List Coloring of Cograph

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Abstract

List coloring is to color each vertex v of graph G from its color set L(v). If any two adjacent vertices have different colors, G is colored properly. We are interested in the smallest size of L(v) for every vertex v such that each L(v) has the same size and the graph G is colored properly. Additionally, the smallest size is called the list chromatic number of G. This paper gives a lower and upper bounds of the list chromatic number for cographs. We show that the bounds we proposed are reachable. Besides, we also give an algorithm for finding the upper and lower bounds of the list chromatic of any cograph.

Keywords: cograph; list coloring; list chromatic number; colored properly.

1 Introduction

List coloring was introduced independently by Vizing [20] in 1976 and by Erdös, Rubin and Taylor [3] in 1979. The problem is able to be applied in processor assignment, frequency channel assignment and so on. Some applications of list coloring were mentioned in [32]. For processor assignment problem, every vertex of a graph is represented as an operation and each color we can use to assign to one vertex is regard as a processor. An edge between two vertices is indicated that the corresponding operations cannot be assign to the same processor. The list coloring problem is equal to satisfy the following two conditions: one is that only compatible operations are assigned to the same processor, the other is an assigned processor can execute the operation.

Then, we introduce the definition of list coloring. Each vertex v of a graph G is with the list $L(v)$ which is the set of allowed colors. A *list assignment* L of G is a collection of the list of each vertex v. G is called *L-list colorable* if there is a coloring c of vertices with:

$$c(u) \neq c(v) \text{ for } uv \in E(G),$$

$$c(v) \in L(v) \text{ for all } v \in V(G).$$

We say that G is *k-choosable* if it is L-list colorable for every list assignment L satisfying $|L(v)| = k$ (also be denoted by *k-list*) for all vertex v. The *list chromatic number* of G (also called *choice number*, $ch(G)$), denoted by $\chi_l(G)$, is the smallest k such that G is k-choosable.

In [3], almost 2-choosable graphs were showed. Thomassen proved that every planar graph is 5-choosable in [17]. An example of a planar graph which is not 4-choosable presented by Voigt [21, 23]. The problem, which is to determine whether a given planar graph is 3-, or 4-choosable, is proved that it is NP-hard [6]. However, there exists the sufficient condition for a planar graph to be 3-choosable ([11, 15, 18, 26, 27, 33, 34]). And, there exists the condition for a planar graph which is not 3-choosable ([4, 6, 12, 13, 22, 24, 25]). In addition to planar graph, cograph we are interested was talked about in [9]. In this article, Jansen and Scheffler proved that the decision problem LICOL is NP-complete and the enumeration problem #LICOL is NP-hard for cographs. If the list of every vertex (the size of $L(v)$ does not need to be the same) is given, LICOL is to consider whether a proper list coloring exists or not and #LICOL is to calculate how many the number of proper list colorings are. The two excellent articles [19] and [28] survey the background of list coloring.

This paper is organized as follows. There are some definitions and related work in Section 2. A lower and upper bounds of the list chromatic number of any cograph be given in Section 3. Section 4 give an algorithm and an example to calculate the upper and lower bounds of the list chromatic of any cograph we gave in previous section. Finally, we make some conclusion. Moreover, the graph we study in this paper is finite, simple and undirected.

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2 Preliminary

In this Section, we introduce some basic definitions in graph theory and some related work about our topical subjects in Part A and Part B, respectively.

A. Basic Definition

In this Part, some definitions are given. First, there are definitions about basis terminology of graph theory we use in this thesis and some basic graphs.

Definition 2.1. [31] A *simple graph G* is an ordered pair ($V(G)$, $E(G)$), where $V(G)$ is the *vertex set*, $E(G)$ is the *edge set* and $E(G) \subseteq V(G) \times V(G) \setminus \{(v, v) \mid v \in V(G)\}$. An element in $V(G)$ is called *vertex* and an element in $E(G)$ is called *edge*. If $uv \in E(G)$ for u, $v \in V(G)$, u, v are called the *end-vertices* of the edge uv, and we say u, v are *adjacent*, or u is *adjacent* to v. Beside, a graph is called to be *finite* if the number of vertices $|V(G)|$ and the number of edges $|E(G)|$ are finite, and a graph is called to be *empty* if $|E(G)|$ is 0.

Definition 2.2. [31] The complement G^c of a simple graph G is the simple graph with the vertex set $V(G^c) = V(G)$, and the edge $e \in E(G^c) \Leftrightarrow e \notin E(G)$.

Definition 2.3. [31] A graph $H = (V(H), E(H))$ is called a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a nonempty subgraph of $V(G)$. The *induced subgraph* of G by S, denoted by $G[S]$, is a subgraph of G whose vertex set is S and whose edge set is the set of those edges of G that have both end-vertices in S.

Definition 2.4. [31] The *degree* of a vertex v in a simple graph G, denoted by $d_G(v)$, is the number of edges incident with v. The parameter $\Delta(G) = \max\{ d_G(v) \mid v \in V(G)\}$ is the *maximum degree* of G.

Definition 2.5. [14] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. A function $f : V_1 \to V_2$ is called a *graph isomorphism* if

1. f is ono-to-one and onto;
2. for any $u_1, u_2 \in V_1$, $u_1u_2 \in E_1$ if and only if $f(u_1)f(u_2) \in E_2$.

When such a function exists, G_1 and G_2 are called *isomorphic*, write $G_1 \cong G_2$.

Definition 2.6. [31] A *complete graph* with n vertices, denoted by K_n, is one in which every pair of distinct vertices are adjacent by exactly one edge.

Definition 2.7. [31] A *bipartite graph* $G = (U \cup V, E)$ is one whose vertex set can be partitioned into two subsets U and V, and each edge has one end-vertex in U and the other in V. A *complete bipartite graph* is a bipartite graph in which each vertex of U is joined by exactly one edge to each vertex of V. If $|U| = m$ and $|V| = n$, it is denoted by $K_{m,n}$.

Definition 2.8. [8] A *rooted tree T* is a finite set of one or more vertices such that:

1. There is a specially designated vertex r called the *root*;
2. The remaining vertices are partitioned into $n \geq 0$ disjoint sets $T_1, \ldots, T_n$, where each of these graphs is a tree, and r is adjacent to every root of $T_1, \ldots, T_n$. We call $T_1, \ldots, T_n$ the *subtree* of the root.

In a rooted tree, a vertex with degree one, unless it is the root, is a *leaf*. A vertex that has subtrees is the *parent* of the roots of the subtrees, and the roots of the subtrees are the *children* of the vertex.

Definition 2.9. [8] We recursively define a *binary tree* as follows:

1. A trivial graph, which is isomorphic to K_1, is a binary tree;
2. A rooted tree consists of a root and two disjoint binary trees called the *left subtree* and the *right subtree* is a binary tree.

The two children of a vertex in binary tree is the *left child* and *right child* of the vertex.

Then, the following definitions are related to the graphs discussing in this paper.

Definition 2.10. [31] The smallest number of colors needed to assign every vertex of G such that no two adjacent vertices share the same color is called the *chromatic number*, $\chi(G)$.

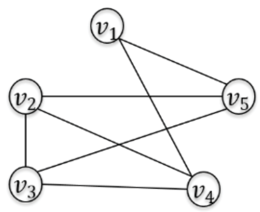
Definition 2.11. [1] The *coloring number* col(G) of a graph G is the smallest integer d for which there exists an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G such that each vertex v_i has at most $d - 1$ neighbors among the vertices $v_1, \ldots, v_{i-1}$.

Definition 2.12. [9] Let $G_i = (V_i, E_i)$ with $i = 1, 2$ be two graphs and $V_1 \cap V_2 = \phi$. $G_1 \cup G_2$ with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$ is called the *union* of G_1 and G_2. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, obtains from the union of G_1 and G_2 adding the edge uv for any $u \in V_1$ and $v \in V_2$

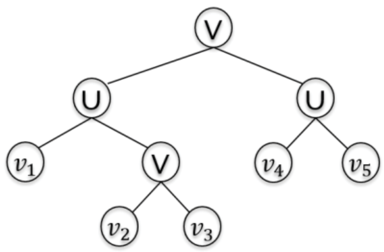
Definition 2.13. [8] The *postorder traversal* of binary tree visits two children of a vertex before it visits the vertex. This means that the children of the vertex will be output before the vertex. And, the left child of the vertex will be output before the right child.

Definition 2.14. [9] A graph is called a *cograph* if it satisfies the following rules:

1. K_1 is a cograph;
2. If G_1 and G_2 are cographs, their union $G_1 \cup G_2$ is a cograph;
3. If G_1 and G_2 are cographs, their join $G_1 \vee G_2$ is a cograph.



(a) G



(b) T_G

Figure 2.1 A cograph G and its cotree T_G.

Definition 2.15. [9] A binary tree T_G is called a *cotree* of the cograph G. According to the definition of cograph, T_G satisfies the following rules:

1. T_{K_1} only contains one node corresponding to the vertex of K_1;
2. Let T_{G_i} be the cotree of G_i for i = 1, 2. If $G = G_1 \cup G_2$, T_G obtains from T_{G_1} and T_{G_2} rooted by a union node;
3. If $G = G_1 \vee G_2$, T_G obtains from T_{G_1} and T_{G_2} rooted by a join node.

There is an example to show the cotree T_G corresponding to G in Figure 2.1.

B. Related Works

In this Part, we introduce some previous results related to list coloring. First, there are two useful properties and they are used frequently in this paper.

Theorem 2.16. [19] *The inequalities*

$$\chi(G) \leq \chi_l(G) \leq col(G) \leq \Delta(G) + 1$$

are valid for every graph G.

Property 2.17. *Let H be a subgraph of graph G, then* $\chi_l(H) \leq \chi_l(G)$.

Then, the list chromatic numbers of some basic graphs are given in the following.

Proposition 2.18. [10] *For* $n \geq 2$, *the list chromatic number of* P_n *is* 2.

The following theorem holds for any graphs, so someone may guess a similar result on list coloring.

Theorem 2.19. [16] *If* G_1 *and* G_2 *are graphs,*

$$\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2).$$

However, there is no similar result for the list chromatic number in general ([2, 5, 29]). There exists an example to show that actually, $\chi_l(G_1 \vee G_2) \neq \chi_l(G_1) + \chi_l(G_2)$ for some graphs G_1, G_2.

Example 2.20. [19] The complete bipartite graph $K_{2,4}$ with the lists {1, 2} and {3, 4} in the first vertex subset and {1, 3}, {1, 4}, {2, 3}, {2, 4} in the second vertex subset (in Figure 2.2) is not L-list colorable. Hence, it is not 2-choosable. On the other hand, it is easy to show that col($K_{2,4}$) = 3. Thus, by Theorem 2.16, $\chi_l(K_{2,4}) = 3$ holds. □

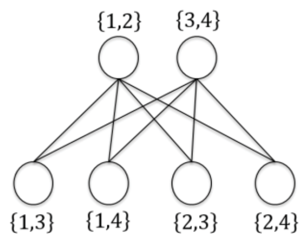


Figure 2.2: The complete bipartite graph $K_{2,4}$ with a 2-list assignment.

Because the list chromatic number of an empty graph is 1, we have $\chi_l(K_m^c) + \chi_l(K_n^c) = 2$. In the other hand, $K_{m,n} \cong K_m^c \vee K_n^c$. So, it is easy to see that $\chi_l(K_{2,4}) = 3 \neq \chi_l(K_m^c) + \chi_l(K_n^c) = 2$. Actually, we have the following theorem.

Theorem 2.21. [7] $\chi_l(K_{m,n}) = m + 1$ *if* $n \geq m^m$.

3 Main Results

According to the definition of cograph, the list chromatic number of a cograph G is calculated by the list chromatic number of K_1, $G_1 \cup G_2$ and $G_1 \vee G_2$ for any two cographs G_1, G_2 which are subgraphs of G. In this section, we present the list chromatic numbers of K_1 and $G_1 \cup G_2$, and a lower and upper bounds of $\chi_l(G_1 \vee G_2)$.

Theorem 3.1. *For graph* K_1 *with one vertex and no edges,* $\chi_l(K_1) = 1$.

Proof. It is easy to see that K_1 is colored properly if given any 1-list to the vertex. Also, the list chromatic number is a positive integer. Therefore, the list chromatic number of K_1 is 1. □

Theorem 3.2. *If G_1 and G_2 are graphs, $\chi_l(G_1 \cup G_2) = \max\{\chi_l(G_1), \chi_l(G_2)\}$.*
Proof. We have these two cases according to the relation between $\chi_l(G_1)$ and $\chi_l(G_2)$.

Case 1. $\chi_l(G_1) = \chi_l(G_2)$
Suppose $\chi_l(G_1) = \chi_l(G_2) = k$, we have the list chromatic number of $G_1 \cup G_2$ is k easily by the definition of union operation.

Case 2. $\chi_l(G_1) \neq \chi_l(G_2)$
Without loss of generality, assume $k_1 = \chi_l(G_1) < \chi_l(G_2) = k_2$. It implies that G_1 is k_2 -choosable. In other words, given any k_2 -list to the vertex of $G_1 \cup G_2$, the graph is k_2-choosable. And, G_2 cannot be colored properly if $\chi_l(G_1 \cup G_2) < k_2$ for some k_2 -list assignment. Therefore, $\chi_l(G_1 \cup G_2) = k_2 = \max\{\chi_l(G_1), \chi_l(G_2)\}$. □

Corollary 3.3 *If $G_1, G_2, \cdots, G_r$ are graphs for some positive integer r, $\chi_l(G_1 \cup G_2 \cup \cdots \cup G_r) = max\{\chi_l(G_i) \mid i = 1, 2, \ldots, r\}$.*

In the following, we focus on the list chromatic number of join of graphs. According to Example 2.20, we already know $\chi_l(G_1 \vee G_2)$ does not equal to $\chi_l(G_1) + \chi_l(G_2)$ for some graphs G_1 and G_2. Therefore, we want to know whether

$$\chi_l(G_1 \vee G_2) > \chi_l(G_1) + \chi_l(G_2) \qquad (3.1)$$

holds for any two graphs G_1 and G_2. Unfortunately, we find there exists an example to show that $\chi_l(G_1 \vee G_2) < \chi_l(G_1) + \chi_l(G_2)$ as Example 3.5. Therefore, we present a lower and upper bounds of $\chi_l(G_1 \vee G_2)$ for any two graphs G_1 and G_2.

Lemma 3.4. [30] *Let $G \cong K_{2,n}$ with $n \leq 4$ and with the bipartition (U, V), where $U = \{u_1, u_2\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. Let L be a 3-list assignment of G, then there exist $c(u_i) \in L(u_i)$ for $i = 1$ and 2 such that $|L(v_j) - \{c(u_1), c(u_2)\}| \geq 2$ for all $j \in \{1, 2, \ldots, n\}$.*

Example 3.5. *The list chromatic number of $K_1 \vee K_{2,4}$ is 3.*
Proof. Let the vertex of K_1 be denoted by u_1, $U = \{v_1, v_2\}$ and $V = \{v_3, v_4, v_5, v_6\}$ for (U, V) is the bipartition of $K_{2,4}$. Besides, $K_{2,4}$ is the subgraph of $K_1 \vee K_{2,4}$ and $\chi_l(K_{2,4}) = 3$, which is described in Example 2.20. So, the list chromatic number of $K_1 \vee K_{2,4}$ is equal to or greater than 3.
Given any 3-list to the vertex of $K_1 \vee K_{2,4}$, then we assign colors $c(v_1) \in L(v_1)$ and $c(v_2) \in L(v_2)$ to v_1 and v_2, respectively, such that $|L(v_j) - \{c(v_1), c(v_2)\}| \geq 2$ for $j = 1, 2, 3, 4$ by using Lemma 3.4. Next, color the vertex u_1 of $V(K_1)$ by $c(u_1) \in L(u_1)$ because $|L(u_1) - \{c(v_1), c(v_2)\}| \geq 1$. Hence, there exists at least one available color to color v_j from $L(v_j) - \{c(u_1), c(v_1), c(v_2)\}$ for $j = 1, 2, 3, 4$. It implies that $K_1 \vee K_{2,4}$ is 3-choosable. Therefore, $\chi_l(K_1 \vee K_{2,4}) = 3$. □

The list chromatic number of $K_1 \vee K_{2,4}$ is 3 and $\chi_l(K_1) + \chi_l(K_{2,4}) = 1 + 3 = 4$ by the above example. So, $\chi_l(K_1 \vee K_{2,4}) < \chi_l(K_1) + \chi_l(K_{2,4})$. That shows the equation (3.1) does not always hold. Hence, we have the following results.

Corollary 3.6. *There exists some graphs G_1, G_2, G_3 and G_4 such that $\chi_l(G_1 \vee G_2) > \chi_l(G_1) + \chi_l(G_2)$, and $\chi_l(G_3 \vee G_4) < \chi_l(G_3) + \chi_l(G_4)$.*

Next, we show a lower and upper bounds of $\chi_l(G_1 \vee G_2)$ for any two graphs G_1 and G_2 in the following theorems respectively.

Theorem 3.7. *For any two graphs G_1 and G_2, $\chi_l(G_1 \vee G_2) \geq \chi(G_1) + \chi(G_2)$.*
Proof. For any two graphs G_1 and G_2, we have

$$\chi_l(G_1 \vee G_2) \geq \chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2).$$

by Theorem 2.16 and 2.19. □

Theorem 3.8. *For any two graphs G_1, G_2, the graph $G_1 \vee G_2$ is k-choosable for $k = min\{\chi_l(G_1) + |V(G_2)|, \chi_l(G_2) + |V(G_1)|\}$.*
Proof. Let $V(G_1) = \{u_1, u_2, \ldots, u_m\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_n\}$ with m, n are positive integers. Suppose $k = \chi_l(G_1) + |V(G_2)|$, without loss of generality, and given any k-list to the vertex of $G_1 \vee G_2$. Because $\chi_l(G_2) \leq \Delta(G_2) + 1 \leq |V(G_2)| \leq k$ (using Theorem 2.16) and $|L(v_j)| = k$, there exists available color from $L(v_j)$ to assign v_j for $j = 1, 2, \ldots, n$. Then, $|L(u_i) - \{c(v_1), c(v_2), \ldots, c(v_n)\}| \geq k - |V(G_2)| = \chi_l(G_1)$. So, u_i is colored properly for $i = 1, 2, \ldots, m$. Hence, $G_1 \vee G_2$ is k-choosable. □

Because of for any complete graph K_n, $\chi_l(K_n) = n$, the number of vertex (Consider when $L(v)$ are all in the same for every vertex v in $V(K_n)$). So, we know that the bounds of Theorem 3.7 and 3.8 are attainable.

Corollary 3.9. *The bounds of Theorem 3.7 and 3.8 are tight.*

4 Algorithm

The following algorithm calculate the upper and lower bounds of the list chromatic number of any cograph by using the theorems proposed in Section 3.

Algorithm 4.1. List chromatic number on cograph algorithm
Input: Binary cotree T of the cograph G rooted by r.
Output: An upper and lower bounds of $\chi_l(T)$, ub and lb, respectively.

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1.  for all v ∈ V(T) in postorder traversal do
2.      ub(v) := 0, lb(v) := 0;
3.      if v is a leaf then
4.          ub(v) := 1, lb(v) := 1;
5.      else if v is a union node then
6.          ub(v) := max{ub(left child), ub(right child)};
7.          lb(v) := max{lb(left child), lb(right child)};
8.      else if v is a join node then
9.          ub(v) := min{ub(left child) + the number of leaf in right subtree, ub(right child) + the number of leaf in left subtree};
10.         lb(v) := lb(left child) + lb(right child);
11. endfor
12. return ub(r), lb(r);
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There is an example to calculate the list chromatic number of G in Figure 2.1.

Example 4.1. In order to calculate $\chi_l(G)$ in Figure 2.1, we need to take the cotree T_G to determine which equations we have to use.

Step 1. Node v_2 and v_3 are rooted by a join node, called x. By line 4, $\text{ub}(v_2) = \text{ub}(v_3) = \text{lb}(v_2) = \text{lb}(v_3) = 1$. Using line 9 and 10, we have $\text{ub}(x) = \min\{1 + 1, 1 + 1\} = 2$ and $\text{lb}(x) = \text{lb}(v_2) + \text{lb}(v_3) = 2$.

Step 2. x and v_1 are rooted by a union node, called y. By line 6 and 7, we have $\text{ub}(y) = \max\{1, 2\} = 2$ and $\text{lb}(y) = \max\{\text{lb}(x), \text{lb}(v_1)\} = \max\{1, 2\} = 2$.

Step 3. Node v_4 and v_5 are rooted by a union node, called z. Similarly to Step 2, $\text{ub}(z) = \max\{1, 1\} = 1$ and $\text{lb}(z) = \max\{1, 1\} = 1$.

Step 4. T obtains from T_y and T_z rooted by a join node r. Therefore, the list chromatic number of G is calcula0ted by line 9 and 10, we have $\text{ub}(r) = \min\{\text{ub}(y) + 2, \text{ub}(z) + 3\} = \min\{4, 4\} = 4$ and $\text{lb}(r) = \text{lb}(y) + \text{lb}(z) = 2 + 1 = 3$. Therefore, we have $3 \le \chi_l(G) \le 4$.

5 Conclusion

In this paper, we proved that $\chi_l(K_1) = 1$, $\chi_l(G_1 \cup G_2) = \max\{\chi_l(G_1), \chi_l(G_2)\}$ and $\chi(G_1) + \chi(G_2) \le \chi_l(G_1 \vee G_2) \le \min\{\chi_l(G_1) + |V(G_2)|, \chi_l(G_2) + |V(G_1)|\}$ for any two graphs G_1, G_2. We also give an algorithm and show an example to calculate the upper and lower bounds of the list chromatic of any cograph.

To find what kind of cograph will reach the upper bound or lower bound will be the interesting future work.

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