# A Lower Bound on the Substar Reliability of Star Networks*

Tzu-Liang Kung†
Department of Computer Science and Information Engineering
Asia University, Wufeng, Taichung City 413, Taiwan
tlkung@asia.edu.tw

Chun-Nan Hung
Department of Information Management
Da-Yeh University, Changhua County 515, Taiwan
chunnan.hung@gmail.com, spring@mail.dyu.edu.tw

Hon-Chan Chen
Department of Information Management
National Chin-Yi University of Technology, Taiping, Taichung City 411, Taiwan
chenhc@ncut.edu.tw

Jui-Yu Hung, Lih-Hsing Hsu
Department of Computer Science and Information Engineering
Providence University, Shalu, Taichung City 433, Taiwan
juiyu1029@gmail.com, lihhsing@gmail.com

**Abstract**

*The star is one of the most fundamental architectures for interconnecting a large number of components in a network system. Based on the popularity of the star, the star network has been a good alternative to the hypercube-based topologies. Under the probability fault model, Wu and Latifi (2008) established an upper bound on the substar reliability for the class of star networks. However, a lower-bounded reliability plays a more informative role in maintaining system availability. For this reason, this paper is aimed at deriving an analytic lower bound on the substar reliability of star networks by means of combinatorial approach. In addition, numerical comparisons are presented for validating the proposed formulation.*

**Keywords:** Availability; reliability; probability fault model; star network.

*This work is supported in part by the Ministry of Science and Technology, Taiwan, Republic of China, under Grand No. MOST 105-2221-E-468-015.

†Corresponding author.

## 1 Introduction

The interconnection network is of great significance for a parallel and distributed computing system. It is usually multi-objected and complicated to design an interconnection network. The star is one of the most fundamental architectures for interconnecting a large number of components in a network system. Based on the popularity of the star, Akers and Krishnamurthy [1] proposed the star network as a viable alternative to the hypercube-based topologies. An immediate advantage of the star network is that it is able to connect more nodes with less connection links and less communication delay than the hypercube [8]. The promising features of the star network include low degree of nodes, small diameter, node transitivity, link symmetry, and high degree of fault tolerance, and so on. Many researchers had investigated the algebraic properties of the star network in terms of various performance metrics, such as connectivity, diameter, fault diameter, surface area, diagnosability [4, 5, 6, 10, 12, 14, 15, 16, 17, 18, 21, 24].

The underlying topology of an interconnection network is modeled as a graph, in which vertices and edges correspond to nodes and connection links between nodes. In general, as the size of a system grows, the likelihood of fault occurrences in the system increases. Reliability is extensively applied to quantify the impact of system failures. The reliability of a network system is defined as the probability that the system is able to perform its required functions for a given time session [11]. A wide range of reliability models have been proposed to measure the network reliability and availability [3, 7, 9, 13, 20, 23]. An explicit formula of the subcube reliability of the hypercube-based network was formulated by Das and Kim [7] under the random fault model, which assumes that there are $f$ faults distributed randomly in the hypercube. Later, Chang and Bhuyan [3] proposed a more efficient computing model, namely the probability fault model, for assessing the subcube reliability of the hypercube. Under the probability fault model, Wu and Latifi [22] analyzed the substar reliability in star networks, and Lin et al. [19] calculated the subgraph reliability of the arrangement graph. However, only upper-bounded reliability is addressed in [22] and [19]. It is intuitive to see that a lower-bounded reliability may play a more informative role in achieving system availability. Therefore, this paper is aimed at deriving an analytic lower bound on the substar reliability of star networks. Numerical comparisons between lower- and upper-bounded reliability are also presented.

The rest of this paper is structured as follows. Section 2 introduces the foundation of the probability fault model and the topological properties of star networks. In Section 3, an analytic lower bound on the substar reliability of star networks is derived. Section 4 presents numerical comparisons between lower- and upper-bounded substar reliability. Finally, Section 5 concludes this paper.

## 2 Background

Throughout this paper, graphs (interchangeably, networks) are finite, simple, and undirected. Fundamental graph-theory definitions and notations are introduced in advance. For those not defined here, we follow the standard terminology given by Bondy and Murty [2]. An *undirected graph* $G$ consists of a vertex set $V(G)$ and an edge set $E(G)$, where $V(G)$ is a finite set, and $E(G)$ is a subset of the set of all unordered pairs of distinct elements in $V(G)$. In this paper, we use $\{u, v\}$ to denote an unordered pair of two elements

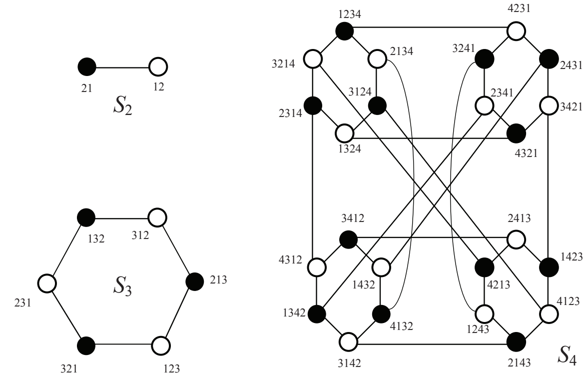


Figure 1: Illustrating $S_2$, $S_3$, and $S_4$.

$u, v$. Two vertices $u$ and $v$ of $G$ are *adjacent* if $\{u, v\} \in E(G)$. A graph $H$ is a *subgraph* of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For any nonempty subset $S$ of $V(G)$, the subgraph of $G$ *induced* by $S$ is a graph whose vertex set is $S$ and whose edge set consists of all the edges of $G$ joining any two vertices in $S$.

For any positive integer $n$, let $\langle n \rangle$ denote the set of all positive integers from 1 to $n$; i.e., $\langle n \rangle = \{1, 2, \ldots, n\}$. A permutation over $\langle n \rangle$ is an order sequence containing each element of $\langle n \rangle$ once, and only once. The vertex set of the $n$-star network $S_n$ is the set of all permutations over $\langle n \rangle$. In $S_n$, two permutations are linked by an $i$-edge if one can be obtained from the other by swapping the 1st and the $i$th digits, $2 \leq i \leq n$. Figure 1 illustrates $S_2$, $S_3$, and $S_4$. Obviously, $S_n$ is bipartite and $(n-1)$-regular for $n \geq 2$. Moreover, it is not only vertex-transitive but also edge-transitive [1]. For any vertex $v$ of $S_n$, its $i$th digit is denoted by $(v)_i$, and for $1 \leq k \leq n-1$, let $S_n^{(i_1:x_1, i_2:x_2, \ldots, i_k:x_k)}$ be the subgraph of $S_n$ induced by $\{v \in V(S_n) \mid (v)_{i_1} = x_1, (v)_{i_2} = x_2, \ldots, (v)_{i_k} = x_k\}$, where $\{i_1, i_2, \ldots, i_k\}$ and $\{x_1, x_2, \ldots, x_k\}$ are $k$-element subsets of $\{2, 3, \ldots, n\}$ and $\langle n \rangle$, respectively. Then, $S_n^{(i_1:x_1, i_2:x_2, \ldots, i_k:x_k)}$ is isomorphic to an $(n-k)$-star network $S_{n-k}$. As $S_n^{(i:x)}$ is an $S_{n-1}$-subgraph of $S_n$ for any $i \in \{2, 3, \ldots, n\}$ and $x \in \langle n \rangle$, there are $n-1$ different ways to partition $S_n$ into $n$ disjoint $S_{n-1}$-subgraphs. Thus, the total number of distinct $S_{n-1}$-subgraphs in $S_n$ is $n(n-1)$.

Wu and Latifi [22] first analyzed the substar reliability of star networks under the probability fault model. In the probability fault model, every node has a homogeneous node reliability $p$, which is defined as the probability that a single node is fault-free at time $t$. Given the homogeneous node reliability $p$, then $R_{n,n-1}(p)$ stands for the probability that there exists at least one fault-free $S_{n-1}$-subgraph in $S_n$. Because there are $n(n-1)$ distinct $S_{n-1}$-subgraphs in $S_n$, we denote them by $S^1_{n-1}$, $S^2_{n-1}$, ..., $S^{n(n-1)}_{n-1}$. Moreover, let $\xi^i_{n-1}$ denote the probabilistic event that $S^i_{n-1}$ is fault-free in $S_n$ for $1 \le i \le n(n-1)$. Then, $Pr\left(\xi^i_{n-1}\right) = p^{(n-1)!}$ and $R_{n,n-1}(p) = Pr\left(\bigcup_{i=1}^{n(n-1)} \xi^i_{n-1}\right)$, where $Pr(\cdot)$ is the probability of a stochastic event. According to the inclusion-exclusion principle, $R_{n,n-1}(p)$ is further decomposed:

$$\begin{aligned}
& R_{n,n-1}(p) \\
= & \sum_{i=1}^{n(n-1)} Pr\left(\xi^i_{n-1}\right) - \sum_{i<j} Pr\left(\xi^i_{n-1} \cap \xi^j_{n-1}\right) \\
& + \sum_{i<j<k} Pr\left(\xi^i_{n-1} \cap \xi^j_{n-1} \cap \xi^k_{n-1}\right) \\
& - \sum_{i<j<k<l} Pr\left(\xi^i_{n-1} \cap \xi^j_{n-1} \cap \xi^k_{n-1} \cap \xi^l_{n-1}\right) \\
& \cdots \\
& +(-1)^{n(n-1)-1} Pr\left(\bigcap_{i=1}^{n(n-1)} \xi^i_{n-1}\right). \qquad (1)
\end{aligned}$$

Wu and Latifi [22] suggested a simple approximation of $R_{n,n-1}(p)$ as follows:

$$R_{n,n-1}(p) \approx 1 - \left(1 - p^{(n-1)!}\right)^{n(n-1)} \qquad (2)$$

Furthermore, an upper bound on $R_{n,n-1}(p)$ has been established by neglecting all after the third term in Eq. (1):

$$\begin{aligned}
& R_{n,n-1}(p) \\
\le & \sum_{i=1}^{n(n-1)} Pr\left(\xi^i_{n-1}\right) - \sum_{i<j} Pr\left(\xi^i_{n-1} \cap \xi^j_{n-1}\right) \\
& + \sum_{i<j<k} Pr\left(\xi^i_{n-1} \cap \xi^j_{n-1} \cap \xi^k_{n-1}\right) \\
= & n(n-1)p^{(n-1)!} - \binom{n}{2} \Big[(2n-3)p^{2(n-1)!} \\
& +2\binom{n-1}{2} p^{2(n-1)!-(n-2)!}\Big] \\
& +\binom{n}{3} \Big[(2n-4)p^{3(n-1)!} \\
& +(6n-6)p^{3(n-1)!-(n-2)!} \\
& +3(2n-5)(n-1)p^{3(n-1)!-2(n-2)!} \\
& +6\binom{n-1}{3} p^{3(n-1)!-3(n-2)!+(n-3)!}\Big] \qquad (3)
\end{aligned}$$

## 3 Lower bound on $R_{n,n-1}(p)$

A lower bound on $R_{n,n-1}(p)$ can be formed from the first four terms of Eq. (1):

$$\begin{aligned}
& R_{n,n-1}(p) \\
\ge & \sum_{i=1}^{n(n-1)} Pr\left(\xi^i_{n-1}\right) - \sum_{i<j} Pr\left(\xi^i_{n-1} \cap \xi^j_{n-1}\right) \\
& + \sum_{i<j<k} Pr\left(\xi^i_{n-1} \cap \xi^j_{n-1} \cap \xi^k_{n-1}\right) \\
& - \sum_{i<j<k<l} Pr\left(\xi^i_{n-1} \cap \xi^j_{n-1} \cap \xi^k_{n-1} \cap \xi^l_{n-1}\right).
\end{aligned}$$

Let

$$\delta(n,p) \triangleq \sum_{i<j<k<l} Pr\left(\xi^i_{n-1} \cap \xi^j_{n-1} \cap \xi^k_{n-1} \cap \xi^l_{n-1}\right).$$

In the rest of this section, we will derive the following analytic formula:

$$\begin{aligned}
& \delta(n,p) \\
= & \left[(n-1)\binom{n}{4} + n\binom{n-1}{4}\right] p^{4(n-1)!} \\
& + \Big[2(n-3)\binom{n-1}{2}\binom{n}{3} + 8\binom{n-1}{4}\binom{n}{2} \\
& +6\binom{n-1}{2}\binom{n}{3} + 6\binom{n-1}{3}\binom{n}{2}\Big] p^{4(n-1)!-3(n-2)!} \\
& + \Big[6\binom{n-1}{2}\binom{n}{3} + 6\binom{n-1}{3}\binom{n}{2} \\
& +\binom{n-1}{2}\binom{n}{2}\Big] p^{4(n-1)!-2(n-2)!} \\
& + \Big[\binom{n-1}{2}\binom{n}{2}\binom{n-2}{2} + 9\binom{n-1}{3}\binom{n}{3} \\
& +6\binom{n-1}{4}\binom{n}{2}\Big] p^{4(n-1)!-4(n-2)!} + \Big[36\binom{n-1}{3}\binom{n}{4} \\
& +36\binom{n-1}{4}\binom{n}{3}\Big] p^{4(n-1)!-5(n-2)!+2(n-3)!} \\
& +24\binom{n-1}{4}\binom{n}{4} p^{4(n-1)!-6(n-2)!+4(n-3)!-(n-4)!} \\
& +36\binom{n-1}{3}\binom{n}{3} p^{4(n-1)!-4(n-2)!+(n-3)!}.
\end{aligned}$$

To compute the probability of event $\xi^i_{n-1} \cap \xi^j_{n-1} \cap \xi^k_{n-1} \cap \xi^l_{n-1}$, we have to enumerate all possible union of $S^i_{n-1}$, $S^j_{n-1}$, $S^k_{n-1}$, and $S^l_{n-1}$. For the sake of clarity, we associate $S^i_{n-1}, S^j_{n-1}, S^k_{n-1}, S^l_{n-1}$ with $S_n^{(i_1:x_1)}, S_n^{(i_2:x_2)}, S_n^{(i_3:x_3)}, S_n^{(i_4:x_4)}$, respectively.

**Case 1**: All of $i_1$, $i_2$, $i_3$, and $i_4$ are the same; that is, $i_1 = i_2 = i_3 = i_4$. Obviously, we have $|\{x_1,x_2,x_3,x_4\}| = 4$, so $S_n^{(i_1:x_1)}$, $S_n^{(i_2:x_2)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$ are mutually disjoint. Then, $Pr\left(\xi_{n-1}^i \cap \xi_{n-1}^j \cap \xi_{n-1}^k \cap \xi_{n-1}^l\right) = p^{4(n-1)!}$. Figure 2(a) illustrates the union of $S_n^{(i_1:x_1)}$, $S_n^{(i_2:x_2)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$. Accordingly, there are $\binom{n-1}{1}\binom{n}{4}$ groups of four $S_{n-1}$-subgraphs corresponding to this union type.

**Case 2**: At least two of $i_1$, $i_2$, $i_3$, and $i_4$ are different. Without loss of generality, we assume $i_1 = s$ is different from $i_4 = t$.

**Subcase 2.1**: Both $i_2$ and $i_3$ are in $\{s,t\}$; that is, $i_2, i_3 \in \{s,t\}$.

- $|\{x_1,x_2,x_3,x_4\}| = 4$.
  - Position $s$ or $t$ corresponds to three identifying codes; that is, either $i_1 = i_2 = i_3 = s$ or $i_2 = i_3 = i_4 = t$. For instance, if $i_2 = i_3 = i_4$, then $S_n^{(i_2:x_2)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$ are mutually disjoint. However, $S_n^{(i_1:x_1)}$ overlaps with each of $S_n^{(i_2:x_2)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$ (see Figure 2(b)). Clearly, $Pr\left(\xi_{n-1}^i \cap \xi_{n-1}^j \cap \xi_{n-1}^k \cap \xi_{n-1}^l\right) = p^{4(n-1)!-3(n-2)!}$. There are $\binom{n-1}{2}\binom{2}{1}\binom{n}{3}\binom{n-3}{1}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.
  - Both positions $s$ and $t$ correspond to two identifying codes. Without loss of generality, we assume that $i_1 = i_2$ and $i_3 = i_4$. Accordingly, $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$ are disjoint; $S_n^{(i_3:x_3)}$ and $S_n^{(i_4:x_4)}$ are disjoint. However, both $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$ overlap with $S_n^{(i_3:x_3)}$ and $S_n^{(i_4:x_4)}$ (see Figure 2(c)). Clearly, $Pr\left(\xi_{n-1}^i \cap \xi_{n-1}^j \cap \xi_{n-1}^k \cap \xi_{n-1}^l\right) = p^{4(n-1)!-4(n-2)!}$. There are $\binom{n-1}{2}\binom{n}{2}\binom{n-2}{2}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.
- $|\{x_1,x_2,x_3,x_4\}| = 3$.
  - Position $s$ or $t$ corresponds to three identifying codes; that is, either $i_1 = i_2 = i_3 = s$ or $i_2 = i_3 = i_4 = t$. Suppose that $i_2 = i_3 = i_4$. Then, $x_2$, $x_3$, and $x_4$ are distinct, so $S_n^{(i_2:x_2)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$ are mutually disjoint. Since $|\{x_1,x_2,x_3,x_4\}| = 3$, we have $x_1 \in \{x_2,x_3,x_4\}$. Without loss of generality, we assume $x_1 = x_2$. Thus, $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$ are disjoint, and $S_n^{(i_1:x_1)}$ overlaps with both $S_n^{(i_3:x_3)}$ and $S_n^{(i_4:x_4)}$ (see Figure 2(d)). Clearly, $Pr\left(\xi_{n-1}^i \cap \xi_{n-1}^j \cap \xi_{n-1}^k \cap \xi_{n-1}^l\right) = p^{4(n-1)!-2(n-2)!}$. There are $\binom{n-1}{2}\binom{2}{1}\binom{n}{3}\binom{3}{1} = 6\binom{n-1}{2}\binom{n}{3}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.
  - Both positions $s$ and $t$ correspond to two identifying codes. Without loss of generality, we assume $i_1 = i_2$ and $i_3 = i_4$. Accordingly, $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$ are disjoint; $S_n^{(i_3:x_3)}$ and $S_n^{(i_4:x_4)}$ are disjoint. Since $|\{x_1,x_2,x_3,x_4\}| = 3$, we have $\{x_1,x_2\} \cap \{x_3,x_4\} \neq \emptyset$. Without loss of generality, we further assume that $x_2 = x_4$. Thus, $S_n^{(i_2:x_2)}$ and $S_n^{(i_4:x_4)}$ are also disjoint. Moreover, $S_n^{(i_1:x_1)}$ overlaps with both $S_n^{(i_3:x_3)}$ and $S_n^{(i_4:x_4)}$, and $S_n^{(i_3:x_3)}$ overlaps with both $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$. See Figure 2(e). Clearly, $Pr\left(\xi_{n-1}^i \cap \xi_{n-1}^j \cap \xi_{n-1}^k \cap \xi_{n-1}^l\right) = p^{4(n-1)!-3(n-2)!}$. There are $\binom{n-1}{2}\binom{n}{3}\binom{3}{1} \times 2! = 6\binom{n-1}{2}\binom{n}{3}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.
- $|\{x_1,x_2,x_3,x_4\}| = 2$. Let $\{a,b\} = \{x_1,x_2,x_3,x_4\}$. Without loss of generality, we assume $i_1 = i_2$ and $i_3 = i_4$, so we obtain $\{x_1,x_2\} = \{x_3,x_4\} = \{a,b\}$. We further assume that $x_1 = x_3 = a$ and $x_2 = x_4 = b$. Thus, $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$ are disjoint; $S_n^{(i_1:x_1)}$ and $S_n^{(i_3:x_3)}$ are disjoint; $S_n^{(i_3:x_3)}$ and $S_n^{(i_4:x_4)}$ are disjoint; $S_n^{(i_2:x_2)}$ and $S_n^{(i_4:x_4)}$ are disjoint. However, $S_n^{(i_1:x_1)}$ overlaps with $S_n^{(i_4:x_4)}$, and $S_n^{(i_2:x_2)}$ overlaps with $S_n^{(i_3:x_3)}$. See Figure 2(f). Clearly, $Pr\left(\xi_{n-1}^i \cap \xi_{n-1}^j \cap \xi_{n-1}^k \cap \xi_{n-1}^l\right) = p^{4(n-1)!-2(n-2)!}$. There are $\binom{n-1}{2}\binom{n}{2}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.

**Subcase 2.2**: Only one of $i_2$ and $i_3$ is in $\{s,t\} = \{i_1,i_4\}$. Without loss of generality, we assume that $i_2 \notin \{s,t\}$ and $i_3 = i_4 = t$. Thus, $x_3$ is different from $x_4$, so $S_n^{(i_3:x_3)}$ and $S_n^{(i_4:x_4)}$ are disjoint.

- $|\{x_1,x_2,x_3,x_4\}| = 4$. Obviously, both $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$ overlap with the others (see Figure 2(g)). Clearly, $Pr\left(\xi_{n-1}^{i}\cap\xi_{n-1}^{j}\cap\xi_{n-1}^{k}\cap\xi_{n-1}^{l}\right) = p^{4(n-1)!-5(n-2)!+2(n-3)!}$. Then, there are $\binom{n-1}{3}\binom{3}{1}\binom{n}{4}\binom{4}{2}\times 2! = 36\binom{n-1}{3}\binom{n}{4}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.

- $|\{x_1,x_2,x_3,x_4\}| = 3$.
  - $\{x_1,x_2\}\cap\{x_3,x_4\}=\emptyset$. Then, we have $x_1 = x_2$ so that $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$ are disjoint. Moreover, both $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$ overlap with $S_n^{(i_3:x_3)}$ and $S_n^{(i_4:x_4)}$. This union type is the same with that in Figure 2(c), and clearly, $Pr\left(\xi_{n-1}^{i}\cap\xi_{n-1}^{j}\cap\xi_{n-1}^{k}\cap\xi_{n-1}^{l}\right) = p^{4(n-1)!-4(n-2)!}$. There are $\binom{n-1}{3}\binom{3}{1}\binom{n}{3}\binom{3}{2} = 9\binom{n-1}{3}\binom{n}{3}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.
  - $\{x_1,x_2\}\cap\{x_3,x_4\}\neq\emptyset$. Without loss of generality, we assume that $x_2 = x_3$. Then, we have $x_1\notin\{x_3,x_4\}$. Thus, $S_n^{(i_2:x_2)}$ and $S_n^{(i_3:x_3)}$ are disjoint, but $S_n^{(i_2:x_2)}$ overlaps with $S_n^{(i_4:x_4)}$. Furthermore, $S_n^{(i_1:x_1)}$ overlaps with each of $S_n^{(i_2:x_2)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$. See Figure 2(h). Clearly, $Pr\left(\xi_{n-1}^{i}\cap\xi_{n-1}^{j}\cap\xi_{n-1}^{k}\cap\xi_{n-1}^{l}\right) = p^{4(n-1)!-4(n-2)!+(n-3)!}$. There are $\binom{n-1}{3}\binom{3}{1}\binom{n}{3}\binom{3}{2}\binom{2}{1}\times 2! = 36\binom{n-1}{3}\binom{n}{3}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.
- $|\{x_1,x_2,x_3,x_4\}| = 2$.
  - $x_1 = x_2$. Since $x_3\neq x_4$, we assume that $x_1 = x_2 = x_3$. Thus, $S_n^{(i_1:x_1)}$, $S_n^{(i_2:x_2)}$, and $S_n^{(i_3:x_3)}$ are mutually disjoint. Moreover, $S_n^{(i_4:x_4)}$ overlaps with both $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$. See Figure 2(d). Clearly, $Pr\left(\xi_{n-1}^{i}\cap\xi_{n-1}^{j}\cap\xi_{n-1}^{k}\cap\xi_{n-1}^{l}\right) = p^{4(n-1)!-2(n-2)!}$. There are $\binom{n-1}{3}\binom{3}{1}\binom{n}{2}\binom{2}{1} = 6\binom{n-1}{3}\binom{n}{2}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.
  - $x_1\neq x_2$. Without loss of generality, we assume that $x_1 = x_3$ and $x_2 = x_4$. Then, $S_n^{(i_1:x_1)}$ and $S_n^{(i_3:x_3)}$ are disjoint; $S_n^{(i_2:x_2)}$ and $S_n^{(i_4:x_4)}$ are disjoint. Figure 2(e) illustrates this union type. Clearly, $Pr\left(\xi_{n-1}^{i}\cap\xi_{n-1}^{j}\cap\xi_{n-1}^{k}\cap\xi_{n-1}^{l}\right) = p^{4(n-1)!-3(n-2)!}$. There are $\binom{n-1}{3}\binom{3}{1}\binom{n}{2}\times 2! = 6\binom{n-1}{3}\binom{n}{2}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.

**Subcase 2.3**: None of $i_2$ and $i_3$ is in $\{s,t\} = \{i_1,i_4\}$; that is, every two of $i_1$, $i_2$, $i_3$, and $i_4$ are different.

- $|\{x_1,x_2,x_3,x_4\}| = 4$. Obviously, each of $S_n^{(i_1:x_1)}$, $S_n^{(i_2:x_2)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$ overlaps with the others (see Figure 2(i)). Clearly, $Pr\left(\xi_{n-1}^{i}\cap\xi_{n-1}^{j}\cap\xi_{n-1}^{k}\cap\xi_{n-1}^{l}\right) = p^{4(n-1)!-6(n-2)!+4(n-3)!-(n-4)!}$. There are $\binom{n-1}{4}\binom{n}{4}\times 4! = 24\binom{n-1}{4}\binom{n}{4}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.
- $|\{x_1,x_2,x_3,x_4\}| = 3$. Without loss of generality, we assume that $x_3 = x_4$. Thus, $S_n^{(i_3:x_3)}$ and $S_n^{(i_4:x_4)}$ are disjoint. However, $S_n^{(i_1:x_1)}$ overlaps with each of $S_n^{(i_2:x_2)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$; $S_n^{(i_2:x_2)}$ overlaps with each of $S_n^{(i_1:x_1)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$. This union type is the same with that in Figure 2(g), and clearly, $Pr\left(\xi_{n-1}^{i}\cap\xi_{n-1}^{j}\cap\xi_{n-1}^{k}\cap\xi_{n-1}^{l}\right) = p^{4(n-1)!-5(n-2)!+2(n-3)!}$. There are $\binom{n-1}{4}\binom{n}{3}\binom{3}{1}\binom{4}{2}\times 2! = 36\binom{n-1}{4}\binom{n}{3}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.
- $|\{x_1,x_2,x_3,x_4\}| = 2$. For convenience, let $\{x_1,x_2,x_3,x_4\} = \{a,b\}$.
  - One identifying code, $a$ or $b$, is associated with three positions. Without loss of generality, we assume that $x_1 = a$ and $x_2 = x_3 = x_4 = b$. Consequently, $S_n^{(i_2:x_2)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$ are mutually disjoint, and $S_n^{(i_1:x_1)}$ overlaps with every of $S_n^{(i_2:x_2)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$. This union type is identical to that in Figure 2(b), and $Pr\left(\xi_{n-1}^{i}\cap\xi_{n-1}^{j}\cap\xi_{n-1}^{k}\cap\xi_{n-1}^{l}\right) = p^{4(n-1)!-3(n-2)!}$. There are $\binom{n-1}{4}\binom{n}{2}[\binom{4}{1}+\binom{4}{3}] = 8\binom{n-1}{4}\binom{n}{2}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.
  - Both identifying codes $a$ and $b$ are associated with two positions. Without loss of generality, we assume that

$x_1 = x_2 = a$ and $x_3 = x_4 = b$. Thus, $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$ are disjoint; $S_n^{(i_3:x_3)}$ and $S_n^{(i_4:x_4)}$ are disjoint. Furthermore, both $S_n^{(i_1:x_1)}$ and $S_n^{(i_2:x_2)}$ overlap with $S_n^{(i_3:x_3)}$ and $S_n^{(i_4:x_4)}$. This union type is identical to that in Figure 2(c), and $Pr\left(\xi_{n-1}^i \cap \xi_{n-1}^j \cap \xi_{n-1}^k \cap \xi_{n-1}^l\right) = p^{4(n-1)!-4(n-2)!}$. There are $\binom{n-1}{4}\binom{n}{2}\frac{4!}{2!2!} = 6\binom{n-1}{4}\binom{n}{2}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.

- $|\{x_1, x_2, x_3, x_4\}| = 1$. Obviously, $S_n^{(i_1:x_1)}$, $S_n^{(i_2:x_2)}$, $S_n^{(i_3:x_3)}$, and $S_n^{(i_4:x_4)}$ are mutually disjoint (see Figure 2(a)). There are $\binom{n-1}{4}\binom{n}{1}$ distinct groups of four $S_{n-1}$-subgraphs leading to this union type.

All possible scenarios for four out of $n(n-1)$ $S_{n-1}$-subgraphs in $S_n$ have been categorized. Denote by $g(n,p)$ the right-hand side of Eq. (3). A lower bound of $R_{n,n-1}(p)$ is summarized in the following theorem.

**Theorem 1.** *Given a homogeneous node reliability $p$ of $S_n$, a lower bound on $R_{n,n-1}(p)$ is $g(n,p) - \delta(n,p)$. That is,*

$$R_{n,n-1}(p) \quad \geq \quad g(n,p) - \delta(n,p). \tag{4}$$

## 4 Numerical comparisons

Wu and Latifi [22] adopted a node failure distribution such that the number of faulty nodes increases over time. Their model assumes that the expected number of faulty nodes at the moment $t$, denoted by $f(t)$, increases as time passes with a constant failure rate $\lambda$. Then, $f(t)$ is given as follows:

$$f(t) = n! \times (1 - e^{-\lambda t}).$$

Therefore, the node reliability function at the moment $t$, denoted by $p(t)$, can be expressed by the following formula:

$$p(t) = 1 - \frac{f(t)}{n!} = e^{-\lambda t}. \tag{5}$$

Figure 3 plots the three estimates, upper bound, lower bound, and approximation of $R_{n,n-1}(p)$, for a variety of $n$ and $p$. According to our numerical results, the curves of these estimates are in reasonable agreement. In particular, the lower and upper bounds get close to each other rapidly as time goes by. This implies that the proposed lower bound of $R_{n,n-1}(p)$ really plays a more informative role than the upper bound of $R_{n,n-1}(p)$ in achieving system availability, especially when a specific substar is available for a user's request to execute his/her programs in the current star network system.

## 5 Concluding remarks

In this paper, an analytic lower bound on the substar reliability of star networks is derived by means of combinatorial approach. The lower bound on $R_{n,n-1}(p)$ plays a more informative role than the upper bound in evaluating the degree of system availability, especially when a specific substar is available for a user to execute his/her programs in the current star network system. Numerical comparisons are presented for validating the proposed formulation.

## References


[1] S. B. Akers and B. Krishnamurthy, A group theoretic model for symmetric interconnection networks, *IEEE Trans. Computers*, Vol. 38, No. 4, pp. 555-566, Apr. 1989.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, London, 2008.

[3] Y. Chang and L. N. Bhuyan, A combinatorial analysis of subcube reliability in hypercube, *IEEE Trans. Computers*, Vol. 44, No. 7, pp. 952-956, Jul. 1995.

[4] E. Cheng and M. J. Lipman, On the day-tripathi orientation of the star graphs: Connectivity, *Inform. Process. Lett.*, vol. 73, no. 1-2, pp. 5-10, 2000.

[5] E. Cheng and M. J. Lipman, Vulnerability issues of star graphs, alternating group graphs and split-stars: Strength and toughness, *Discrete Applied Mathematics*, Vol. 118, No. 3, pp. 163-179, 2002.

[6] E. Cheng, M. J. Lipman, and H. A. Park, Super connectivity of star graphs, alternating group graphs and split-stars, *Ars Combinatoria*, Vol. 59, pp. 107-116, 2001.

[7] C. R. Das and J. Kim, A unified task-based dependability model for hypercube computers, *IEEE Trans. Parallel and Distributed*

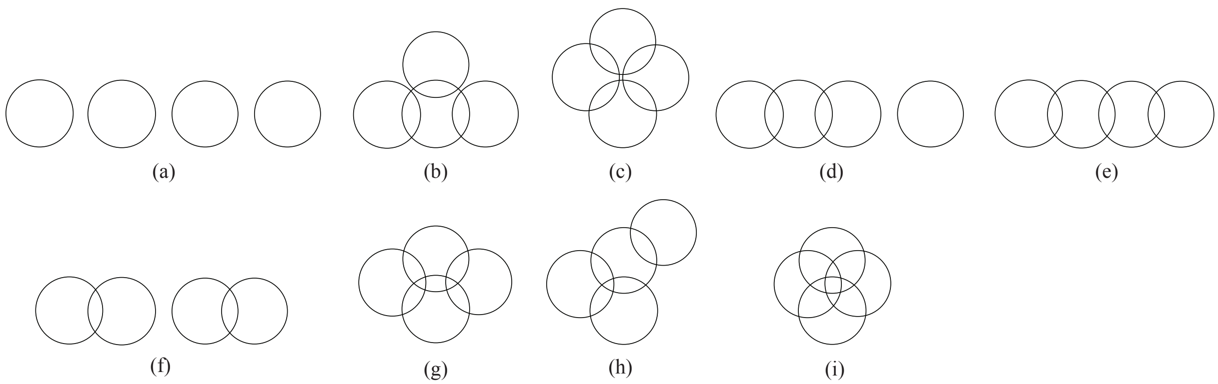


Figure 2: Union types of four out of $n(n-1)$ $S_{n-1}$-subgraphs in $S_n$.

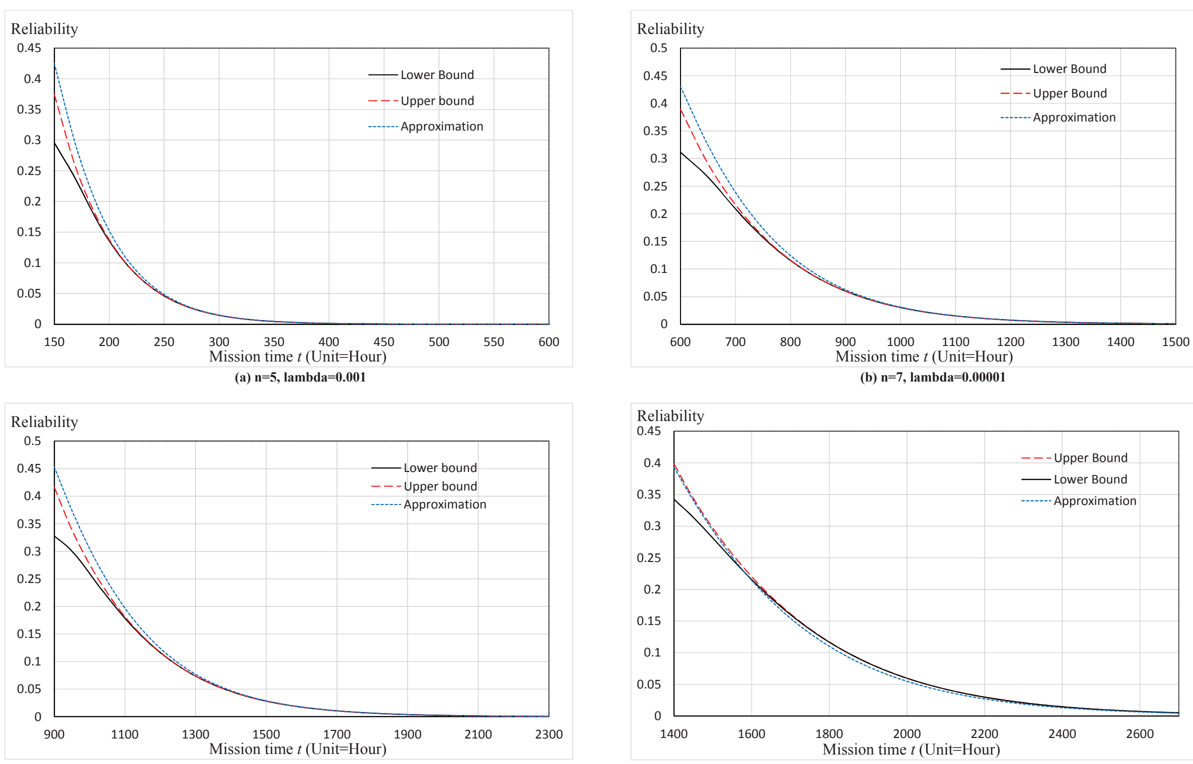


Figure 3: Reliability estimations of $R_{n,n-1}(p)$ for $S_n$, where $n = 5, 7, 8, 10$.

*Systems*, Vol. 3, No. 3, pp. 312-324, Mar. 1992.

[8] K. Day and A. Tripathi, A comparative study of topologies properties of hypercubes and star networks, *IEEE Trans. Parallel and Distributed Systems*, Vol. 5, No. 1, pp. 31-38, 1994.

[9] K. Fitzgerald and S. Latifi, Reliability modeling and assessment of the star-graph networks, *IEEE Trans. Reliability*, Vol. 51, No. 1, pp. 49-57, Mar. 2002.

[10] S.-Y. Hsieh, Embedding longest fault-free paths onto star graphs with more vertex faults, *Theoretical Computer Science*, Vol. 337, pp. 370-378, 2005.

[11] P. Kales, *Reliability: For Technology, Engineering, and Management*, Prentice-Hall, New Jersey, 1998.

[12] T.-L. Kung, C.-K. Lin, T. Liang, J. J. M. Tan, and L.-H. Hsu, Fault-free mutually independent hamiltonian cycles of faulty star graphs, *International Journal of Computer Mathematics*, Vol. 88, No. 4, pp. 731-746, 2011.

[13] S.-Y. Kuo, C.-Y. Huang, and M.-R. Lyu, Framework for modeling software reliability using various testing-efforts and fault-detection rates, *IEEE Trans. Reliability*, Vol. 50, No. 3, pp. 310-320, Mar. 2001.

[14] S. Latifi, On the fault-diameter of the star network, *Information Processing Letters*, Vol. 46, No. 3, pp. 143-150, 1993.

[15] S. Latifi, A study of fault tolerance in star graph, *Information Processing Letters*, Vol. 102, No. 5, pp. 196-200, 2007.

[16] S. Latifi, E. Saberinia, and X. L. Wu, Robustness of star graph network under link failure, *Information Sciences*, Vol. 178, pp. 802-806, 2008.

[17] T.-K. Li, J. J. M. Tan, and L.-H. Hsu, Hyper hamiltonian laceability on edge fault star graph, *Information Sciences*, Vol. 165, No. 1-3, pp. 59-71, 2004.

[18] C.-K. Lin, T.-L. Kung, J. J. M. Tan, Conditional-fault diagnosability of multiprocessor systems with an efficient local diagnosis algorithm under the PMC model, *IEEE Trans. Parallel and Distributed Systems*, Vol. 22, No. 10, pp. 1669-1680, Oct. 2011.

[19] L. Lin, L. Xu, S. Zhou, D. Wang, The reliability of subgraph in the arrangement graph, *IEEE Trans. Reliability*, Vol. 64, No. 2, pp. 807-818, Jun. 2015.

[20] S. Soh, S. Rai, and J. L. Trahan, Improved lower bounds on the reliability of hypercube architectures, *IEEE Trans. Parallel and Distributed Systems*, Vol. 5, No. 4, pp. 364-378, Apr. 1994.

[21] S. Sur and P.K. Srimani, Topological properties of star graph, *Computers and Mathematics with Applications*, Vol. 25, No. 12, pp. 87-98, 1993.

[22] X. Wu and S. Latifi, Substar reliability analysis in star networks, *Information Sciences*, Vol. 178, pp. 2337-2348, 2008.

[23] S. Zarezadeh and M. Asadi, Network reliability modeling under stochastic process of component failures, *IEEE Trans. Reliability*, Vol. 62, No. 4, pp. 917-929, Dec. 2013.

[24] J. Zheng, S. Latifi, E. Regentova, K. Luo, and X. L. Wu, Diagnosability of star graphs under the comparison diagnosis model, *Information Processing Letters*, Vol. 93, No. 1, pp. 29-36, 2005.