The k-Canadian Traveller Problem on Equal-Weight Graphs *

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Abstract

In the k-Canadian traveller problem (k-CTP), a traveller is given a positively weighted graph G and two vertices s,t. At most k edges in G are blocked. The traveller only learns the blocking of an edge when he reaches one of its endpoints. The goal of the traveller is to minimize the total length of the route from s to t in G. It is shown by Westphal that no deterministic online algorithm can obtain a competitive ratio less than 2k + 1 and there is a deterministic algorithm achieving the competitive ratio 2k + 1. Furthermore, Westphal showed that no randomized online algorithm can achieve a competitive ratio smaller than k + 1.

In this paper, we study the lower bounds of competitiveness of the k-Canadian traveller problem on equal-weight graphs (k-EWCTP). Previous lower bounds for k-CTP cannot be applied to this special class of weighted graphs. For deterministic and randomized online algorithms, we show that the lower bounds on competitive ratios are at least 2k + 1 and at least k + 1, respectively.

1 Introduction

Given an undirected graph G = (V, E) of nonnegative weights, and two vertices s, t, the shortest path problem is to determine a shortest path from s to t. Let us consider the following online version of this problem. We assume that some edges are blocked. An online algorithm learns the blocking of ah edge only when it reaches one of its endpoints. If the traveller reaches a blocked edge in his currently planned route, then he cannot pass this blocked edge. Hence the traveller must choose another route to reach the destination t. This online problem on graphs of at most k blocked edges is called the k-Canadian Traveller Problem (k-**CTP**) which is introduced by Papadimitriou and Yannakakis [3].

The performance of a deterministic algorithm for k-CTP is based on the competitive ratio defined as follows. Let σ denote the input (G, s, t)where G is a graph and s and t are two vertices of G. For any deterministic algorithm ALG for k-CTP and any input σ , let $|ALG(\sigma)|$ denote the total length of the s - t-path produced by ALG on input σ . Let OPT denote the optimal offline algorithm k-CTP. Note that, for any input $\sigma = (G, s, t), |\mathsf{OPT}(\sigma)|$ is exactly the shortest length of s - t-path in G with the blocked edges removed. A deterministic algorithm ALG is ccompetitive if, for input $\sigma = (G, s, t), |\mathsf{ALG}(\sigma)| \leq$ $c \cdot |\mathsf{OPT}(\sigma)|$. The smallest value c such that ALG is *c*-competitive is called the competitive ratio of ALG. The competitive ratio of a randomized algorithm ALG for k-CTP is the smallest value csuch that the expected total length $\mathbb{E}(|\mathsf{ALG}(\sigma)|) \leq$ $c \cdot |\mathsf{OPT}(\sigma)|$ for any input σ .

Westphal [4] gave the first comprehensive study of lower bounds of the competitive ratio for k-Canadian Traveller Problem. In [4], it is shown that no deterministic online algorithm can achieve a competitive ratio smaller than 2k + 1 for k-**CTP**. Furthermore, Westphal showed that this bound is optimal by providing an easy algorithm called **BackTrack** which obtains the competitive ratio 2k+1. For lower bounds of the competitive ratio of randomized online algorithms, Westphal [4] showed that no randomized online algorithm can achieve a competitive ratio less than k + 1 for k-**CTP** by using Yao's principle [5].

Upper bounds of the competitive ratio of randomized online algorithms are also studied recently. For vertex-disjoint graphs, Bender and Westphal [1] proposed a randomized version of the algorithm BackTrack which obtains competitive ratio k + 1. For arbitrary graphs, Demaine et al. [2] proposed a polynomial-time randomized

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algorithm which obtains the competitive ratio less than 2k + 1 by an o(1) factor. Furthermore, they showed a super-polynomial randomized algorithm can achieve the competitive ratio $(1 + \sqrt{2}/2)k + 1$.

In this paper, we consider a special class of graphs called equal-weight graphs. Let k-**EWCTP** denote the k-Canadian Traveller Problem on equal-weight graphs. We study the lower bounds of competitive ratios of deterministic and randomized algorithms for k-**EWCTP**. Note that Westphal's lower bound provided in [4] does not imply the lower bounds for k-**EWCTP**. We show that the competitive ratio of any deterministic online algorithm for k-**EWCTP** is at least 2k + 1. This bound is tight since the algorithm BackTrack can obtain this bound. Furthermore, we also show that the competitive ratio of any randomized online algorithm for k-**EWCTP** is at least k + 1.

2 Optimal competitive ratio of deterministic online algorithms for *k*-EWCTP

First, we consider the upper bound. In [4], Westphal proposed the algorithm BackTrack described as follows: on input G = (V, E), s, t, BackTrack finds shortest s - t-path P_1 . If there is a blocked edge $e_1 \in P_1$, the algorithm goes back to s and sets $E_1 = E \setminus \{e_1\}$. Next, BackTrack finds the shortest s - t-path P_2 in $G_1 = (V, E_1)$. If there is a blocked edge $e_2 \in P_2$, BackTrack goes back to s and sets $E_2 = E_1 \setminus \{e_2\}$. Then BackTrack finds the shortest s - t-path P_3 in $G_2 = (V, E_2)$ and so on.

Lemma 1. ([4]) For any input $\sigma = (G, s, t)$, the competitive ratio of BackTrack is at most 2k + 1.

Theorem 1. For k-EWCTP, no deterministic online algorithm can have its competitive ratio less than 2k + 1.

Proof. The proof roadmap is as follows. Given any deterministic online algorithm ALG, we design an equal-weight graph G and its two vertices s, t as its input $\sigma = (G, s, t)$. Then we show that ALG must have its competitive ratio at least 2k + 1 for this particular input σ . The input $\sigma = (G, s, t)$ is illustrated in the Fig. 1. Note that there are k + 1 vertex-disjoint paths $s \to v_i \to t$ for $1 \le i \le k + 1$. We can view each deterministic algorithm ALG as a permutation on these k + 1 paths where each permutation corresponds to the order of paths $\{s \to v_i \to t : 1 \le i \le k + 1\}$ which ALG takes



Figure 1: The input $\sigma = (G, s, t)$ for the proof of the lower bound.

while searching a route from s to t. Without loss of generality, we assume that edges (v_i, t) are blocked for $1 \le i \le k$ and the last path which ALG chooses is $s \to v_{k+1} \to t$. The cost of ALG is 2km + m + 1 while the cost of the optimal offline algorithm is m + 1. Hence, the competitive ratio of ALG is at least

$$\frac{2km+m+1}{m+1} = \frac{2k+1+1/m}{1+1/m} \to 2k+1.$$

as m is sufficiently large. Therefore, the competitive ratio of ALG will be close to 2k + 1.

By Lemma 1, the algorithm BackTrack obtains the competitive ratio 2k + 1. Hence there is no deterministic online algorithm with competitive ratio less than 2k + 1.

2.0.1 Lower bounds of the competitiveness of randomized online algorithm for *k*-EWCTP

In this subsection, we consider the competitive ratio of randomized online algorithms for k-**EWCTP**. We will use the known Yao's minimax principle. For completeness, we include the definitions, the statement, and its proof here.

Definition 1. Let \mathcal{A} denote the set of deterministic algorithms and let \mathcal{X} denote the set of inputs. Given $a \in \mathcal{A}$ and $x \in \mathcal{X}$, let c(a, x) be the cost of the algorithm a on input x.

Let p and q be distributions on \mathcal{A} and \mathcal{X} , respectively. Let \mathbf{A} be the random algorithm chosen according to p and let \mathbf{X} be the random input chosen according to q.

Lemma 2. (Yao's Minimax Principle [5]) $\min_{a \in \mathcal{A}} \mathbb{E}[c(a, \mathbf{X})] \leq \max_{x \in \mathcal{X}} \mathbb{E}[c(\mathbf{A}, x)].$

Proof. Let $C = \max_{x \in \mathcal{X}} \mathbb{E}[c(\mathbf{A}, x)]$. For any $x \in \mathcal{X}$,

$$\sum_{a \in \mathcal{A}} p_a \cdot c(a, x) = \mathbb{E}[c(\mathbf{A}, x)] \le C.$$

This implies that

$$\min_{a \in \mathcal{A}} \mathbb{E}[c(a, \mathbf{X})] = \min_{a \in \mathcal{A}} \sum_{x \in \mathcal{X}} q_x \cdot c(a, x)$$

$$\leq \sum_{a \in \mathcal{A}} p_a \sum_{x \in \mathcal{X}} q_x \cdot c(a, x)$$

$$= \sum_{x \in \mathcal{X}} q_x \sum_{a \in \mathcal{A}} p_a \cdot c(a, x)$$

$$\leq \sum_{x \in \mathcal{X}} q_x \cdot C$$

$$= C.$$

Next, we show the lower bound of competitive ratio of randomized online algorithm for k-**EWCTP** by using Lemma 2.'

Theorem 2. There is no randomized online algorithm against an oblivious adversary with competitive ratio less than k + 1.

Proof. The proof roadmap is as follows. Here we define the cost $c(A, \sigma)$ of an algorithm A on an input $\sigma = (G, s, t)$ as the total length which A walks from s to t in the equal-weight graph G with at most k blocked edges. To prove the lower bound of the competitiveness of a randomized algorithm, we construct a deterministic algorithm ALG and an input distribution \mathbf{X} and show that the cost of ALG on the input distribution \mathbf{X} is at least k + 1. Then, by Yao's minimax principle, the theorem follows.

We consider the following k + 1 inputs $\sigma_i = (G_i, s, t)$ where each G_i is the same as the graph shown in Fig. 1 in which the edges (v_j, t) are blocked for all $j \neq i$. The input distribution **X** is defined by $\mathbf{X} = \sigma_i$ with probability 1/(k + 1). Note that ALG corresponds to a permutation on k+1 paths it searches for a way from s to t. Without loss of generality, we assume the corresponding searching order is $(1, 2, \ldots, k + 1)$. For this ALG, its expected competitive ratio with respect to **X** is

$$\frac{E[c(ALG, \mathbf{X})]}{m+1} = \frac{1}{(k+1)(m+1)} \sum_{i=1}^{k+1} (2(i-1)m + (m+1)) \\
= \frac{1}{(k+1)(m+1)} \sum_{i=1}^{k+1} (2i-1)m + 1) \\
= \frac{(k+1)(k+2)m + (1-m)(k+1)}{(k+1)(m+1)} \\
= \frac{(k+1)(k+1)(m+1) - (k+1)k}{(k+1)(m+1)} \\
= k+1 - \frac{k}{m+1}.$$

The last term is close to k+1 as m is large enough. By Lemma 2, the competitive ratio of any randomized algorithm for k-EWCTP is at least k + 1.

References

- M. Bender and S. Westphal, "An optimal randomized online algorithm for the k-Canadian traveller problem on node-disjoint paths," *Journal of Combinatorial Optimization 2013*, published online.
- [2] E. D. Demaine, Y. Huang, C.-S. Liao, and K. Sadakane, "Canadians Should Travel Randomly," in *Proceedings of the 41st International Colloquium on Automata, Languages* and Programming (ICALP 2014), pages 380– 391, 2014.
- [3] C.H. Papadimitriou and M. Yannakakis, "Shortest paths without a map," *Theoret. Comput. Sci*, 84(1), pages 127–150, 1991.
- [4] S. Westphal, "A note on the k-Canadian traveller problem," *Information Processing Let*ters, 106, pages 87–89, 2008.
- [5] A. C.-C. Yao, "Probabilistic computations: Toward a unified measure of complexity," in Proceedings of the 18th Annual Symposium on Foundations of Computer Science, (FOCS 1977), pages 222–227, 1977.