Finding paired dominating sets

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Abstract

A dominating set of an undirected graph $G = (V, E)$ is a set $D \subseteq V$ such that each vertex not in D has at least one neighbor in D. A paired-dominating set is a dominating set whose induced subgraph contains at least one perfect matching. This paper shows that the minimum paired-dominating set problem has a polynomial-time $2\lceil \log |V| \rceil$-approximation algorithm.

1 Introduction

A graph is an ordered pair $G = (V, E)$ consisting of a finite nonempty set V of vertices and a set E of edges, where each edge is an unordered pair of vertices. A dominating set of G is a set $D \subseteq V$ such that each vertex not in D has at least one neighbor in D. A paired-dominating set is a dominating set whose induced subgraph contains at least one perfect matching [1].

Raz and Safra prove that the dominating set problem has no polynomial-time $(c \log |V|)$-approximation algorithms for some $c>0$ unless P = NP [3].

Lin and Tu design an $O(|E| + |V|)$-time algorithm for interval graphs and an $O(|E|(|E|+|V|))$-time algorithm for circular-arc graphs, for the minimum paired-dominating set problem [2].

Let $f \colon \mathbb{N} \to \mathbb{N}$ be any function. If, given any graph $G = (V, E)$, an algorithm A outputs a paired-dominating set of G whose size is at most $f(|V|)$ times the minimum, then A is said to be $f(|V|)$-approximate for the minimum paired-dominating set problem.

By modifying a well-known approximation algorithm for set covering, this paper obtains a polynomial-time $O(\log |V|)$-approximation algorithm for the minimum paired-dominating set problem.

The following lemma is a consequence of line 2 of the algorithm in Fig. 1.

1: $\mathcal{D} \leftarrow \emptyset$;
2: **while** $\bigcup_{v \in \mathcal{D}} N[v] \neq V$ **do**
3: Among the edges in E not having an endpoint in $\mathcal{D}$, pick an edge (a, b) that maximizes $|(N[a] \cup N[b]) \cap (V \setminus \bigcup_{v \in \mathcal{D}} N[v])|$, breaking ties arbitrarily;
4: $\mathcal{D} \leftarrow \mathcal{D} \cup \{a, b\}$;
5: **end while**
6: **return** $\mathcal{D}$;

Figure 1: A greedy algorithm

Lemma 1. *Whenever line 3 of the algorithm in Fig. 1 is executed,*

$$\left| V \setminus \bigcup_{v \in \mathcal{D}} N[v] \right| > 0.$$

Let D^* be a smallest dominating set of $G = (V, E)$.

Lemma 2. *Whenever line 3 of the algorithm in Fig. 1 is executed, there exists $u \in D^* \setminus \mathcal{D}$ satisfying*

$$\left| N[u] \cap \left(V \setminus \bigcup_{v \in \mathcal{D}} N[v] \right) \right| \geq \frac{1}{|D^*|} \cdot \left| V \setminus \bigcup_{v \in \mathcal{D}} N[v] \right| \quad (1)$$

and that $N(u) \not\subseteq \mathcal{D}$.

Proof. Because D^* is a dominating set,

$$\bigcup_{v \in D^*} N[v] = V.$$

Consequently,

$$V \setminus \bigcup_{v \in \mathcal{D}} N[v] \subseteq \bigcup_{v \in D^*} N[v].$$

So by the averaging argument, there exists $u \in D^*$ satisfying inequality (1). It is not hard to verify that $u \notin \mathcal{D}$ and $N(u) \not\subseteq \mathcal{D}$. □

Corollary 1. *Right after each execution of line 3 of the algorithm in Fig. 1,*

$$\left|(N[a] \cup N[b]) \cap \left(V \setminus \bigcup_{v \in \mathcal{D}} N[v]\right)\right| \geq \frac{1}{|D^*|} \cdot \left|V \setminus \bigcup_{v \in \mathcal{D}} N[v]\right|.$$

By elementary calculus, $(1 - 1/n)^n < 1/e$ for all $n \in \mathbb{Z}^+$.

After $|D^*| \cdot \lceil \log |V| \rceil$ iterations,

$$\left|V \setminus \bigcup_{v \in \mathcal{D}} N[v]\right| \leq \left(1 - \frac{1}{|D^*|}\right)^{|D^*| \cdot \lceil \log |V| \rceil} |V| < 1$$

by repeatedly invoking Corollary 1, implying

$$\left|V \setminus \bigcup_{v \in \mathcal{D}} N[v]\right| = 0.$$

When the algorithm halts, $|\mathcal{D}|$ is simply twice number of iterations. As a result, the algorithm outputs a set $\mathcal{D}$ of size at most $2 \cdot |D^*| \cdot \lceil \log |V| \rceil$. Summarizing the above gives our main theorem, stated below.

Theorem 1. *The minimum paired-dominating set problem has a polynomial-time* $2\lceil \log |V| \rceil$*-approximation algorithm.*

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References

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