On tree-4-colorable graphs

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Abstract

A k-coloring of an undirected graph G is said to be a tree-k-coloring if the subgraph of G induced by any two color classes is a tree. This paper proves an upper bound on the number of tree-4-colorable n-vertex graphs.

1. Introduction

In this paper all graphs considered are finite, simple and undirected. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a simple undirected graph G, respectively.

A k-coloring of G is a function $c: V(G) \to \{1,2,\ldots,k\}$ such that no two adjacent vertices have the same color, where each vertex $v \in V(G)$ is assigned the color $c(v)$ [1]. A color class means the set of vertices of a particular color. If a graph G has a k-coloring, it is k-colorable. A k-coloring $c: V(G) \to \{1, 2, \ldots, k\}$ is said to be a tree-k-coloring if the subgraph of G induced by $\{v \in V(G) \mid c(v) \in \{i, j\}\}$ is a tree for all distinct i, $j \in \{1, 2, \ldots, k\}$ [2].

Cayley's formula says that the number of n-vertex trees is n^{n-2}. Examples are shown in Figs. 1–2.

In this paper, we prove an upper bound on the number of tree-4-colorable n-vertex graphs.

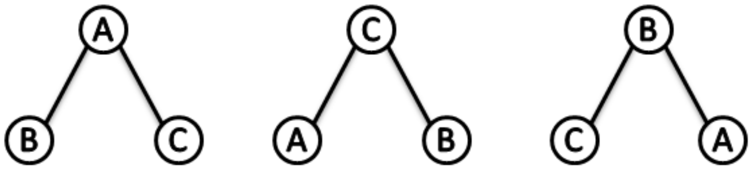

Fig. 1. Three vertices constitute 3^{3-2} different trees.

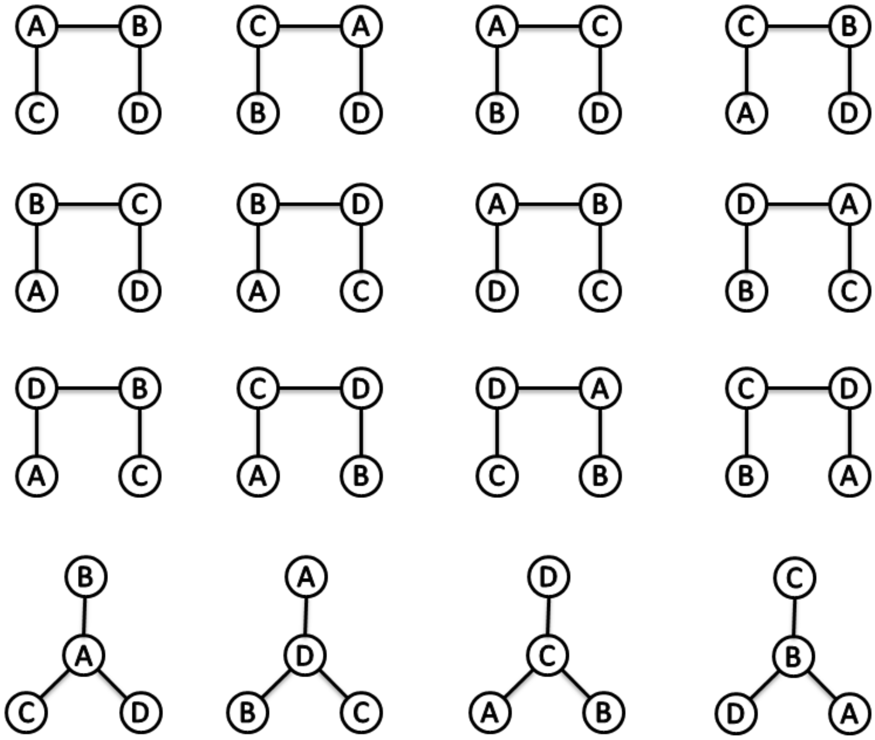

Fig. 2. Four vertices constitute 4^{4-2} different trees.

2. The number of tree-4-colorable n-vertex graphs

Recall that a tree-k-coloring of G is a k-coloring $c: V(G) \to \{1,2,\dots,k\}$ such that the subgraph of G induced by $\{v \in V(G) \mid c(v) \in \{i,j\}\}$ is a tree for all distinct $i,\ j \in \{1,2,\dots,k\}$.

Theorem 1. For all $n \in \mathbb{Z}^+$, there are at most

$$\sum_{t=1}^{n-1} (t^{t-2}) \binom{n}{t} (n-t)^{(n-t-2)} \binom{t(n-t)}{2n-4}$$

tree-4-colorable n-vertex graphs.

Proof. Suppose A, B, C and D are four different colors. Take any tree-4-colorable n-vertex graph G and a tree-k-coloring of G.

For any two distinct colors $X, Y \in \{A, B, C, D\}$, let $G_{X,Y}$ be the subgraph of G induced by the vertices of colors in $\{X, Y\}$. Denote by t the number of vertices of $G_{A,B}$. Then $1 \leq t \leq n-1$. So the vertex set of $G_{A,B}$ is one of the $\binom{n}{t}$ subsets of $V(G)$ of size t. By Cayley's formula, $G_{A,B}$ is one of the t^{t-2} trees formed by its vertices and $G_{C,D}$ is one of the $(n-t)^{n-t-2}$ trees formed by its vertices.

For each $X \in \{A, B, C, D\}$, let $V(X)$ be the set of vertices of color X. For all distinct $X, Y \in \{A, B, C, D\}$,

$$|V(X)| + |V(Y)| = |V(G_{X,Y})|.$$

Therefore,

$$|V(G_{A,C})| + |V(G_{A,D})| + |V(G_{B,C})| + |V(G_{B,D})| = 2(|V(A)| + |V(B)| + |V(C)| + |V(D)|) = 2n. \quad (1)$$

For every tree T, $|E(T)| = |V(T)| - 1$. So by equation (1),

$$|E(G_{A,C})| + |E(G_{A,D})| + |E(G_{B,C})| + |E(G_{B,D})| = 2n - 4. \quad (2)$$

As $|V_{A,B}| = t$ and $|V_{C,D}| = n - t$, $V_{A,B} \times V_{C,D}$ contains $t(n-t)$ pairs. So by equation (2), the set of edges of G in $V_{A,B} \times V_{C,D}$ is one of the $\binom{t(n-t)}{2n-4}$ subsets of $V_{A,B} \times V_{C,D}$. ■

Acknowledgments

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References

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