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The (n, k)-star graph: A generalized star graph

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1. Introduction

The n-star graph [1] is an attractive alternative to the n-cube. It has significant advantages over the n-cube, such as a lower degree and a smaller diameter. However, a major practical difficulty with the n-star graph is the restriction on the number of nodes: $n!$ for an n-star graph. Since there is a large gap between $n!$ and $(n+1)!$, one may face the choice of either too few or too many available nodes.

The objective of this paper is to propose a new topology, called the (n, k)-star graph, such that it removes the restriction of the number of nodes $n!$ in the n-star graph, and preserves many attractive properties of the n-star graph such as node symmetry, hierarchical structure, maximal fault tolerance, and simple shortest routing.

The (n, k)-star graph is a generalized version of the n-star graph. The two parameters n and k can be tuned to make a suitable choice for the number of nodes in the network and for the degree/diameter tradeoff. This allows more flexibility in designing network topology than the star graph. The (n, k)-star graph is regular of degree $n-1$, the number of nodes $n!/(n-k)!$, and diameter $2k-1$ for $k \leqslant \lfloor n/2 \rfloor$ and $\lfloor (n-1)/2 \rfloor + k$ for $k \geqslant \lfloor n/2 \rfloor + 1$. In addition, the $(n, n-1)$-star graph is isomorphic to the n-star graph, and hence, all these properties can be derived for the n-star graph as it is a special case of the (n, k)-star graph. Moreover, many parallel algorithms [2] for the n-star graph may adapt to the (n, k)-star graph with a slight modification.

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2. Network topology and basic properties

For simplicity, denote $\langle n \rangle = \{1, 2, \ldots, n\}$.

Definition 1. An *(n, k)-star graph*, denoted by $S_{n,k}$, is specified by two integers n and k, where $1 \leqslant k < n$. The node set of $S_{n,k}$ is denoted by $\{p_1 p_2 \ldots p_k \mid p_i \in \langle n \rangle \text{ and } p_i \neq p_j \text{ for } i \neq j\}$. The adjacency is defined as follows: $p_1 p_2 \ldots p_i \ldots p_k$ is adjacent to (1) $p_i p_2 \ldots p_1 \ldots p_k$ through an edge of dimension i, where $2 \leqslant i \leqslant k$ (swap p_1 with p_i), and (2) $x p_2 \ldots p_k$ through dimension 1, where $x \in \langle n \rangle - \{p_i \mid 1 \leqslant i \leqslant k\}$. The edges of type (1) are referred to as i-edges, and the edges of type (2) are referred to as 1-edges. A $(4, 2)$-star graph is shown in Fig. 1.

Definition 2. A graph is *node symmetric* if and only if for its any pair of nodes u and v, there exists an automorphism of the graph that maps u to v.

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Theorem 3. *The (n,k)-star graph is node symmetric.*

Proof. We need to show that given any two nodes in $S_{n,k}$ there exists an automorphism that maps one node into the other. Let $p = p_1p_2\dots p_k$ and $q = q_1q_2\dots q_k$ be the two nodes in $S_{n,k}$, $P = \{p_1, p_2, \dots, p_k\}$, $Q = \{q_1, q_2, \dots, q_k\}$, and $|X|$ denote the number of elements in the set X.

Define the one-to-one onto mapping μ_1 in $\langle n\rangle$:

- $\mu_1(p_i) = q_i$, for $1 \leqslant i \leqslant k$, i.e., for $p_i \in P$;
- $\mu_1(x) = y$, one-to-one mapping for $x \in Q - P$ and $y \in P - Q$ (since $|P| = |Q|$, $|Q - P| = |Q| - |P \cap Q| = |P| - |P \cap Q| = |P - Q|$);
- $\mu_1(z) = z$, for $z \in \langle n\rangle - P \cup Q$.

Let M_1 be the one-to-one onto mapping in $S_{n,k}$:

$$M_1(t) = \mu_1(t_1)\mu_1(t_2)\dots\mu_1(t_k)$$

for all $t = t_1t_2\dots t_k$ in $S_{n,k}$.

Clearly M_1 maps the node p into the node q. Further, M_1 is an automorphism, for if two nodes s and t are adjacent in $S_{n,k}$, then $M_1(s)$ and $M_1(t)$ are adjacent. More precisely, let $s = s_1s_2\dots s_i\dots s_k$, then $t = s_is_2\dots s_1\dots s_k$ (swap s_1 with s_i) or $t_1s_2\dots s_i\dots s_k$ ($t_1 \in \langle n\rangle - \{s_i \mid 1 \leqslant i \leqslant k\}$). Consider

$$M_1(s) = \mu_1(s_1)\mu_1(s_2)\dots\mu_1(s_i)\dots\mu_1(s_k)$$

and

$$M_1(t) = \mu_1(t_1)\mu_1(s_2)\dots\mu_1(s_i)\dots\mu_1(s_k) \quad \text{or}$$
$$\mu_1(s_i)\mu_1(s_2)\dots\mu_1(s_1)\dots\mu_1(s_k),$$

so $M_1(s)$ and $M_1(t)$ are adjacent. □

Note that $S_{n,k}$ could not be edge symmetric. For instance, in Fig. 1, each 2-edge belongs to a cycle of length at least 6, but each 1-edge may be within a cycle of length 3.

Lemma 4. *The $(n, n-1)$-star graph $S_{n,n-1}$ is isomorphic to the n-star graph S_n.*

Proof. To prove that $S_{n,n-1}$ and S_n are isomorphic, we remove the last symbol in all nodes of S_n, and obtain an $S_{n,n-1}$ by Definition 1. That is, we define a bijection M_2 from the nodes of S_n to those of $S_{n,n-1}$ by:

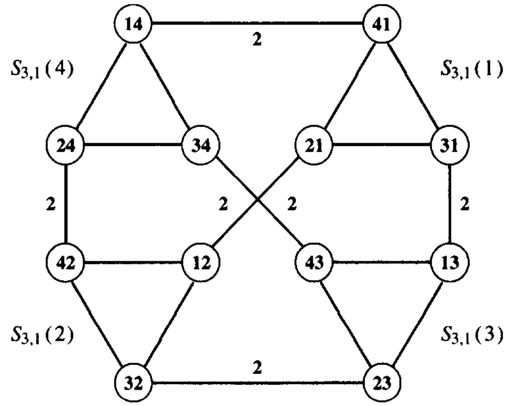

Fig. 1. A (4,2)-star graph.

$$M_2(p_1p_2\dots p_{n-1}p_n) = p_1p_2\dots p_{n-1}$$

for a node $p = p_1p_2\dots p_{n-1}p_n$ of S_n.

Moreover, M_2 preserves adjacency. Let p and q are two nodes joined with an i-edge in S_n. If $2 \leqslant i \leqslant n-1$, then $M_2(p)$ and $M_2(q)$ are also joined with an i-edge in $S_{n,n-1}$; if $i = n$, then $M_2(p)$ and $M_2(q)$ are joined with a 1-edge in $S_{n,n-1}$. □

3. Hierarchical structure and fault tolerance

Definition 5. Let $S_{n-1,k-1}(i)$ denote a subgraph of $S_{n,k}$ induced by all the nodes with the same last symbol i, for some $1 \leqslant i \leqslant n$.

Lemma 6. *$S_{n,k}$ can be decomposed into n subgraphs $S_{n-1,k-1}(i)$, $1 \leqslant i \leqslant n$, and each subgraph $S_{n-1,k-1}(i)$ is isomorphic to $S_{n-1,k-1}$.*

Proof. If we remove the last symbol of all the nodes in $S_{n-1,k-1}(n)$, we obtain an $S_{n-1,k-1}$. That is, $S_{n-1,k-1}(n)$ is isomorphic to $S_{n-1,k-1}$. So, we only need to show $S_{n-1,k-1}(i)$, $1 \leqslant i \leqslant n-1$, and $S_{n-1,k-1}(n)$ are isomorphic. We define the one-to-one mapping μ_3 in $\langle n\rangle$:

$$\mu_3(i) = n, \quad \mu_3(n) = i,$$
$$\mu_3(x) = x \text{ for } x \in \langle n\rangle - \{i, n\}.$$

We also define a bijection M_3 by:

$M_3(p_1p_2\dots p_k) = \mu_3(p_1)\mu_3(p_2)\dots\mu_3(p_k)$
for a node $p = p_1p_2\dots p_k$ in $S_{n,k}$.

Obviously, M_3 transforms the nodes of $S_{n-1,k-1}(i)$ into those of $S_{n-1,k-1}(n)$ and preserves adjacency. □

Since $S_{n,k}$ can be partitioned into n mutually disjoint subgraphs $S_{n-1,k-1}(i)$, $1 \leqslant i \leqslant n$, $S_{n,k}$ is hierarchical. Fig. 1 shows that $S_{4,2}$ can be viewed as an interconnection of four $S_{3,1}$'s through 2-edges.

Lemma 7. *There are $(n-2)!/(n-k)!$ k-edges between any two subgraphs $S_{n-1,k-1}(i)$ and $S_{n-1,k-1}(j)$; each of these nodes of $S_{n-1,k-1}(i)$ is connected to exactly one node in $S_{n-1,k-1}(j)$.*

Proof. For any pair of two nodes $jp_2\dots p_{k-1}i \in S_{n-1,k-1}(i)$ and $ip_2\dots p_{k-1}j \in S_{n-1,k-1}(j)$, $i \neq j$, there is a k-edge connecting them. Each $p_2p_3\dots p_{k-1}$ is a unique permutation of $k-2$ distinct symbols chosen out of the $n-2$ symbols of $\langle n\rangle - \{i,j\}$. Thus, we get the result. □

Definition 8. The *fault tolerance* of a graph G is defined as the maximum number f such that if any f nodes are deleted from G, the resulting subgraph is still connected.

Theorem 9. *The fault tolerance of the (n,k)-star graph is $n-2$.*

Proof. Let $f(S_{n,k})$ denote the fault tolerance of $S_{n,k}$. First, we claim $f(S_{n,k}) \geqslant f(S_{n-1,k-1}) + 1$ for $2 \leqslant k \leqslant n$. For brevity, let $f = f(S_{n-1,k-1})$. We prove it by showing that $S_{n,k}$ remains connected even if $f+1$ of its nodes are faulty. We consider two cases. In the first case, suppose that all of the faults are within a subgraph $S_{n-1,k-1}(i)$. Since the fault tolerance of $S_{n-1,k-1}$ is f, $S_{n-1,k-1}(i)$ could be disconnected because of the $f+1$ faults. But each nonfaulty node in $S_{n-1,k-1}(i)$ has a k-edge to the other copy $S_{n-1,k-1}$. All other $S_{n-1,k-1}$'s remain connected since they have no faults. Thus, $S_{n,k}$ remains connected.

Proceeding to the second case, suppose that the $f+1$ faults are distributed among more than one subgraph $S_{n-1,k-1}$ in $S_{n,k}$. Since there are at most f faults in any subgraph, each subgraph $S_{n-1,k-1}(i)$ remains connected. We merely need to prove that any two of n subgraphs $S_{n-1,k-1}(i)$, $1 \leqslant i \leqslant n$, remain connected to each other. By Lemma 7, each $S_{n-1,k-1}(i)$ has $(n-2)!/(n-k)!$ adjacent nodes in each of other $S_{n-1,k-1}(j)$'s, $j \neq i$. If we regard each $S_{n-1,k-1}(i)$ of $S_{n,k}$ as a supernode, then the resulting graph is a complete graph of n nodes. To disconnect it, we would have to remove at least $n-1$ nodes. So, if $S_{n,k}$ could be disconnected because of the $f+1$ faults, then $f+1 \geqslant (n-1)!/(n-k)!$. But $f+1 < \deg(S_{n-1,k-1}) + 1 = (n-2)+1 = n-1 \leqslant (n-1)!/(n-k)!$, for $2 \leqslant k \leqslant n$. This is a contradiction. Therefore, $S_{n,k}$ also remains connected.

Based on the preceding discussion, we have the following recurrence relation: $f(S_{n,k}) \geqslant f(S_{n-1,k-1}) + 1$, and the initial condition: $f(S_{n-k+1,1}) = n-k-1$ since $S_{n-k+1,1}$ is isomorphic to a complete graph of $n-k+1$ nodes. Thus, $f(S_{n,k}) \geqslant n-2$. The degree of $S_{n,k}$ is $n-1$, which implies $f(S_{n,k}) \leqslant n-2$. Therefore, $f(S_{n,k}) = n-2$. □

4. Routing path and diameter

Due to node symmetry of the (n,k)-star graph, any node can be mapped to the identity node $I_k = 12\dots k$ by renaming the symbols. For the routing between any two arbitrary nodes s and t, the renaming function M maps the destination node to the identity node, i.e., $M(t) = I_k$. Then all the paths between nodes s and t in the original graph are isomorphic to those between $M(s)$ and the identity I_k in the renamed graph. So, without loss of generality, considering the problem of the routing between two nodes in the (n,k)-star graph, the destination node is always assumed to be the identity node I_k.

Before solving the problem of routing between an arbitrary node p and the identity node I_k, we define a cycle representation for the label of each node in (n,k)-star graphs similar to the well-known cycle structure of permutation for star graphs [1].

To clarify our presentation, we call a symbol $\in \langle n\rangle - \langle k\rangle$ an external symbol since it is not used in the label of the identity node (destination). On the contrary, a symbol $\in \langle k\rangle$ is called internal. Unless stated otherwise, we use C_i (C'_i) to denote an internal (external) cycle and m_i (m'_i) to denote the number of elements in C_i (C'_i).

Let $p = p_1 p_2 \dots p_k$, where p_i denotes the symbol in the position i. The symbol in its correct position, i.e., $p_i = i$, is called invariant. In the following discussion, we omit all invariants in the cycle representation since they will not be moved during the shortest routing.

We construct the cycle representation of node p as follows. First, for each external symbol $x_{m_i'}$ in p we construct an external cycle $C_i' = (x_1, x_2, \dots, x_{m_i'})$ such that the desired position of x_j in p is held by x_{j+1} for $1 \leqslant j \leqslant m_i' - 1$. Additionally, for the cycle C_i' we define the desired symbol $d_{C_i'}$ whose desired position is held by the first element, x_1, of the cycle. When we have constructed the external cycles for all external symbols in p, the rest are all internal symbols. Then we construct the internal cycles for the rest as the same as those in the star graph. An internal cycle $C_i = (x_1, x_2, \dots, x_{m_i})$ of p means that the position of x_{j+1} in p is desired by x_j for $1 \leqslant j \leqslant m_i - 1$. For ease of illustration, if there exists a cycle containing p_1, we specially choose p_1 as the first element of the cycle since any cyclic shift of the sequence of symbols within each cycle is allowed.

In the cycle representation of a node p, cycles can appear in any order. So, the cycle representation of a node p with α internal cycles and β external symbols can be denoted as

$$C(p) = C_1 C_2 \dots C_\alpha C_1' C_2' \dots C_\beta' \quad \text{and} \quad \beta \geqslant 0.$$

We now demonstrate the cycle representation of a node in the (n, k)-star graph through an example. Consider node $p = 2968134$ in $S_{9,7}$. The cycle representation of p is $C(p) = C_1 C_1' C_2'$ where the cycles $C_1 = (3,6), C_1' = (2,9,1)$, and $C_2' = (4,8)$. The desired symbols of C_1' and C_2' are $d_1 = 5$ and $d_2 = 7$, respectively.

The routing from an arbitrary node p to the identity node I_k can be achieved by moving internal symbols to their correct positions and exchanging external symbols with desired symbols. Here, we correct the cycles of $C(p)$ one by one. If there exists a cycle containing p_1, we certainly correct it first.

In general, we correct an internal cycle $C_i = (x_1, x_2, \dots, x_{m_i})$ as follows. If $x_1 = p_1$, we directly move x_1 to its correct position (held by x_2), while x_2 is swapped to the first position. Then x_2 is taken to its correct position (held by x_3) and so on until $x_{m_i - 1}$ is taken to its correct position. Note that each element of a cycle except for the first element of the cycle will be moved to the first position as a result of the correction of the previous element in the cycle representation. Since $x_{m_i} = 1$, its correction can be considered as a result of the correction of $x_{m_i - 1}$. If $x_1 \neq p_1$, it requires an additional step to take x_1 into the first position of the label since a symbol must be in the first position before taken to its correct position. Then, we move $x_1, x_2, \dots, x_{m_i}$ to their correct positions in the same order. So, the number of steps required to correct each internal cycle C_i of length m_i, $1 \leqslant i \leqslant \alpha$, is

$$\begin{cases} m_i - 1 & \text{if } p_1 \in C_i, \\ m_i + 1 & \text{if } p_1 \notin C_i. \end{cases}$$

On the other hand, in order to correct an external cycle, we move the internal symbols into their correct positions by using the same argument as the preceding. But, when the only external symbol is taken to the first position, we exchange it with the desired symbol of any other uncorrected external cycle (if exists) or itself (otherwise). More precisely, let C_1' denote the first corrected but not completed external cycle. Only when all external cycles other than C_1' have been corrected completely, the external symbol in the first position is exchanged with d_1. Through this strategy, all the external cycles are corrected consecutively such that the additional steps to swap the first element of these cycles other than C_1' to the first position are reduced. Thus, the number of steps required to correct the β external cycles is

$$\begin{cases} m' + \beta - 1 & \text{if } p_1 \in \text{some } C_j', \\ m' + \beta + 1 & \text{if } p_1 \notin \text{any } C_j', \end{cases}$$

where $m' = \sum_{j=1}^{\beta} m_j'$.

Example. Consider the correction of node $p = 2968134$ in $S_{9,7}$. Following the above strategy, we first correct the external cycles $C_1' C_2' = (2,9,1)(4,8)$ along the path:

$$2968134 \to_2 9268134 \to_1 7268134 \to_7 4268137 \to_4 8264137 \to_1 5264137 \to_5 1264537.$$

Then the internal cycle $C_1 = (3,6)$ is corrected along the path:

$$1264537 \to_6 3264517 \to_3 6234517 \to_6 1234567.$$

Note that the subscripts indicate the dimensions of passed edges.

Definition 10. The *distance* between two nodes in G is the length of a shortest path joining them. The *diameter* of G is the largest distance between nodes of G.

Let c denote the total number of cycles of $C(p)$, where all the external cycles is regarded as a cycle in count. So, if $\beta > 0$, then $c = \alpha + 1$; if $\beta = 0$, then $c = \alpha$. Let e denote the number of external symbols in p, i.e., $e = \beta$. Let m denote the total number of misplaced symbols of p with respect to the identity I_k, i.e., $m = \sum_{i=1}^{\alpha} m_i + \sum_{j=1}^{\beta} m'_j = \sum_{i=1}^{\alpha} m_i + m'$.

Theorem 11. *The distance $d(p)$ from a node p to the identity node I_k in $S_{n,k}$ is given by*

$$d(p) = \begin{cases} c+m+e & \text{if } p_1 = 1, \\ c+m+e-2 & \text{if } p_1 \neq 1. \end{cases}$$

Proof. (a) $p_1 = 1$. There is no cycle containing p_1. If $\beta = 0$, then $d(p) = \sum_{i=1}^{\alpha}(m_i+1) = \alpha+m = c+m+e$. If $\beta > 0$, then $d(p) = \sum_{i=1}^{\alpha}(m_i+1) + \{\sum_{i=1}^{\beta}(m'_i+1)+1\} = \alpha+m+\beta+1 = c+m+e$.

(b) $p_1 \neq 1$. There exists a cycle containing p_1. If $\beta = 0$, then $p_1 \in$ some C_j, and $d(p) = (m_j - 1) + \{\sum_{i=1}^{\alpha}(m_i+1) - (m_j+1)\} = \alpha + m - 2 = c+m+e-2$. If $\beta > 0$ and $p_1 \in$ some C_j, then $d(p) = (m_j-1) + \{\sum_{i=1}^{\alpha}(m_i+1) - (m_j+1)\} + \{\sum_{i=1}^{\beta}(m'_i+1)+1\} = m+\alpha+\beta-1 = c+m+e-2$. If $\beta > 0$ and $p_1 \in C'_j$, it takes $\{\sum_{i=1}^{\beta}(m'_i+1)-1\}$ to correct all external cycles of p. Hence, $d(p) = \sum_{i=1}^{\alpha}(m_i+1)+\{\sum_{i=1}^{\beta}(m'_i+1)-1\} = \alpha+m+\beta-1 = c+m+e-2$. □

Corollary 12. *The distance $d(p)$ from a node p to the identity node I_k in S_n is given by*

$$d(p) = \begin{cases} c+m & \text{if } p_1 = 1, \\ c+m-2 & \text{if } p_1 \neq 1. \end{cases}$$

Proof. Based on Lemma 4, the distance from $p = p_1p_2\ldots p_n$ to $I_n = 12\ldots n$ in S_n is equivalent to the distance from $p' = p_1p_2\ldots p_{n-1}$ to $I_{n-1} = 12\ldots(n-1)$ in $S_{n,n-1}$. Let $c(c')$, $m(m')$, and $e(e')$ denote the numbers of cycles, misplaced symbols, and external symbols in $p(p')$ with respect to $I_n(I_{n-1})$.

(a) $p_1 = 1$. If $p_n = n$, then $c' = c, m' = m$, and $e' = 0$; thus, $d(p) = d(p') = c+m$. If $p_n \neq n$, then $c' = c, m' = m-1$, and $e' = 1$; thus, $d(p) = d(p') = c+m$.

(b) $p_1 \neq 1$. If $p_n = n$, then $c' = c, m' = m$, and $e' = 0$; thus, $d(p) = d(p') = c+m-2$. If $p_n \neq n$, then $c' = c, m' = m-1$, and $e' = 1$; thus, $d(p) = d(p') = c+m-2$. □

From the above, the distributed routing algorithm may be described by using repeatedly one of the following three rules until I_k is reached:

(R1) if symbol 1 is in the first position, then interchange it with any symbol not in its correct position,

(R2) if symbol i ($1 < i \leqslant k$) is in the first position, then move it to its correct position, and

(R3) if the external symbol of cycle C'_i is in the first position, then exchange it with the desired symbol, d_j, of any other uncorrected cycle C'_j.

Note that in rule 2, when an internal symbol is in the first position, we can move it to its correct position or interchange it with any symbol within another internal cycle. In addition, in rule 3, all the external cycles other than the cycle containing p_1 can be corrected in any order. So, for any symbol i, $1 \leqslant i \leqslant n$, in the first position, we has more than one choice for routing. This turns out to be an especially useful rule for routing in the presence of faults.

Theorem 13. *The diameter $D(S_{n,k})$ of $S_{n,k}$ is given by*

$$D(S_{n,k}) = \begin{cases} 2k-1 & \text{if } 1 \leqslant k \leqslant \left\lfloor \frac{n}{2} \right\rfloor, \\ k + \left\lfloor \frac{n-1}{2} \right\rfloor & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leqslant k \leqslant n-1. \end{cases}$$

Proof. The diameter $D(S_{n,k})$ is $max\{d(p) \mid p \in S_{n,k}\}$. We consider two cases as follows.

(a) For $1 \leqslant k \leqslant \lfloor n/2 \rfloor$: The diameter is obtained from one of the following two: (1) $p_1 = 1, c = 1, m = k-1$ and $e = k-1$, and (2) $p_1 \neq 1, c = 1, m = k$ and $e = k$. Therefore, $D(S_{n,k}) = 2k-1$.

(b) For $\lfloor n/2 \rfloor + 1 \leqslant k \leqslant n-1$: If n is odd, the diameter is obtained for $p_1 = 1$, $c = (2k-n-1)/2+1$, $m = k-1$, and $e = n-k$; thus, $D(S_{n,k}) =$

Table 1
Comparison of network topologies

Topology	Size	Degree	Diameter	Cost factor
S_n	$n!$	$n-1$	$\lfloor \frac{3(n-1)}{2} \rfloor$	$\approx \frac{3(n-1)^2}{2}$
AG_n	$\frac{n!}{2}$	$2(n-2)$	$\lfloor \frac{3(n-2)}{2} \rfloor$	$\approx 3(n-2)^2$
$A_{n,k}$	$\frac{n!}{(n-k)!}$	$k(n-k)$	$\lfloor \frac{3k}{2} \rfloor$	$\approx \frac{3}{2}k^2(n-k)$
$S_{n,k}$	$\frac{n!}{(n-k)!}$	$n-1$	$\begin{cases} 2k-1 & \text{if } 1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor \\ k+\lfloor \frac{n-1}{2} \rfloor & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leqslant k \leqslant n-1 \end{cases}$	$\begin{cases} \approx (n-1)(2k-1) \\ \approx (n-1)k + \frac{1}{2}(n-1)^2 \end{cases}$

$k+\lfloor (n-1)/2 \rfloor$. If n is even, the diameter can be obtained from one of the following two: (1) $p_1 = 1$, $c = (2k-n)/2$, $m = k-1$, $e = n-k$, and (2) $p_1 \neq 1$, $c = (2k-n)/2+1$, $m = k$, $e = n-k$. Therefore, $D(S_{n,k})$ is also $k+\lfloor (n-1)/2 \rfloor$. □

Corollary 14. *The diameter $D(S_n)$ of the n-star graph is $\lfloor 3(n-1)/2 \rfloor$.*

Proof. Since S_n is isomorphic to $S_{n,n-1}$, $D(S_n) = D(S_{n,n-1})$. □

5. Performance comparison

Finally, we compare the (n,k)-star graph with three well-known topologies: (1) the n-star graph (S_n) [1], (2) the alternating group graph (AG_n) [5], and (3) the (n,k)-arrangement graph ($A_{n,k}$), another generalized version of the star graph proposed in [4]. The numbers of nodes, degrees, and diameters of S_n, AG_n, $A_{n,k}$, and $S_{n,k}$ are presented in Table 1.

The number of parameters that one may vary in order to specify a graph provides a crude feeling for whether there are large gaps in the number of nodes of successive graphs in the family. From this viewpoint, $A_{n,k}$ and $S_{n,k}$ are more flexible than S_n and AG_n. The cost factor (diameter × degree of a node) is a good criterion to measure the performance of a network [3]. We list the approximate cost factors of these four topologies in the last column of Table 1. The results show that $S_{n,k}$ is better than $A_{n,k}$.

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