# Maximal singly connected digraphs*

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**Abstract**

*A digraph $G$ is singly connected if each pair of vertices is connected by at most one simple path, and maximal singly connected if no other singly connected digraph with the same set of vertices contains the edges of $G$. In this paper, we explore the properties of a maximal singly connected digraph and extend the concept to subgraphs of a digraph. In particular, we give a class of such kind of subgraphs in unidirectional hypercubes.*

**Keywords:** singly connected digraph; strongly connected digraph; unidirectional hypercube;

## 1 Introduction

Let $G$ be a directed graph (digraph for short) with $n$ vertices and $m$ edges. If $G$ contains no cycles, then it is called a *directed acyclic graph* (DAG for short). A directed path in $G$ is a *simple path* if it contains no repeating vertices. A digraph $G$ is *singly connected* if for each pair of vertices $u$ and $v$, there is at most one simple $u,v$-path. A Bayesian network is a probabilistic graph model that represents a set of random variables and their conditional dependencies via a DAG. As mentioned in the textbook of Neapolitan and Naimipour [6, p. 416], although the problem of probabilistic inference in a Bayesian network is NP-hard, polynomial-time algorithms have been found for the subclass of instances in which the DAG is singly connected (see also Neapolitan [5, p. 142] for a developed algorithm).

Originally, the problem of testing whether or not a digraph $G$ is singly connected was emerged in the textbook by Corment et al. [3, p. 485]. Based on the technique of DFS trees and a reduction of contracting each strongly connected component into a single vertex to produce a reduced graph $G^r$, several algorithms have been proposed for solving this problem in $O(n^2)$ time (e.g., see works of Buchsbaum and Carliske [1], Khuller [9] and Karlin [8], where the latter was mentioned in the addendum [10]). In addition, Khuller [9] posed a challenging question of designing an algorithm to solve this problem in linear time, and however, so far it remains open.

Recently, using a similar approach mentioned above, Dietzdelbinger and Jaberi [4] presented a refined version of the algorithm with running time $O(s \cdot t + m)$ for testing whether $G$ is singly connected, where $s$ and $t$ are the numbers of sources (i.e., vertices with indegree 0) and sinks (i.e., vertices with outdegree 0), respectively, in the reduced graph $G^r$. Moreover, they studied two optimization problems related to singly connectivity of graphs as follows. Given a digraph $G = (V, E)$, the ESC problem asks to find an edge set $F \subseteq E$ of minimum size such that the digraph $(V, E \setminus F)$ is singly connected. By contrast, the VSC problem asks to find a vertex set $U \subseteq V$ of minimum cardinality such that the digraph $G - U$ (i.e., the digraph obtained from $G$ by removing all vertices of $U$ and their incident arcs) is singly connected. A result given in [4] also shows that both ESC and VSC are NP-hard.

In this paper, inspired by the ESC problem, we explore the properties of a *maximal singly connected digraph*, formally defined in Section 2. In particular, we give a class of such kind of subgraphs in a class of interconnection networks called *unidirectional hypercubes*, which is formally defined in Section 3.

## 2 Properties of a maximal singly connected digraph

The digraph under consideration is assumed to be simple, i.e., with neither loop nor multiple edges. Notice that the underlying graph may con-

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tain multiple edges. The set of vertices and edges of a digraph $G$ are denoted by $V(G)$ and $E(G)$, respectively. When there is no danger of misinterpretation, the set of vertices is denoted by $G$ instead of $V(G)$. A simple path from vertex $u$ to $v$ in the digraph under consideration is denoted by $u \rightsquigarrow v$. If the simple $u, v$-path consists of only vertices in a set $U$, we use $u \underset{U}{\rightsquigarrow} v$. An edge incident from $u$ to $v$ is denoted by the pair $(u, v)$. We use $G - e$ to denote the spanning subgraph of $G$ with edge set $E(G) \setminus \{e\}$.

**Definition 1** *A digraph $G = (V, E)$ is maximal singly connected if $G$ is singly connected and there is no other singly connected digraph $G' = (V', E')$ such that $V' = V$ and $E \subsetneq E'$ (i.e., $E$ is a proper subset of $E'$).*

**Proposition 1** *If digraph $G$ is singly and strongly connected, then $G$ is maximal singly connected.*

**Proof.** Let $G = (V, E)$. Suppose to the contrary that $G$ is not maximal singly connected. There is a singly connected digraph $G'$ with vertex set $V$ and edge set $E'$ such that $E \subsetneq E'$. Let $e \in E' \setminus E$, and let $e = (u, v)$. We claim that there are two simple paths from $u$ to $v$. Clearly, $G' - e$ is strongly connected. There is a $u, v$-path in $G' - e$. Along with the edge $e$, we have two simple paths from $u$ to $v$ in $G'$. This contradicts that $G'$ is singly connected. □

**Proposition 2** *Let $G$ be a digraph with vertex set $V$ and edge set $E$. Given that there is a bipartition $\{V_1, V_2\}$ of $V$ such that $G[V_1]$ and $G[V_2]$ are both strongly and singly connected, we have that $G$ is maximal singly connected if and only if $\{(u, v) \in E\colon u \in V_i, v \in V_{3-i}, i \in \{1, 2\}\} = \{(x, y), (y, x)\}$ for some $x \in V_1$ and $y \in V_2$.*

**Proof.** Let $X = \{(u, v) \in E\colon u \in V_i, v \in V_{3-i}, i \in \{1, 2\}\}$. For necessity, we prove the contrapositive. If $X = \emptyset$, then clearly the graph $(V, E \cup \{(x, y)\})$, where $x \in V_1$ and $y \in V_2$, is singly connected. Similarly, for $X = \{(x, y)\}$ the graph $(V, E \cup \{(y, x)\})$ is singly connected. For $|X| \geq 2$ and $X \neq \{(x, y), (y, x)\}$ for $\{x, y\} \subseteq V$, then there exist two elements of $X$, say $(x_1, y_1)$ and $(x_2, y_2)$, which share at most one vertex. If $\{x_1, x_2\} \subseteq V_i$ for some $i \in \{1, 2\}$, then there are two simple $x_1, y_1$-paths, namely the edge $(x_1, y_1)$ itself and $x_1 \underset{V_i}{\rightsquigarrow} x_2 \rightsquigarrow y_2 \underset{V_{3-i}}{\rightsquigarrow} y_1$. Otherwise, $\{x_1, y_2\} \in V_i$ for some $i \in \{1, 2\}$ so there are two simple $x_1, y_2$-paths, namely $x_1 \rightsquigarrow y_1 \rightsquigarrow x_2 \rightsquigarrow y_2$ and $x_1 \underset{V_i}{\rightsquigarrow} y_2$.

For sufficiency, it is not difficult to verity that $G$ is singly connected. Suppose to the contrary that $G$ is not maximal singly connected. Then there is a digraph $G' = (V, E')$ with $E \subseteq E'$ and $E \neq E'$. Let $(x', y') \in E' \setminus E$. Clearly, $\{x', y'\} \not\subseteq V_i$ for $i \in \{1, 2\}$ since otherwise there are two simple $x', y'$-paths in $G[V_i]$. Therefore, $|\{x, y\} \cap \{x', y'\}| \leq 1$. However, similar to the analysis above, there are more than one simple paths between two vertices, and we have a contradiction. □

**Definition 2** *Given a digraph $G$, a subgraph $H$ of $G$ is a maximal singly connected subgraph if $H$ is singly connected, and there is no singly connected subgraph $H'$ of $G$ such that $V(H) \subsetneq V(H')$ or $E(H) \subsetneq E(H')$.*

Clearly, if $H$ is a maximal singly connected subgraph of $G$, then $V(H) = V(G)$. Therefore, to find a maximal singly connected subgraph of a graph, it suffices to consider its spanning subgraphs.

**Corollary 3** *Let $G$ be a digraph whose underlying graph is connected with no multiple edges, and let $H$ be a spanning subgraph of $G$. Given that there is a bipartition $\{V_1, V_2\}$ of $V(H)$ such that $H[V_1]$ and $H[V_2]$ are both strongly and singly connected, we have that $H$ is a maximal singly connected subgraph of $G$ if and only if $|\{(u, v) \in E(H)\colon u \in V_i, v \in V_{3-i}, i \in \{1, 2\}\}| = 1$.*

## 3 Maximal singly connected subgraphs of a unidirectional hypercube

In this section, we give a class of maximal singly connected subgraphs of a specific digraph, called *unidirectional hypercube*, defined as follows. The *$n$-dimensional hypercube* (*$n$-cube* for short), denoted by $Q_n$, is a graph with $2^n$ vertices such that each vertex $v$ is represented by a distinct binary string $v = v_{n-1}v_{n-2} \cdots v_1 v_0$ and two vertices are adjacent if they differ in exactly one position [11]. In particular, if two adjacent vertices differ at $i$th position, the edge between them is called an *$i$-edge.* For $Q_n$, the *Hamming weight* $h(v)$ of a vertex $v$ is defined by $h(v) = v_{n-1} + v_{n-2} + \cdots + v_1 + v_0$. Obviously, given an $i$-edge between two vertices $u$ and $v$, there is exactly one of $h(u) + i$ and $h(v) + i$ is even. If the former (respectively, the latter) is even, then $u$ (respectively, $v$) is call the *even vertex* with respect to the $i$-edge, otherwise it is the

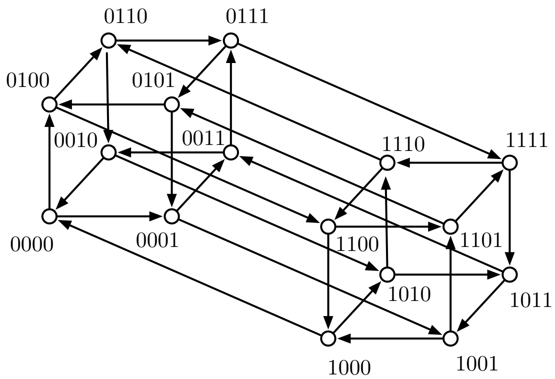


Figure 1: $\mathrm{UQ}_4$

*odd vertex.* Unidirectional hypercube, denoted by $\mathrm{UQ}_n$, was proposed by Chou and Du [2] as a directed version of $n$-cube. It serves as the basis for high speed networking. Formally, $\mathrm{UQ}_n$ is obtained from $Q_n$ by orienting every edge from the even vertex to the odd vertex. For example, Fig 1 depicts $\mathrm{UQ}_4$. Recently, Hung et al. [7] studied the fault Hamiltonicity in $\mathrm{UQ}_n$. In particular, they mentioned that so far only less properties and research results are known for unidirectional hypercubes.

An important property of $\mathrm{UQ}_n$ is that it can be decomposed recursively, as indicated by Chou and Du [2].

**Property 4 (See [2])** *$UQ_n$ can be decomposed into two $UQ_{n-1}$ and $2^{n-1}$ edges between them. The sets of vertices $\{v\colon v_{n-1} = 0\}$ and $\{v\colon v_{n-1} = 1\}$ induce the requested two $UQ_{n-1}$, respectively.*

In the following, we give a class of spanning subgraphs $S(\mathrm{UQ}_n)$ of $\mathrm{UQ}_n$, for $n \geq 0$, which are maximal singly connected. Let $\mathrm{UQ}^i_{n-1}$ be the subgraph of $\mathrm{UQ}_n$ induced by $\{v\colon v_{n-1} = i\}$, and analogously let $\mathrm{UQ}^{ij}_{n-2}$ be the subgraph of $\mathrm{UQ}_n$ induced by $\{v\colon v_{n-1} = i, v_{n-2} = j\}$. In $\mathrm{UQ}_n$, the set $X_a$ of cross edges is defined as $\{(u, v)\colon u_{n-1} \neq v_{n-1}\}$. A 4-cycle with vertex set $\{u, v, x, y\}$ with $u_{n-1}u_{n-2} = 00$, $v_{n-1}v_{n-2} = 01$, $x_{n-1}x_{n-2} = 11$, and $y_{n-1}y_{n-2} = 10$ is called a cross cycle. Let the set of cross cycles be $X_c$. For $n = 0$, we let $S(\mathrm{UQ}_n) = \{K_1\}$, For $n \geq 1$, a member $S$ in $S(\mathrm{UQ}_n)$ is defined as

$$S = \begin{cases} S_0 \cup S_1 \cup A, & \text{if } n \text{ is odd,} \\ S_{00} \cup S_{01} \cup S_{11} \cup S_{10} \cup C, & \text{otherwise,} \end{cases}$$

where $S_i \in S(\mathrm{UQ}^i_{n-1})$, $S_{ij} \in S(\mathrm{UQ}^{ij}_{n-2})$, $A \in X_a$, and $C \in X_c$. We claim that $S(\mathrm{UQ}_n)$ is a maximal singly connected subgraph of $\mathrm{UQ}_n$, as shown in the following.

**Lemma 5** *For $S \in S(UQ_n)$, if $n$ is even, then $S$ is strongly connected.*

**Proof.** Let $n = 2r$. We prove the lemma by induction on $r$. For $r = 0$, clearly the lemma holds. Suppose that the lemma holds for $r \leq k$. For $r = k+1$, by definition we have $S = S_{00} \cup S_{01} \cup S_{11} \cup S_{10} \cup C$, where $S_{ij} \in S(\mathrm{UQ}^{ij}_{2k})$ and $C \in X_c$. For any two vertices $x$ and $y$ of $S$, if they belong to $S_{ij}$ for $ij \in \{00, 01, 11, 10\}$, then by the induction hypothesis, there is an $x, y$-path in $S_{ij}$. Otherwise, let $u$ and $v$ be the vertices of $C$ that belong to the subgraph containing $x$ and that containing $y$, respectively. By the induction hypothesis, there is an $x, u$-path and a $v, y$-path. Along with $C$, we have an $x, y$-path in $S$. $\square$

**Lemma 6** *For $S \in S(UQ_n)$, if $n$ is even, then $S$ is singly connected.*

**Proof.** Let $n = 2r$. We prove the lemma by induction on $r$. For $r = 0$, clearly the lemma holds. Suppose that the lemma holds for $r \leq k$. For $r = k+1$, by definition we have $S = S_{00} \cup S_{01} \cup S_{11} \cup S_{10} \cup C$, where $S_{ij} \in S(\mathrm{UQ}^{ij}_{2k})$ and $C \in X_c$. For any two vertices $x$ and $y$ of $S$, we show that there is at most one simple $x, y$-path in $S$. If $x$ and $y$ belong to $S_{ij}$ for $ij \in \{00, 01, 11, 10\}$, then any simple $x, y$-path contains no edge of $C$. Therefore, any simple $x, y$-path lies in $S_{ij}$, and by the induction hypothesis we have that there is at most one simple $x, y$-path in $S$. Otherwise, assume that $x$ and $y$ are vertices of $S_{ij}$ and $S_{st}$, respectively. Any simple $x, y$-path is of the form

$$x \underset{S_{ij}}{\rightsquigarrow} u \underset{C}{\rightsquigarrow} v \underset{S_{st}}{\rightsquigarrow} y.$$

By the induction hypothesis and that $C$ is singly connected, there is at most one simple $x, y$-path in $S$. This proves the lemma. $\square$

**Theorem 7** *For $n \geq 0$, any member of $S(UQ_n)$ is a maximal singly connected subgraph of $UQ_n$.*

**Proof.** Let $S \in S(\mathrm{UQ}_n)$. For even $n$, by Lemmas 5 and 6, we have that $S$ is strongly and singly connected. Then by Proposition 1, $S$ itself is maximal singly connected. Since $S$ is a subgraph of $\mathrm{UQ}_n$, the theorem holds when $n$ is even. For odd

$n$, by definition $S = S_0 \cup S_1 \cup A$. Hence, by Corollary 3, $S$ is a maximal singly connected subgraph of $\mathrm{UQ}_n$, and the theorem is proved. □

## 4 Concluding remarks

For $S \in \mathcal{S}(\mathrm{UQ}_n)$, we are interested in whether

$|E(S)| = \max\{|E(H)| : H$ is a singly connected subgraph of $\mathrm{UQ}_n\}$.

The ESC problem on $\mathrm{UQ}_n$ is solved if the equation holds. We have verified that the equation holds for $n \leq 4$ by enumerating all the cases. It remains open if the equation holds for $n \geq 5$. An appropriate analysis is expected and will be conducted as a future work.

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