Weak Roman Domination of Cartesian Products

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Abstract

For $G=(V,E)$ and a cost function $f:V\to\{0,1,2\}$, define $V_i=\{v\in V \mid f(v)=i\}$, where $0\le i\le 2$, then V_0,V_1,V_2 is a vertex partition of $V(G)$. A vertex u is said to be under protection if u is not in V_0 or it is adjacent to at least one vertex in $V_1\bigcup V_2$. f is a guard function of G if every vertex of G is under protection. Let $u\in V_0$, and $v\in V_1\bigcup V_2$ is a neighbor of u. A move function $f'_{v\to u}$ based on f is defined as $f'(u)=1$ $f'(v)=f(v)-1$ and $f'(x)=f(x)$ for all $x\in V-\{u,v\}$. For every vertex $u\in V_0$, if there is a move function f' based on the guard function f is a guard function of G, then f is called a weak Roman dominating function, and the cost of f is $\sum_{v\in V} f(v)=|V_1|+2|V_2|$. The foolproof version asked that every vertex $u\in V_0$, if $v\in N(u)\bigcap(V_1\bigcup V_2)$, then the move function $f'_{v\to u}$ is a guard function of G. The minimum cost among all possible (foolproof) weak Roman dominating functions of G, is called the (foolproof) weak Roman domination number of G and is denoted as $\gamma_r(G)$ ($\gamma_r^*(G)$ for foolproof version). This paper established the (foolproof) weak Roman domination number of Cartesian product of complete graph with other graphs.

1. Introduction

Roman dominating function was first motivated by Stewart as a new variety of the domination problem [1][12]. When the Roman Empire is getting weak and was not able to have armies in all towns, they have to protect all the towns by using the least amount of armies. In order not to scarify itself when try to rescue others, they set the rule that a town can protect itself if there is one group of army in that town, and it may protect all its neighboring towns if there are two groups of armies in it. The problem of finding the least amount of armies needed to have all towns either has at least a group of army or is neighboring to a town with two groups of armies, is known as *Roman Domination Problem*. Cockayne et al. transform this problem to a variety of domination problems [4][5]. Henning et al. proposed a new strategy to protect the Roman Empire using even less armies [9]. They reset the rule that a town with only one group of army can also protect its neighbor as long as after sending the army to rescue its neighbor, every town without army in it is still neighboring with a town with at least a group of army. Based on the new strategy, the problem is called the *weak Roman domination problem*. Let us express this problem mathematically. Let G be a simple and undirected graph. Let every vertex be a town and there is an edge between two vertices if the corresponding towns are adjacent to each other. We use $N(v)=\{u \mid uv\in E(G)\}$ to denote the neighboring vertices of v and a function $f:V\to\{0,1,2\}$ to indicate the number of armies stationed in the town. Let $V_i=\{v\in V \mid f(v)=i\}$ where $0\le i\le 2$, then V_0,V_1,V_2 is a vertex partition of $V(G)$. The *weight* of f is $\sum_{v\in V} f(v)=|V_1|+2|V_2|$. A vertex is *protected* if either itself or at least one of its neighbors is in $V_1\bigcup V_2$. Function f is called a *guard function* of G if every vertex of G is under protection. If $V_2=\varnothing$, the guard function is called *domination function* of *G*, and the minimum weight among all domination function of *G* is the *domination number of G*, denoted as $\gamma(G)$. Then the *Roman domination function* of G is a special guard function such that every vertex $u\in V_0$, there is a vertex $v\in N(u)\bigcap V_2$. The minimum weight among all Roman domination function of *G* is the *Roman domination number of G*, denoted as $\gamma_R(G)$. Let $u\in V_0$, and $v\in N(u)\bigcap(V_1\bigcup V_2)$, a *move function* $f'_{v\to u}$ based on f is defined as $f'(u)=1$ $f'(v)=f(v)-1$ and $f'(x)=f(x)$ for all $x\in V-\{u,v\}$. A *weak Roman dominating function* of G is a special guard function f such that for every vertex $u\in V_0$, there is a vertex $v\in N(u)\bigcap(V_1\bigcup V_2)$ such that the move function $f'_{v\to u}$ based on f is also a guard function of G. The minimum weight of all weak Roman dominating functions is the *weak Roman domination number of* G, denoted as $\gamma_r(G)$. A weak Roman domination function *f* of *G* is *optimal* if and only if the weight of *f* is $\gamma_r(G)$. Later on, the foolproof version was proposed in 2004 by [2][3]. The original version of the weak Roman domination is then referred as the *smart version*. The foolproof version asks for a guard function *f* of *G* such that for $u \in V_0$ and $v \in N(u) \bigcap (V_1 \bigcup V_2)$, $f'_{v \to u}$ is a guard function of G. The minimum cost among all possible foolproof weak Roman dominating functions of G, is called the *foolproof weak Roman domination number of* G and is denoted as $\gamma_r^*(G)$. A foolproof weak Roman domination function *f* of *G* is *optimal* if and only if the weight of f is $\gamma_r^*(G)$.

As the relation between different version of domination numbers, we know that for any graph G, $\gamma(G) \le \gamma_r(G) \le \gamma_r^*(G) \le \gamma_R(G) \le 2\gamma(G)$ [4][7][9]. The Roman domination problem on trees is solvable in linear time, but it is NP-complete on split graphs, bipartite graphs, and planar graphs [4]. Cockayne et al. [6] found a general lower bound on the Roman domination number of a graph *G*, involving the order and maximum degree Δ(G) such that $\gamma_R(G) \ge 2n/(\Delta(G)+1)$. They also achieved the exact value of the Roman domination number $\gamma_R(G)$ for several graphs including paths, cycles, complete *k*-partite graphs and the Cartesian product of complete graphs. Weak Roman domination problem has also been proved to be NP-Complete even if the graph is restricted to bipartite or chordal graphs [9]. The polynomial solution to this problem is found for a limited classes of graphs such as complete graph, path, cycle and complete multipartite graph [6][9][11]. The upper bound on $\gamma_r(G)$ is known for some complex graph structures, such as grid and torus [6].

In this paper, we purpose a linear-time algorithm for solving the foolproof weak Roman domination problem on the Cartesian product of complete graphs.

2. Main Result

The Cartesian product of two graphs G_1 and G_2 is denoted as $H = G_1 \square G_2$. The vertex set of *H* is the Cartesian product $V(G_1) \times V(G_2)$, where $V(G_1)$ and $V(G_2)$ are the vertex sets of G_1 and G_2, respectively. Two vertices (u_1, u_2) and (v_1, v_2) of *H* are connected by an edge if and only if $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$ or $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$. Figure 1 illustrates the graph $K_2 \square K_3$.

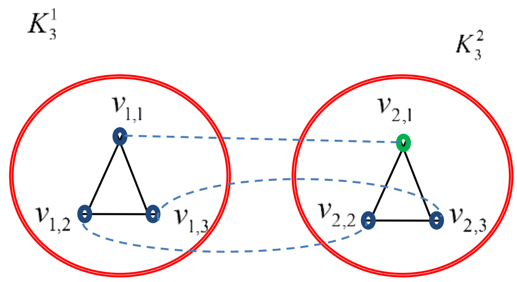

Figure 1: An illustration of $K_2 \square K_3$

As shown in Figure 1, the graph $G = K_n \square K_m$ for $m \ge n$ may be represented by connecting corresponding vertices in n copies of K_m namely $K_m^1, K_m^2, \cdots, K_m^n$ where each vertex in G may be named as $v_{i,j}$ for $1 \le i \le n$, and $1 \le j \le m$.

Lemma 1: *Let G be a graph with m vertices and* $H = G \square K_n$ *for* $n \ge 1$. *Then* $\gamma_r^*(H) \le m$.

Proof: Let $v_{i,j}$ ($1 \le i \le m$, and $1 \le j \le n$) denote the vertex in *H*. Note that *H* may be viewed as *n* copies of *G* (say $G^1, G^2, \cdots, G^n$), and the *i*-th vertex in each copy of G form a complete graph. Define the function $f: V(H) \to \{0,1,2\}$ where $V_2 = \varnothing$, $V_1 = V(G^1) = \{v_{i,1} | 1 \le i \le m\}$, and $V_0 = V(H) - V_1$. Clearly, f is a foolproof weak Roman domination function of H. That is $\gamma_r^*(H) \le |V(G^1)| = m$.

Lemma 2 comes directly from the definition of the Cartesian product.

Lemma 2: *For any two graphs* G_1 *and* G_2, $G_1 \square G_2 \equiv G_2 \square G_1$.

Lemma 3: (From [10]) *Let* $f: V(G) \to \{0,1,2\}$ *be a guard function of G. Then f is a foolproof weak Roman domination function of G if and only if for each* $v \in V_0$, *one of the following holds.*

1. $\sum_{u \in N(v)} f(u) \ge 2$;
2. $\sum_{u \in N(v)} f(u) = 1$ *and* $\{w\} = N(v) \bigcap V_1$, *then* $(N(w) - N[v]) \bigcap V_0 = \varnothing$.

A smart version of Lemma 3 is stated as Lemma 4, which can be showed in a similar way as in [10].

Lemma 4: *Let* $f: V(G) \to \{0,1,2\}$ *be a guard function of G. Then f is a weak Roman domination function of G if and only if for each* $v \in V_0$, *one of the following holds.*

1. $\sum_{u \in N(v)} f(u) \geq 2$;
2. $\sum_{u \in N(v)} f(u) = 1$ *and* $\{w\} = N(v) \cap V_1$. *If C is a clique containing* $\{v, w\}$ *, then f is a guard function for* $G - C$.

Theorem 1: *For graph* $H = G \Box K_n$ *for* $n \geq 4$, *if G is a path or a cycle with m vertices, then* $\gamma_r(H) = \gamma_r^*(H) = m$.

Proof: By Lemma 1 we know $\gamma_r^*(H) \leq m$, hence we just need to show that $\gamma_r(H) \geq m$. Suppose to the contrary that $\gamma_r(H) < m$, let f be an optimal weak Roman domination function of H, then there must exist $V(K_n^i) \subset V_0$ for some i, $1 \leq i \leq m$. Clearly, in order to protect $V(K_n^i) \subset V_0$, $\sum_{1 \leq j \leq n}(f(v_{i-1,j}) + f(v_{i+1,j})) = n$. Consider another guard function g of H such that $g(v_{i,1}) = g(v_{i-1,1}) = g(v_{i+1,1}) = 1$ and $g(v_{i,j}) = g(v_{i-1,j}) = g(v_{i+1,j}) = 0$ for $2 \leq j \leq n$ and $g(u) = f(u)$ for $u \in V(H) - \{V(K_n^{i-1}), V(K_n^i), V(K_n^{i+1})\}$, then g is a weak Roman domination function of H with the weight $n-3$ smaller than the weight of f, which produces a contradiction. Figure 3 shows the illustration of $P_3 \Box K_4$. ■

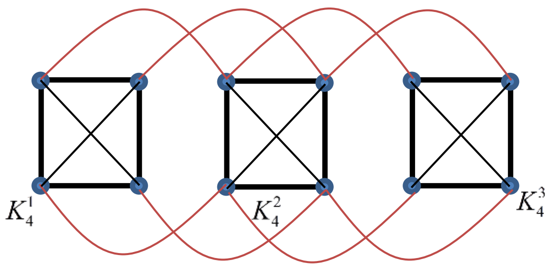

Figure 3: An illustration of $P_3 \Box K_4$

Theorem 2: *Let* $G = K_m \Box K_n$ *for* $m \geq n \geq 1$. *Then* $\gamma_r(G) = \gamma_r^*(G) = n$.

Proof: $\gamma_r^*(G) \leq n$ comes directly from Lemma 1 and Lemma 2. To see that $\gamma_r(G) \geq n$, assume $\gamma_r(G) < n$, let $f: V(G) \to \{0, 1, 2\}$ be an optimal weak Roman domination function of G. Since $\gamma_r^*(G) < n$, there must be a $V(K_m^i) \subset V_0$ for some $i, 1 \leq i \leq n$. Since $m \geq n$, there must be a vertex $v_{x,i} \in V(K_m^i)$ such that $N(v_{x,j}) \subset V_0$, which contradicts to the fact that f is a weak Roman domination function of G. By the fact that $\gamma_r(G) \leq \gamma_r^*(G)$ we know $\gamma_r(G) = \gamma_r^*(G) = n$. ■

Theorem 3: *Let* $G = K_{r,s} \Box K_n$ *for* $1 \leq r \leq s$ *and* $n \geq 2$. *Then the (smart proof) weak Roman domination number of G is*

$$\gamma_r(G) = \begin{cases} \min\{n+r, 4n\}, & \text{for } n \leq r \leq s; \\ \min\{r+s, 2n\}, & \text{for } r < n \leq r+s; \\ r+s, & \text{for } r+s < n. \end{cases}$$

Proof: Without loss of generality, let $V(K_{r,s}) = \{v_i | 1 \leq i \leq r+s\}$ where v_1 to v_r are in a partite set and v_{r+1} to v_{r+s} are in the other partite set. Then $V(G) = \{v_{i,j} | 1 \leq i \leq r+s, 1 \leq j \leq n\}$ where $v_{i,j}$ is adjacent to $v_{i,k}$ for $j \neq k$ and if $i \leq r$, $v_{i,j}$ is adjacent to $v_{t,j}$ for $r+1 \leq t \leq r+s$, otherwise ($r+1 \leq i \leq r+s$) $v_{i,j}$ is adjacent to $v_{t,j}$ for $1 \leq t \leq r$.

Case 1: $n \leq r \leq s$. First we show the upper bound. Define a guard function $f: V(G) \to \{0, 1, 2\}$ such that $V_2 = \{v_{i,i+1} | 1 \leq i \leq n\}$, $V_1 = \{v_{i,1} | n+1 \leq i \leq r\}$ and $V_0 = V(G) - V_1 - V_2$. Since every vertex $v_{i,j}$ for $r+1 \leq i \leq r+s$, and $1 \leq j \leq n$ has a neighbor in V_2, they are all protected. Since every vertex $v_{i,j}$ for $1 \leq i \leq r$ and $1 \leq j \leq n$ is included in a clique with some vertex in $V_1 \cup V_2$, by Lemma 4, they are also protected. Hence f is a weak Roman domination function of G, that is

$\gamma_r(G) \leq 2n + r - n = n + r$ ------------ (1).

Consider another guard function $f': V(G) \to \{0, 1, 2\}$ such that $V_1 = \varnothing$, $V_2 = \{v_{i,j} | r \leq i \leq r+1, 1 \leq j \leq n\}$, and $V_0 = V(G) - V_2$. Since all vertex $v_{i,j}$ for $1 \leq i \leq r-1$ is adjacent to vertex $v_{r+1,j}$ which is in V_2 and all vertex $v_{i,j}$ for $r+2 \leq i \leq r+s$ is adjacent to vertex $v_{r,j}$ which is in V_2, they are all protected. Hence f' is a weak Roman domination function of G, that is

$\gamma_r(G) \leq 4n$ ------------ (2)

By (1) and (2) we have

$\gamma_r(G) \leq \min\{n+r, 4n\}$ --------------------(3).

Next we show that $\gamma_r(G) \geq \min\{n+r, 4n\}$. Suppose to the contrary, $\gamma_r(G) < \min\{n+r, 4n\}$. Let $g: V(G) \to \{0,1,2\}$ be an optimal weak Roman domination function of G. Then there exists some K_n^i such that $V(K_n^i) \subset V_0$.

Subcase 1.1: $\nexists\, r+1 \leq i \leq r+s$ such that $V(K_n^i) \subset V_0$. Without loss of generality, assume $V(K_n^1) \subset V_0$. In order to protect $v \in V(K_n^1)$, by Lemma 4, we shall have $\sum_{u \in N(v)} g(u) \geq 2$ for each $v \in V(K_n^1)$. Then the weight of g is at least

$2n+s-n=s+n \geq r+n$, which is a contradiction.

Subcase 1.2: $\nexists 1 \leq i \leq r$ such that $V(K_n^i) \subset V_0$. Without loss of generality, assume $V(K_n^{r+1}) \subset V_0$. In order to protect $v \in V(K_n^{r+1})$, by Lemma 4, we shall have $\sum_{u \in N(v)} g(u) \geq 2$ for each $v \in V(K_n^{r+1})$. Then the weight of g is at least $2n+r-n=r+n$, which is a contradiction.

Subcase 1.3: $\exists 1 \leq i_1 \leq r$ and $\exists r+1 \leq i_2 \leq r+s$ such that $\left(V(K_n^{i_1}) \bigcup V(K_n^{i_2})\right) \subset V_0$ Without loss of generality, let $\left(V(K_n^r) \bigcup V(K_n^{r+1})\right) \subset V_0$. In this case we need to have $\sum_{u \in N(v)} g(u) \geq 2$ for each $v \in V(K_n^r) \bigcup V(K_n^{r+1})$, hence the weight of g is at least $4n$, which is a contradiction.

By above subcases we have

$$\gamma_r(G) \geq \min\{n+r, 4n\} \quad \text{-----------------(4).}$$

By (3) and (4) we have $\gamma_r(G) = \min\{n+r, 4n\}$, which completes the proof of case 1.

Case 2: $r < n \leq r+s$. First consider a guard function f of G such that $V_2 = \varnothing, V_1 = \{v_{i,1} | 1 \leq i \leq r+s\}$, and $V_0 = V(G) - V_1$. Since every copy of K_n^i has a vertex in V_1, by Lemma 4, f is a weak Roman domination function of G, which implies

$$\gamma_r(G) \leq r+s \quad \text{------------(5).}$$

Next consider another guard function f' such that $V_2 = \{v_{i,i} | 1 \leq i \leq r\} \bigcup \{v_{i,i+r} | r+1 \leq i \leq n\}$, $V_1 = \varnothing$ and $V_0 = V(G) - V_2$. Since every vertex in V_0 has at least a neighbor in V_2, by Lemma 4, f' is a weak Roman domination function of G, which implies

$$\gamma_r(G) \leq 2n \quad \text{----------------(6)}$$

By (5) and (6) we have $\gamma_r(G) \leq \min\{r+s, 2n\}$.

To see that $\gamma_r(G) \geq \min\{r+s, 2n\}$, assume to the contrary that $\gamma_r(G) < \min\{r+s, 2n\}$, then there must exist $V(K_n^i) \subset V_0$ some i, $1 \leq i \leq r+s$. Since $r < n \leq r+s$, similar to subcase 1.1, subcase1.2, and subcase 1.3, we have $\gamma_r(G) \geq \min\{2n, 4n\}$, which produce a contradiction. Therefore, we have $\gamma_r(G) = \min\{r+s, 2n\}$.

Case 3: $r+s < n$. The guard function f of G in case 2 give us $\gamma_r(G) \leq r+s$. The subcase 1.1, subcase1.2, and subcase 1.3 showed that $\gamma_r(G) \geq r+s$. ■

Theorem 4: *Let $G = K_{r,s} \square K_n$ for $1 \leq r \leq s$ and $n \geq 2$. Then the foolproof weak Roman domination number of G is*

$$\gamma_r^*(G) = \begin{cases} \min\{2r, 4n\}, & \text{for } n \leq r \leq s; \\ \min\{r+s, 2n\}, & \text{for } r < n \leq r+s; \\ r+s, & \text{for } r+s < n. \end{cases}$$

Proof: The case for $r < n$ is true since the upper bound can be showed by the guard functions f and f' in case 2 of the proof of Theorem 3 are also foolproof weak Roman domination function of G (by Lemma 3), and the lower bound come from the fact that $\gamma_r(G) \leq \gamma_r^*(G)$. So we only have to take care of the case for $n \leq r \leq s$. First we show that $\gamma_r^*(G) \leq \min\{2r, 4n\}$. Consider a guard function $g: V(G) \to \{0,1,2\}$ such that $V_2 = \{v_{i,i} | 1 \leq i \leq n\} \bigcup \{v_{i,1} | n+1 \leq i \leq r\}$, $V_1 = \varnothing$, and $V_0 = V(G) - V_2$. Since every vertex in V_0 is adjacent to a vertex in V_2, by Lemma 3, g is a foolproof weak Roman domination function of G, which implies

$$\gamma_r^*(G) \leq 2r \quad \text{----------------(7).}$$

Since the guard function f' in case 1 of the proof of Theorem 3 are also foolproof weak Roman domination function of G (by Lemma 3), we know

$$\gamma_r^*(G) \leq 4n \quad \text{-----------------(8)}$$

By (7) and (8) we have $\gamma_r^*(G) \leq \min\{2r, 4n\}$. Next we show that $\gamma_r^*(G) \geq \min\{2r, 4n\}$. Let g' be an optimal foolproof weak Roman domination function of G. Suppose $\gamma_r^*(G) < 2r$, then there must have a copy of K_n^i such that $V(K_n^i) \bigcap V_2 = \varnothing$ for some $1 \leq i \leq r$ and a copy of K_n^j such that $V(K_n^j) \bigcap V_2 = \varnothing$ for some $r+1 \leq i \leq r+s$. Without loss of generality, let $(V(K_n^1) \bigcup V(K_n^{r+1})) \bigcap V_2 = \varnothing$. In order to protect $V(K_n^1)$, by Lemma 3, it requires at least weight of $2n$ from $V(K_n^i)$ for $r+1 \leq i \leq r+s$; similarly, in order to protect $V(K_n^{r+1})$, by Lemma 3, it requires at least weight of $2n$ from $V(K_n^i)$ for $1 \leq i \leq r$. That implies the weight of g' is at least $2n+2n=4n$. Hence we have $\gamma_r^*(G) \geq \min\{2r, 4n\}$. Therefore $\gamma_r^*(G) = \min\{2r, 4n\}$ for $n \leq r \leq s$. That completes the proof.

3. Conclusion

This paper showed that the Cartesian product of a complete graph with a path, a cycle, and a complete graph has the same weak Roman domination number for both smart version and foolproof version, but on Cartesian product of a complete graph with a complete bipartite graph, the value of weak Roman domination number for smart version and foolproof version, which are provided in the paper, are different.

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