v of a graph $G = (V, E)$ *dominates* a vertex w if $v \in N_G[w]$. A vertex v of G *totally dominates* a vertex w if $v \in N_G(w)$. A subset $D \subseteq V$ *dominates* (respectively, *totally dominates*) a vertex v if v is dominated (respectively, totally dominated) by some vertex in D, i.e., $|D \cap N_G[v]| \geq 1$ (respectively, $|D \cap N_G(v)| \geq 1$). A *dominating* (respectively, *total dominating*) *set* of a graph $G = (V, E)$ is a subset D of V such that D dominates (respectively, totally dominates) every vertex v in V.

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For a positive integer k, a *k-total dominating set* of a graph $G = (V, E)$ is a subset D of V such that $|D \cap N_G(v)| \geq k$ for every $v \in V$. Clearly, G does not have any k-total dominating set if $k > \delta(G)$.

Two sets A and B are *disjoint* if $A \cap B = \emptyset$. A collection $P = \{S_1, S_2, \ldots, S_\ell\}$ forms a *partition* of a set S if $S = S_1 \cup S_2 \cup \cdots \cup S_\ell$, and $S_i \cap S_j = \emptyset$ for any two distinct sets S_i and S_j in P.

A partition $P = \{V_1, V_2, \ldots, V_\ell\}$ of V is a *domatic* (respectively, *total domatic*) *partition* of a graph $G = (V, E)$ if V_i is a dominating (respectively, total dominating) set of G for $i = 1, 2, \ldots, \ell$. The *domatic* (respectively, *total domatic*) *partition problem* is to find a domatic (respectively, total domatic) partition of G of maximum size. The *domatic* (respectively, *total domatic*) *number* of G, denoted by $d(G)$ (respectively, $d_t(G)$), is the size of a maximum domatic (respectively, total domatic) partition of G. For any positive integer k, the k-domatic (respectively, total k-domatic) partition problem is to find a domatic (respectively, total domatic) partition P of G with $|P| = k$.

The domatic number of a graph was introduced by Cockayne and Hedetniemi [8]. The concept of the total domatic number was introduced by the same authors and Dawes [6]. They proved that $d(G) \leq \delta(G) + 1$ and $d_t(G) \leq \delta(G)$ for any graph G. The domatic partition problem has been widely studied from the algorithmic point of view [2, 3, 7, 9, 12, 13, 15, 16, 18]. The total domatic number of a graph has been investigated in [1, 3, 6, 9, 19, 20, 21, 22]. However, few results exist in the literature about the algorithmic complexity of the total domatic partition problem in graphs. From the algorithmic point of view, the total k-domatic partition problem on the 2-section graph of the order-interval hypergraph of a finite poset is NP-complete for any fixed positive integer $k \geq 3$ [3], and the total 2-domatic partition problem on bipartite graphs is NP-complete [11].

The results motivate us to consider the algorithmic complexity of the total domatic partition problem for other classes of graphs.

In this paper, we concentrate on bipartite graph, circular-arc graphs, and graphs with balanced adjacency matrices. For any fixed integer $k \geq 3$, we show that the total k-domatic partition problem is NP-complete, even when restricted to bipartite graphs and circular-arc graphs. For any positive integer $k \leq \delta(G)$, we show that the k-total domatic partition problem is polynomial-time solvable for any graph G with a balanced adjacency matrix. Furthermore, we show that the total k-domatic partition problem is linear-time solvable for chordal bipartite graphs.

2 The NP-completeness results

In this section, we present NP-completeness results for bipartite graphs and circular-arc graphs. Before presenting the NP-completeness results, we restate the k-domatic partition problem and the total k-domatic partition problem as decision problems.

(1) **The k-domatic partition problem:**

Instance: A graph $G = (V, E)$ and a positive integer $k \leq \delta(G) + 1$.

Question: Is $d(G) \geq k$?

(2) **The total k-domatic partition problem:**

Instance: A graph $G = (V, E)$ and a positive $k \leq \delta(G)$.

Question: Is $d_t(G) \geq k$?

Theorem 1. *For any fixed integer $k \geq 3$, the total k-domatic partition problem on bipartite graphs is NP-complete.*

Proof. It is obvious that total k-domatic partition problem is a member of NP since we can guess at a partition of the vertices of a graph G in polynomial time and verify if each set of the partition is a total dominating set of G in polynomial time.

It has been shown [11] that the total 2-domatic partition problem on bipartite graphs is NP-complete. In the following, we show the NP-completeness of the total k-domatic partition problem on bipartite graphs for any fixed positive integer $k \geq 3$ by a polynomial-time reduction from the total $(k-1)$-domatic partition problem on bipartite graphs.

Let $G = (A, B, E)$ be a bipartite graph. We create two new vertices x and y. Let $A' = A \cup \{x\}$ and let $B' = B \cup \{y\}$. We connect x to y and all the vertices in B, and then connect y to all the vertices in A. Let H be the resulting graph. Clearly, the graph H is a bipartite graph with two disjoint sets of vertices A' and B'.

Let $d_t(G) = \ell$ and let $\mathcal{D} = \{D_1, D_2, \ldots, D_\ell\}$ be a total domatic partition of G. Clearly, $D_1, D_2, \ldots, D_\ell$ are also disjoint total dominating sets of H. Let $D_{\ell+1} = \{x, y\}$. By the construction of H, the set $D_{\ell+1}$ is a total dominating set of H. Then, $D_1, D_2, \ldots, D_{\ell+1}$ are disjoint total dominating sets of H. We have $d_t(H) \geq d_t(G) + 1$.

Let $d_t(H) = \kappa$ and let $\mathcal{S} = \{S_1, S_2, \ldots, S_\kappa\}$ be a total domatic partition of H. We consider the following two cases.

Case 1: The vertices x and y are in the same set of $\mathcal{S}$. Let S_i be the set containing the vertices x and y. Note that $V(H) = V \cup \{x, y\}$. Let $S_j \in \mathcal{S} \setminus \{S_i\}$ and let $S'_j = S_j \cup S_i \setminus \{x, y\}$. Clearly, the collection $P = (\mathcal{S} \setminus \{S_i, S_j\}) \cup \{S'_j\}$ is a total domatic partition of G. Therefore, $d_t(G) \geq d_t(H) - 1$.

Case 2: The vertices x and y are in different sets of $\mathcal{S}$. Suppose that S_i and S_j are distinct sets of $\mathcal{S}$, and $x \in S_i$ and $y \in S_j$. By the construction of H, the vertex x (respectively, y) dominates every vertex in B (respectively, A), and the set $S_i \cap B$ (respectively, $S_j \cap A$) dominates every vertex in A (respectively, B). Let $S'_i = (S_i \setminus \{x\}) \cup (S_j \setminus \{y\})$. Then, S'_i is a total dominating set of G and the collection $P = (\mathcal{S} \setminus \{S_i, S_j\}) \cup \{S'_i\}$ is a total domatic partition of G. Therefore, $d_t(G) \geq d_t(H) - 1$.

Following the discussion above, we know that $d_t(H) \leq d_t(G) + 1$ and $d_t(H) \geq d_t(G) + 1$. Therefore, $d_t(H) = d_t(G) + 1$. It implies that $d_t(H) \geq k$ for any fixed integer $k \geq 3$ if and only if $d_t(G) \geq k - 1$. □

Definition 1. For a given set M of objects, the *intersection graph* G_M of these objects has M as vertex set, and two objects are adjacent in G_M if the intersection of the corresponding objects is nonempty. A *circular-arc graph* is the intersection graph of a set of arcs on the circle. It has one vertex for each arc in the set, and an edge between every pair of vertices corresponding to arcs that intersect.

Theorem 2. *For any fixed integer $k \geq 3$, the total k-domatic partition problem on circular-arc graphs is NP-complete.*

Proof. The total k-domatic partition problem is clearly in NP. It has been shown [2] that the k-domatic partition problem on circular-arc graphs is NP-complete for any fixed positive integer $k \geq 3$. In the following, we show the NP-completeness of the total k-domatic partition problem on circular-arc graphs for any fixed positive integer $k \geq 3$ by a polynomial-time reduction from the k-domatic partition problem on circular-arc graphs.

Let $G = (V, E)$ be a circular-arc graph. For each vertex $v \in V$, we construct a new vertex v' and connect the vertex v' to all vertices in $N_G[v]$. Let H be the resulting graph and let $V' = \{v' \mid v \in V\}$. Then, $V(H) = V \cup V'$ and $E(H) = E \cup \{(u, v') \mid v' \in V' \text{ and } u \in N_G[v]\}$. Clearly, the graph H is a circular-arc graph and the construction of H can be done in polynomial time.

Let $d(G) = \ell$ and let $\mathcal{D} = \{D_1, D_2, \ldots, D_\ell\}$ be a domatic partition of G. By the construction of H, we know that $N_H[v] = N_G[v] \cup \{u' \mid u \in N_G[v]\}$ for every vertex $v \in V$. For any dominating set D of G, $V = \bigcup_{v \in D} N_G[v]$. Then, $V' = \bigcup_{v \in D}\{u' \mid u \in N_G[v]\}$. We have $\bigcup_{v \in D} N_H[v] = \bigcup_{v \in D}(N_G[v] \cup \{u' \mid u \in N_G[v]\}) = V \cup V'$. Therefore, a dominating set of G is also a dominating set of H. Let $D'_i = D_i \cup \{v' \mid v \in D_i\}$ for $1 \leq i \leq \ell$. The sets $D'_1, D'_2, \ldots, D'_\ell$ are disjoint total dominating sets of H. Hence, $d_t(H) \geq d(G)$.

Let $d_t(H) = \kappa$ and let $\mathcal{S} = \{S_1, S_2, \ldots, S_\kappa\}$ be a total domatic partition of H. By the construction of H, the set V' is an independent set of H and $N_H(v') = N_G[v]$ for every vertex $v' \in V'$. For each set S_i of $\mathcal{S}$, we know that $S_i \setminus V' \neq \emptyset$ and S_i contains a vertex in $N_G[v]$ to totally dominate each $v' \in V'$. Therefore, $S_i \setminus V'$ is a dominating set of G. We have $d_t(H) \leq d(G)$.

Following the discussion above, we know that $d_t(H) = d(G)$. It implies that $d(G) \geq k$ for any fixed integer $k \geq 3$ if and only $d_t(H) \geq k$. □

3 Graphs with balanced adjacency matrices

Suppose that $G = (V, E)$ is a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. The *adjacency matrix* of G is the (0,1)-matrix whose entry (i, j) is 1 if $(v_i, v_j) \in E$ and 0 otherwise. The *edge-vertex incidence matrix* of G is the (0,1)-matrix whose entry (i, j) is 1 if v_j is an endvertex of e_i and 0 otherwise. A (0,1)-matrix is *balanced* if it does not contain the edge-vertex incidence matrix of an odd cycle as a submatrix. A (0,1)-matrix is *totally balanced* if it does not contain the edge-vertex incidence matrix of a cycle as a submatrix.

Let V be a finite set of vertices with $|V| = n$ and let $\mathcal{P}(V)$ denote the power set of the set V, i.e., the set of all subsets of V. The pair $H = (V, \mathcal{E})$ is a hypergraph if $\mathcal{E} \subseteq \mathcal{P}(V)$. $\mathcal{E}$ is the set of *hyperedges* of the hypergraph. A hypergraph H is balanced if the hyperedge-vertex incidence matrix of H is balanced. Let k be a positive integer. A *k-fold transversal* S of a hypergraph H is a set of vertices such that every hyperedge of H has at least k vertices in S.

Theorem 3 (Dahlhaus et al. [9]). *A k-fold transversal of a balanced hypergraph can be partitioned into k pairwise disjoint transversals in polynomial time.*

Suppose that $G = (V, E)$ is a graph with the balanced adjacency matrix $A(G)$. Let $V = \{v_1, v_2, \ldots, v_n\}$ and let $H' = (V, \mathcal{E})$ be a hypergraph where $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ and $E_i = N_G(v_i)$ for $i = 1, 2, \ldots, n$. Since $A(G)$ is balanced and each row of $A(G)$ stands for an open neighborhood of a vertex in V, it can be easily verified that H' is balanced. A k-fold transversal of H' is therefore equivalent to a k-total dominating set of G. Following Theorem 3, we have the following result.

Theorem 4. *For any positive integer $k \leq \delta(G)$, a k-total dominating set of a graph G with a balanced adjacency matrix can be partitioned into k pairwise disjoint total dominating sets in polynomial time.*

Theorem 5. *The total domatic partition problem can be solved in polynomial time for graphs with balanced adjacency matrices.*

Proof. Let $G = (V, E)$ be a graph with a balanced adjacency matrix. Note that $\delta(G)$ is the minimum degree of G. We have $|N_G(v) \cap V| \geq \delta(G)$ for every $v \in V$. Clearly, V is a $\delta(G)$-total dominating set of G and can be found in linear time. By Theorem 4, V can be partitioned into $\delta(G)$ pairwise disjoint total dominating sets $V_1, V_2, \ldots, V_{\delta(G)}$ in polynomial time. Hence the total domatic partition problem is polynomial-time solvable for graphs with balanced adjacency matrices. □

Theorem 6 (Dahlhaus et al. [9]). *A k-fold transversal of a totally balanced hypergraph can be partitioned into k pairwise disjoint transversals in $O(\log n)$ time by an optimal parallel algorithm with $O(n + m/\log n)$ processors.*

Corollary 1. *For any positive integer $k \leq \delta(G)$, the total k-domatic partition problem can be solved in linear time for any graph G with a totally balanced adjacency matrix.*

Proof. It follows from Theorems 4 and 6. □

Corollary 2. *The total domatic partition problem is linear-time solvable for chordal bipartite graphs.*

Proof. For any chordal bipartite graph G, the adjacency matrix of G is totally balanced [4]. By Corollary 1 and using the argument similar to those for proving Theorem 5, the corollary holds. □

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