Helly Type Fixed Clique Theorems for Multifunctions *

Rueiher Tsaur
Department of Computer Information and Network Engineering
College of Engineering, Lunghwa University of Science and Technology, Taiwan
rueiher@gm.lhu.edu.tw

Abstract

In this paper, a notion of strong convexity is defined in the intersection graphs of n-dimensional real digital pictures based on the usual Euclidean convex closure operator. It is shown that any (3^n-1)-adjacent n-dimensional digital picture has the strong convexity fixed clique property for weak multifunctions if the strong convexity is Helly.

1 Preliminaries

For a systematic treatment of multifunctions on graphs and simplicial complexes, we view them as mappings into "power graphs" and "power complexes" as is typically done in topology and domain theory.

1.1 Power structures and corresponding multifunctions

By a *graph* G we mean a set $V(G)$ of points (or *vertices*), with a reflexive and symmetric relation $E(G)$ the *edges*. For graphs G and H, a *graph homomorphism* $f : G \to H$ maps points of G to points of H, preserving the edges, i.e., $(f(x), f(y)) \in E(H)$ whenever $(x, y) \in E(G)$.

Let G be a graph. The *power graphs* $\mathcal{P}_w(G)$ and $\mathcal{P}_s(G)$ having the non-empty finite subsets of $V(G)$, $\mathcal{P}^+_{fin}(V(G))$ as points, such that $(A, B) \in E_w(G) \Leftrightarrow \exists x \in A, y \in B, (x, y) \in E(G)$ and $(A, B) \in E_s(G) \Leftrightarrow \forall x \in A, \exists y \in B, (x, y) \in E(G) \& \forall y \in B, \exists x \in A, (x, y) \in E(G)$.

Let G and H be graphs. A *multifunction* f of G into H is a function that assigns to a point x of G a non-empty subset $f(x)$ in $V(H)$. The multifunction $f : G \to H$, is *weak* (resp. *strong*) if $\hat{f} : G \to \mathcal{P}_w(H)$ (resp. $\hat{f} : G \to \mathcal{P}_s(H)$) is a graph homomorphism, where $\hat{f}(x) = f(x) \in \mathcal{P}^+_{fin}(V(H))$.

*This work was supported by the National Science Council, Republic of China, under Contract NSC-104-2633-E-262-001-.

An *abstract simplicial complex* K (briefly a complex) is defined to consist of a collection $V(K)$ of points together with a prescribed collection $\Lambda(K) \subseteq \mathcal{P}^+_{fin}(V(K))$ of finite non-empty subsets of $V(K)$ (the *simplexes* of K), satisfying: (1) $x \in V(K) \Rightarrow \{x\} \in \Lambda(K)$; and (2) $\forall \alpha \in \Lambda(K), \beta \subseteq \alpha, \beta \neq \emptyset \Rightarrow \beta \in \Lambda(K)$. We write $\sigma = \langle v_0, \ldots, v_n \rangle$ to mean that σ is the simplex with points $v_0, \ldots, v_n$. A *simplicial mapping* $\phi : K_1 \to K_2$ from the complex K_1 into the complex K_2 is a function which maps points of K_1 to the points of K_2 preserving the simplex structures.

Let K be a complex. The *weak power complex* $\mathcal{P}_w(K)$ is $(\mathcal{P}^+_{fin}(V(K)), \Lambda_w)$, where $\rho = \langle \rho_i \rangle \in \Lambda_w \Leftrightarrow \exists \sigma \in \Lambda(K)$ s.t. $\rho_i \cap \sigma \neq \emptyset$, for all i. The *strong power complex* $\mathcal{P}_s(K)$ also has the non-empty finite subsets of $V(K)$ as points, and the collection Λ_s of simplexes, where $\rho = \langle \rho_i \rangle_{i \in I} \in \Lambda_s \Leftrightarrow \forall k \in I, \forall v \in \rho_k, \exists \langle v_i \rangle_{i \in I, v_i \in \rho_i}$ s.t. $v_k = v \ \& \ \langle v_i \rangle_{i \in I} \in \Lambda(K)$.

An important example of the complexes is the clique complex $\mathcal{C}(G)$ of a graph G: A graph H is called *complete* if any two points $x, y \in V(H)$ are joined by an edge, i.e., $(x, y) \in E(H)$. A *clique* of a graph is a maximal complete subgraph. The *clique complex* $\mathcal{C}(G)$ of a graph G is the simplicial complex on the point set of G whose simplexes are the finite cliques of G. It is easy to check that, if $f : G \to H$ is a graph homomorphism, then $g : \mathcal{C}(G) \to \mathcal{C}(H), x \mapsto f(x), \forall x \in V(\mathcal{C}(G)) = V(G)$, is a simplicial mapping. Thus, every graph is just a special simplicial complex. For other important features of clique complexes, see [3].

Let K_1 and K_2 be complexes, and $s : K_1 \to K_2$ a multifunction from K_1 to K_2. Then s is *weak simplicial* if $\hat{s} : K_1 \to \mathcal{P}_w(K_2)$ is simplicial, and s is *strong simplicial* if $\hat{s} : K_1 \to \mathcal{P}_s(K_2)$ is simplicial, where $\hat{s}(x) = s(x) \in \mathcal{P}^+_{fin}(V(K_2))$.

1.2 Almost fixed point and fixed clique properties

A structure has the *fixed point property* if for every self-mapping structure homomorphism f, there is a point x with $f(x) = x$. Since an exact fixed point result is in general not possible for graphs and complexes, the following three natural properties are more well-known: Let G be a graph and K a complex, then

(1) G is said to have the *fixed clique property* (*FCP*) if for every self-mapping (single-valued) graph homomorphism f, there is a clique C with $C = f(C)$.

(2) K is said to have the *fixed simplex property* (*FSP*) if for every self-mapping (single-valued) simplicial mapping g, there is a simplex S with $S = g(S)$.

(3) K (resp. G) is said to have the *almost fixed point property* (*AFPP*) if for every self-mapping single-valued simplicial mapping (resp. single-valued graph homomorphism) h, there exists a point x such that $\langle \{x\}, h(x)\rangle \in \Lambda(\mathcal{P}_w(K))$ (resp. $(\{x\}, h(x)) \in E(\mathcal{P}_w(G))$).

C, S and x are said to be the *fixed clique*, *fixed simplex* and *almost fixed point* of f, g and h, respectively.

In considering the FCP (resp. FSP) for multifunctions, the above condition $f(C) = C$ is no longer suitable (just consider the self-mapping multifunction which maps each point to the whole structure). Thus

Definition 1.1. Let $f : G \to G$ be a multifunction, and $g : K \to K$ a simplicial multifunction, then

(1) A point x is said to be a *fixed point* of f (resp. g) if $x \in f(x)$ (resp. $x \in g(x)$).

(2) A clique C is said to be a *fixed clique* of f if $C \subseteq f(C)$.

(3) A simplex S is said to be a *fixed simplex* of g if $S \subseteq g(S)$.

(4) A point x is said to be an *almost fixed point* of f or g if, respectively, $(\{x\}, f(x)) \in E(\mathcal{P}_w(G))$ or $\langle \{x\}, g(x)\rangle \in \Lambda(\mathcal{P}_w(K))$.

Let H_1, H_2 be structures, and ϕ a property of subsets of H_2; then the multifunction $f : H_1 \to H_2$ is said to be a ϕ-multifunction if f sends points of H_1 into subsets of H_2 satisfying ϕ. One of the most well-known examples is the "compact convex-valued" multifunctions in topological vector spaces. Thus we have

Definition 1.2. Let G be a graph and K a complex. Then

(1) G is said to have the *ϕ-fixed clique property* (*ϕ-FCP*) for weak (resp. strong) multifunctions if every ϕ-weak (resp. strong) multifunction of G has a fixed clique.

(2) K is said to have the *ϕ-fixed simplex property* (*ϕ-FSP*) for weak (resp. strong) simplicial multifunctions if every ϕ-weak (resp. strong) simplicial multifunction of K has a fixed simplex.

(3) G is said to have the *ϕ-almost fixed point property* (*ϕ-AFPP*) for weak (resp. strong) multifunctions if every ϕ-weak (resp. strong) multifunction of G has an almost fixed point.

(4) K is said to have the *ϕ-almost fixed point property* (*ϕ-AFPP*) for weak (resp. strong) simplicial multifunctions if every ϕ-weak (resp. strong) simplicial multifunction of K has an almost fixed point.

Note that the graph G has the ϕ-FCP if and only if the clique complex $\mathcal{C}(G)$ has the ϕ-FSP when the multifunctions are weak or strong (or any intermediate possibility between them). It is clear that notion of ϕ-FCP (resp. ϕ-FSP) in Definition 1.2 generalizes the notion of FCP (resp. FSP) for single-valued mappings (just consider ϕ as the singletons).

2 Fixed clique property for weak multifunctions

Some of the material presented in this section can be found in [4].

Let S be a set and $D = \{c_1, \ldots, c_k\}$ a non-empty collection of distinct non-empty subsets of S whose union is S. Then we call G the *intersection graph* of D if $V(G) = D$, with c_i and c_j adjacent whenever $c_i \cap c_j \neq \emptyset$ (note that this definition is different with the one as in [1], where their graphs by convention permit no loops).

Let $\prod_{1\le i\le n}\lceil 0, m_i\rfloor$ be an n-dimensional real digital picture with unit mesh. By $\bigotimes_{1\le i\le n} I_{m_i}$ (I_m^n) we denote the intersection graph induced from $\prod_{1\le i\le n}\lceil 0, m_i\rfloor$ ($\lceil 0, m\rfloor^n$); we may call

$\bigotimes_{1\leq i\leq n} I_{m_i}$ a (3^n-1)-*adjacent* n-*dimensional digital picture* as $\bigotimes_{1\leq i\leq n} I_{m_i}$ reflects the adjacent (neighbourhood) relationships of cells in $\prod_{1\leq i\leq n}\lceil 0, m_i\rfloor$.

In [4], a (possibly weakest) notion of weak convexity was defined in the intersection graphs of n-dimensional real digital pictures. It showed from Kakutani's theorem that any (3^n-1)-adjacent n-dimensional digital picture has the simplicial weak convex almost fixed point property (AFPP), and claimed that this theorem may be considered as a digital version of the Kakutani fixed point theorem for convex-valued multifunctions.

Since the fixed clique property (FCP) is a rather strong condition comparing with the AFPP, in considering the FCP for graphs, it is natural to consider rather strong conditions on graphs or multifunctions (or both). As for a relatively simple example, consider the case that T is a finite non-empty *tree* (connected cycle-free graph). In [2], Nowakowski and Rival had already proved that any finite non-empty tree T has the FCP (of course, the fixed edge property) for single-valued graph homomorphisms. We generalize their result to the FCP for weak multifunctions:

Theorem 2.1. *Let Υ be the set of non-empty subtrees of any finite non-empty tree T (equivalently, of connected subsets of $V(T)$). Then T has the Υ-FCP for weak multifunctions.*

Proof. Let us choose (arbitrarily) a point r to be the root of T. A path is a sequence of distinct points such that each consecutive pair of points are joined by an edge; the distance between two points is the length of the path between them. Given a Υ-weak multifunction f defined on T, we shall say that a point x is a *forward mapping point* if $f(x)$ is a proper subtree of the subtree (of T) with root x. Stated differently, x is a forward mapping point if $x \notin f(x)$, but every path from r to $f(x)$ passes through x.

Given a Υ-weak multifunction f, we may assume that the root r is a forward mapping point (else r is already a fixed point). Also note that not every point is forward mapping, in particular, all leaves are not. Since the set of forward mapping points is non-empty (as r is forward mapping), therefore we may choose a forward mapping point x whose distance from r is maximal.

All descendants (children) of x, say $C = \{y_1,\ldots,y_n\}$, are not forward mapping. Since $f(x)$ is contained in a proper subtree of the (sub)tree rooted at x, therefore $f(x)$ is contained in a subtree of the tree rooted at an unique element of C, say y. Hence we have: $f(x)$ is contained in a subtree (no need to be proper) of the tree rooted at y; $f(y)$ is not contained in a proper subtree of the tree rooted at y; yet there is a point of $f(x)$ adjacent to some point of $f(y)$. Thus, it is easy to check that either y is a fixed point of f, or (x,y) is a fixed edge. □

3 Helly Type Fixed Clique Theorems

Note that the non-empty finite trees are among the simplest (nicest) of graphs, with many good properties. In connection with Theorem 2.1 and problems of the FCP for weak multifunctions, we observe the following features of the non-empty finite tree with its Υ-weak multifunctions:

(a) If $f : T_1 \to T_2$ is a Υ-weak multifunction, then $\hat{f} : \mathcal{C}(T_1) \to \mathcal{C}(T_2), \hat{f}(x) = f(x)$, is Υ-weak simplicial.

(b) Any element of Υ is not only a "weak convex" subset but in fact, a "convex" subset, in any non-empty finite tree T. Although T is in general (unless T is a linear graph I_k) not $\bigotimes_{1\leq i\leq n} I_{m_i}$, it is natural to consider T as an one dimensional space whose convex subsets are simply connected subsets of T (e.g. considering its corresponding topological tree).

Thus by the feature (b) above, in considering a good "approximation" of ϕ-FCP for weak multifunctions for $\bigotimes_{1\leq i\leq n} I_{m_i}$, we would expect that a suitable notion of ϕ-sets should satisfy the following "strong" property:

Definition 3.1. Given any subset C of $\prod_{1\leq i\leq n}\lceil 0, m_i\rfloor$. Then C is said to be a *strong convex subset* of $\prod_{1\leq i\leq n}\lceil 0, m_i\rfloor$ if $\mathrm{conv}(\bigcup C) = \bigcup C$. A given subset C of $\bigotimes_{1\leq i\leq n} I_{m_i}$ is said to be *strong convex* if and only if C^{-1} is strong convex in $\prod_{1\leq i\leq n}\lceil 0, m_i\rfloor$.

It is not difficult to check that a subset of $\bigotimes_{1\leq i\leq n} I_{m_i}$ is strong convex if and only if it is a rectangular block of $\bigotimes_{1\leq i\leq n} I_{m_i}$.

Theorem 3.2. *If each $m_i < \infty$, then $\mathcal{C}(\bigotimes_{1\leq i\leq n} I_{m_i})$ has the* **sc**-*FCP for weak simplicial multifunctions, where* **sc** *stands for strong convexity.*

Proof. Everything is almost the same as with the proof of Theorem 5.3 in [4], except that Lemma 5.7 in [4] is replaced by Lemma 3.3 to follow:

Lemma 3.3. *If F has a fixed point $x^* \in \bigcup\lceil 0, m\rfloor^n$ such that $x^* \in c$ for some cell $c \in \lceil 0, m\rfloor^n$, then c^{-1} is either a fixed point or the collection of points c^{-1} forms a fixed clique of f.*

Proof. Denote by $\mathcal{D}_c$ the subset $\bigcup(f(c^{-1}))^{-1} \subseteq \bigcup\lceil 0, m\rfloor^n$ induced by cells in the image of c. Since Ω is a partition of $\bigcup\lceil 0, m\rfloor^n$, therefore there exists an unique $\omega \in \Omega_i, 0 \leq i \leq n$ satisfying $x^* \in \omega$. Consider the following cases:

(1) $i = n$: Clearly $x^* \in \text{conv}(\mathcal{D}_c) = \mathcal{D}_c$, thus c^{-1} is a fixed point of f.

(2) $i \neq n$: There exist 2^{n-i} mutually adjacent cells $D = \{c_1, c_2, \ldots, c_{2^{n-i}}\}$ satisfying $x^* \in \omega \subseteq \bigcup_{c_j \in D} c_j$ and $F(x^*) = \bigcap_{c_j \in D} \text{conv}(\mathcal{D}_{c_j})$. Therefore we have $x^* \in \text{conv}(\mathcal{D}_{c_j})$ for any $c_j \in D$. It is easy to check that: if $x^* \in \bigcap_{c_j \in D} c_j$, then D^{-1} is a fixed clique of f; otherwise there must exist $c_j \in D$ such that c_j^{-1} is a fixed point of f. □

Therefore by Lemma 5.6 in [4] and 3.3, f has a fixed clique. So we complete the proof of Theorem 3.2. □

For any φ-weak multifunction $f : \bigotimes_{1 \leq i \leq n} I_{m_i} \to \bigotimes_{1 \leq j \leq k} I_{l_j}$, it is easy to check that, by the definition of weak multifunctions, we "at the most" have, for any clique $\Delta \subseteq V(\bigotimes_{1 \leq i \leq n} I_{m_i})$ with $x, y \in \Delta$, $\bigcup f(x)^{-1} \cap \bigcup f(y)^{-1} \neq \emptyset$ (in $\bigcup\lceil 0, m\rfloor^n$). So comparing this with the feature (a) above, how can we have $\bigcap_{z \in \Delta}(\bigcup f(z)^{-1}) \neq \emptyset$? Recall that a family H of subsets of a set X is said to satisfy the 2-*intersection property* if any two elements of H are intersection non-empty (also called pairwise non-disjoint). H is said to satisfy the (2-) *Helly property* if for every subfamily H' of H satisfying the 2-intersection property, H' is intersection non-empty. We may say that the weak multifunction $f : \bigotimes_{1 \leq i \leq n} I_{m_i} \to \bigotimes_{1 \leq j \leq k} I_{l_j}$ is (2-) *Helly* if for any clique $\Delta \subseteq V(\bigotimes_{1 \leq i \leq n} I_{m_i})$, we have $\bigcap_{x \in \Delta} \bigcup f(x)^{-1} \neq \emptyset$. It is easy to check that the weak multifunction f is Helly if and only if $\hat{f} : \mathcal{C}(\bigotimes_{1 \leq i \leq n} I_{m_i}) \to \mathcal{C}(\bigotimes_{1 \leq j \leq k} I_{l_j}), \hat{f}(x) = f(x), \forall x \in V(\mathcal{C}(\bigotimes_{1 \leq i \leq n} I_{m_i})) = V(\bigotimes_{1 \leq i \leq n} I_{m_i})$, is weak simplicial. Also we may say that the strong convexity S (of $\bigotimes_{1 \leq i \leq n} I_{m_i}$) is (2-) *Helly* if the set $\{\bigcup C^{-1} \mid C \in S\}$ satisfies the Helly property (in $\bigcup\lceil 0, m\rfloor^n$). Then the following theorems are direct consequences of Theorem 5.3 in [4] and Theorem 3.2:

Theorem 3.4. *If each $m_i < \infty$, then*

(1) *$\bigotimes_{1 \leq i \leq n} I_{m_i}$ has the* **sc**-*FCP and* **wc**-*AFPP for Helly weak multifunctions.*

(2) *$\bigotimes_{1 \leq i \leq n} I_{m_i}$ has the* **sc**-*FCP for weak multifunctions if the strong convexity (of $\bigotimes_{1 \leq i \leq n} I_{m_i}$) is Helly.*

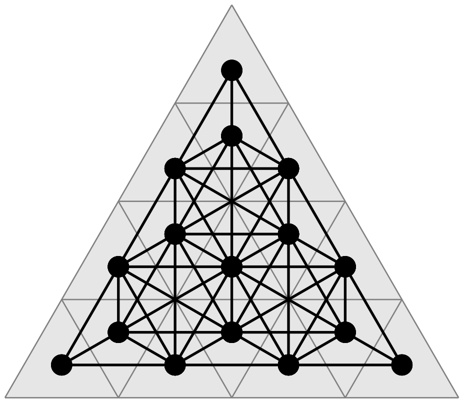

Figure 1: A self-similar equilateral triangle tiling and its intersection graph

4 Concluding remarks

We remark that Theorem 3.2 is **not** the "best" result (for FCP), since we used the Kakutani theorem in deriving it. In [6], we showed that the hypothesis of Theorem 3.2 can be weakened by changing the "strong convex" subsets into "dismantlable (contractible)" subsets by using the Eilenberg-Montgomery fixed point theorem.

Since the strong convex sets of $\bigotimes_{1 \leq i \leq n} I_{m_i}$ are just the rectangular blocks in $\bigotimes_{1 \leq i \leq n} I_{m_i}$, as is well-known in (Euclidean) convex geometry, the set of rectangular blocks of $\mathbb{R}^n$ satisfies the Helly property, thus the "Helly" condition in Theorem 3.4(2) seems redundant. However, what we have in mind here is that Theorem 3.4(2) is extendable to other complex structures and their intersection graphs, where the strong convex subsets of them may not "automatically" satisfy the Helly property: a simple example would be the class of self-similar equilateral triangle tilings and its intersection graphs (see Fig. 1).

References

[1] F. Harary, "Graph Theory," *Addison-Wesley*, 1969.

[2] R. Nowakowski and I. Rival, “Fixed-edge theorem for graphs with loops,” *J. Graph Theory*, vol. 3, pp. 339–350, 1979.

[3] B. Schröder, “Algorithms for the fixed point property,” *Theoret. Comput. Sci.*, vol. 217, pp. 301–358, 1999.

[4] M. B. Smyth and R. Tsaur, “A digital version of the Kakutani fixed point theorem for convex-valued multifunctions,” *Electron. Notes Theor. Comput. Sci.*, vol. 40, 15 pages, 2001.

[5] M. B. Smyth and R. Tsaur, “AFPP vs FPP: the link between almost fixed point properties of discrete structures and fixed point properties of spaces,” *Appl. Categ. Structures*, vol. 11, pp. 95–116, 2003.

[6] R. Tsaur and M. B. Smyth, ““Continuous” multifunctions in discrete spaces with applications to fixed point theory,” *Lecture Notes in Comput. Sci.*, vol. 2243, pp. 75–88, 2001.

[7] R. Tsaur, “The BaillonSimons theorems,” *Discrete Math.*, vol. 293, pp. 251–261, 2005.