Exemplar Longest Common Subsequence

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Abstract—In this paper, we investigate the computational and approximation complexity of the Exemplar Longest Common Subsequence (ELCS) of a set of sequences (ELCS problem), a generalization of the Longest Common Subsequence problem, where the input sequences are over the union of two disjoint sets of symbols, a set of mandatory symbols and a set of optional symbols. We show that different versions of the problem are **APX**-hard even for instances with two sequences. Moreover, we show that the related problem of determining the existence of a feasible solution of the ELCS of two sequences is **NP**-hard. On the positive side, we first present an efficient algorithm for the ELCS problem over instances of two sequences where each mandatory symbol can appear in total at most three times in the sequences. Furthermore, we present two fixed-parameter algorithms for the ELCS problem over instances of two sequences where the parameter is the number of mandatory symbols.

Index Terms—Longest common subsequence, comparative genomics, algorithm design and analysis, combinatorial algorithms, analysis of algorithms, problem complexity.

1 Introduction

Algorithmic studies in comparative genomics have produced powerful tools for the analysis of genomic data that has been successfully applied in several contexts, from gene functional annotation to phylogenomics and whole genome comparison. A main goal in this research field is to explain differences in gene order in two (or more) genomes in terms of a limited number of rearrangement operations.

When there are no duplicates in the considered genomes, the computation of the similarity measure is usually polynomial-time solvable, for example, the number of breakpoints, reversal distance for signed genomes, number of conserved intervals, number of common intervals, maximum adjacency disruption, and summed adjacency disruption [8], [9], [10]. However, except for a few exceptions, several copies of the same gene or several highly homologous genes are usually scattered across the genome and, hence, it is a major problem to handle those duplicates when computing the similarity between two genomes. One approach to overcome this difficulty is based on the concept of *exemplar* [11]: For each genome, an exemplar sequence is constructed by deleting all but one occurrence of each gene family. Another approach is based on *matching* [12]: In this two-step procedure, the two genomes are first made *balanced* (the number of occurrences of genes from the same family must be the same in both genomes) by removing the minimum number of genes and, next, a one-to-one correspondence (among genes of each family) between the genes of the genomes is computed.

Unfortunately, in the presence of duplicates, most similarity measures turn out to be **NP**-hard to compute [12], [13], [14], [15] for both the exemplar and the matching models, so we generally have to rely on approximation algorithms or heuristic approaches. We discuss here one such general heuristic approach, the Exemplar LCS (ELCS) problem, which is basically a constrained string alignment problem. The basic idea of the general framework we propose here is based on the observation that, for most similarity measures and for both the exemplar and the matching models, specific common subsequences may correspond to highly conserved sets of genes. This suggests the following greedy heuristic algorithm: Find a common subsequence of significant length—but compact enough—between the two genomes, replace in the two genomes the substring that contains the common subsequence (the substring that starts at the first character of the common subsequence and ends at the last character of the common subsequence) by a new letter, and continue in a similar way. Observe that, after we have identified a common subsequence of the genomes, we can establish a one-to-one correspondence between the genes of the two genomes.

At each iteration of this simple heuristic algorithm, one, however, has to be cautious about how to choose the common subsequence as bad choices may have a disastrous impact on the rest of the algorithm. Let us take the exemplar model as a very simple explanatory example and suppose that we are searching for a common subsequence between two precise substrings of the two genomes. For one, if one gene family has occurrences elsewhere in the two genomes,

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then taking or not taking one occurrence of this particular gene family in the common subsequence is thus not based on necessity but on the length of the obtained solution. For another, if there do not exist any other occurrences of one gene family except for the one in the two considered substrings, definitively one has to take this occurrence in the common subsequence (observe that, in this case, the obtained common subsequence may not be the longest one). This simple example suggests considering an LCS-like problem that deals with two types of letters (*mandatory* and *optional* symbols) to allow greater flexibility in the searching process.

In this paper, we will formally define such a framework with a simple combinatorial problem that generalizes the well-known LCS problem and we will study its computational and approximation complexity. We show that some different versions of the problem are **APX**-hard even for instances with two sequences and that even determining if a feasible solution exists or not is **NP**-hard. On the positive side, the hardness of the problem can be limited in some cases; in fact, we show that it is possible to efficiently determine a feasible solution provided that each symbol appears at most three times in total in the input sequence. Finally, we present two fixed-parameter algorithms, where the parameter is the number of mandatory symbols.

2 The Problems

The LCS problem is a well-known problem in computational biology. Let $s = s[1], s[2], \ldots, s[m]$ and $t = t[1], t[2], \ldots, t[l]$ be two sequences. s is a subsequence of t if, for some $j_1 < j_2 < \ldots < j_m$, $s[h] = t[j_h]$. Let S be a set of sequences. Then, a *longest common subsequence* of S is the longest possible sequence s that is a subsequence of each sequence in S.

A simple way to informally define a subsequence is by using the notion of a *threading scheme*. First, write the two sequences on two parallel lines. Then, a threading scheme is a set of lines, each one connecting two identical symbols of different sequences so that no two lines are crossing. In this case, a common subsequence consists of symbols connected by the noncrossing lines.

Given a set of sequences S, the LCS problem asks for an *LCS* of S. The complexity of the LCS problem has been deeply studied in the past. In [7], it is shown that the problem is **NP**-hard even for sequences over a binary alphabet. However, when the instance of the problem consists of a fixed number of sequences, the LCS can be solved in polynomial time via dynamic programming algorithms [4], [5], [16].

The ELCS problem is related to the LCS problem. The input of the ELCS problem consists of a set S of sequences over alphabet $A_o \cup A_m$, $A_o \cap A_m = \emptyset$, where A_o is the set of *optional* symbols, and A_m is the set of *mandatory* symbols. The output of the problem is an LCS of all sequences in S that contains all mandatory symbols. Next, we formally state the ELCS problem.

Problem 1. ELCS PROBLEM
Input: a set S of sequences over alphabet $A_o \cup A_m$, where A_o is the set of *optional* symbols, and A_m is the set of *mandatory* symbols. The sets A_o and A_m are disjoint.

TABLE 1
Versions of Exemplar LCS

Problem name	Occurrences mandatory symbols	Occurrences optional symbols
ELCS$(1, \leq 1)$	exactly 1	at most 1
ELCS(1)	exactly 1	unrestricted
ELCS$(\geq 1, \leq 1)$	at least 1	at most 1
ELCS(≥ 1)	at least 1	unrestricted

Output: an LCS of all sequences in S that contains an occurrence of each mandatory symbol in A_m.

Given an instance S of ELCS, by *exemplar common subsequence* we mean a feasible solution of ELCS over S. It is possible to define different versions of the problem according to the number of occurrences of each symbol in the solution, as represented in Table 1. In this paper, we will deal with such different versions of ELCS. First, notice that ELCS(1) and ELCS(≥ 1) are generalizations of the LCS problem. Indeed, the LCS problem can be seen as the restriction of ELCS(1) and ELCS(≥ 1) with an empty set of mandatory symbols. Therefore, all of the hardness results for LCS apply to ELCS(1) and ELCS(≥ 1). Moreover, we will show that the above problems are also hard on instances of only two sequences (whereas the LCS problem can be solved in polynomial time for any fixed number of sequences). When dealing with the restriction of ELCS containing only a fixed number of sequences, we will denote such a restriction by prefixing the problem name with the number of sequences, for example, 2-ELCS$(1, \leq 1)$ is the restriction of ELCS$(1, \leq 1)$ to instances of two sequences.

3 Complexity Results

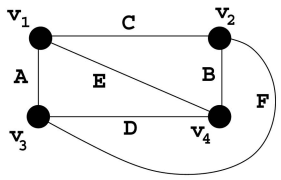
In this section, we investigate the complexity of the 2-ELCS$(1, \leq 1)$ problem and the 2-ELCS$(\geq 1, \leq 1)$ problem. More precisely, we will show that both problems are **APX**-hard even when restricted to instances where each symbol appears at most twice in each input sequence.

3.1 Complexity of 2-ELCS$(1, \leq 1)$

We prove that 2-ELCS$(1, \leq 1)$ is **APX**-hard even when each symbol appears at most twice in each input sequence via an L-reduction from the MAX INDEPENDENT SET problem on a cubic graph (MISC) to 2-ELCS$(1, \leq 1)$ (Fig. 1) since the MISC problem is known to be **APX**-hard [1]. The MISC problem is defined as follows:

Problem 2. MISC PROBLEM
Input: $G = (V, E)$ a cubic graph.
Output: a set $V' \subseteq V$ of maximum size such that no two vertices u, $v \in V'$ are adjacent.

Let $G = (V, E)$ be a cubic graph. Since G is cubic, for each vertex $v_i \in V$, there are exactly three edges incident on it; denote by $e_1(v_i)$, $e_2(v_i)$, and $e_3(v_i)$ these edges. The reduction associates with each vertex v_i a symbol v_i in A_o



$$s_1 = v_1CAEx_1v_2CFBx2v_3AFDx_3v_4EBDx_4$$
$$s_2 = CAEv_1x_1CFBv_2x2AFDv_3x_3EBDv_4x_4$$

Fig. 1. The cubic graph K_4 and its associated instance of 2-ELCS$(1, \leq 1)$.

and a symbol x_i in A_m. Furthermore, the reduction associates with each edge $e_j \in E$ a distinct symbol $s_j \in A_m$.

Let $v_i \in V$ and let $e_1(v_i)$, $e_2(v_i)$, and $e_3(v_i)$ be the edges incident on it. In what follows, we denote by $s(e_1(v_i))$, $s(e_2(v_i))$, and $s(e_3(v_i))$, respectively, the symbols of A_m associated by the reduction with edges $e_1(v_i)$, $e_2(v_i)$, and $e_3(v_i)$. Notice that each edge $e = (v_i, v_j)$ appears in the incidence lists of both v_i and v_j; thus, e will be denoted by $e_x(v_i)$ and $e_y(v_j)$ for some $1 \leq x, y \leq 3$, in the incidence list of v_i and v_j, respectively. Nonetheless, observe that e is mapped to one distinct symbol of A_m, that is, $s(e_x(v_i)) = s(e_y(v_j))$.

Define a *block* associated with a vertex v_i as a string consisting of a vertex symbol v_i, the symbols associated with edges incident to v_i in G, and the symbol x_i. There are two blocks associated with v_i, one contained in s_1 and defined as $b_1(v_i) = v_i s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))x_i$ and the other contained in s_2 and defined as $b_2(v_i) = s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))v_ix_i$. The instance of 2-ELCS$(1, \leq 1)$ consists of the two sequences: $s_1 = b_1(v_1)b_1(v_2)\cdots b_1(v_n)$, that is,

$$s_1 = v_1 s(e_1(v_1))s(e_2(v_1))s(e_3(v_1))x_1v_2\cdots x_{n-1}v_n \\ s(e_1(v_n))s(e_2(v_n))s(e_3(v_n))x_n,$$

and $s_2 = b_2(v_1)b_2(v_2)\cdots b_2(v_n)$, which, just as we have done for s_1, can be expanded to

$$s_2 = s(e_1(v_1))s(e_2(v_2))s(e_3(v_3))v_1x_1v_2\cdots x_{n-1} \\ s(e_1(v_n))s(e_2(v_n))s(e_3(v_n))v_nx_n.$$

Lemma 1. *Each exemplar common subsequence contains the symbol x_i and x_i is taken from blocks $b_1(v_i)$ and $b_2(v_i)$.*

Proof. Observe that each symbol x_i is mandatory; hence, it must appear in any feasible solution of 2-ELCS$(1, \leq 1)$. Furthermore, observe that there is only one occurrence of x_i in s_1 and in s_2. More precisely, x_i occurs in block $b_1(v_i)$ in s_1 and in block $b_2(v_i)$ in s_2. It follows that any symbol x_i in a feasible solution of 2-ELCS$(1, \leq 1)$ over s_1 and s_2 must be taken from blocks $b_1(v_i)$ and $b_2(v_i)$. □

Thus, we can divide an exemplar common subsequence s into n blocks, where block i of s starts after the positions containing the symbol x_{i-1} (or with the first symbol of s if $i = 1$) and ends in the position containing x_i. Observe that, since each x_i must appear in any exemplar common subsequence, each block of s contains at least one symbol of the solution.

Lemma 2. *The ith block of an exemplar common subsequence s contains either the symbol v_i or some symbols in $s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))$.*

Proof. Observe that, by Lemma 1, block i of s can contain only symbols from blocks $b_1(v_i)$ and $b_2(v_i)$. Furthermore, observe that, if symbol v_i is in an exemplar common subsequence s, then s does not contain any symbol of $s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))$ of $b_1(v_i)$ and $b_2(v_i)$; otherwise, it is easy to see that this block of s will not be a subsequence of the ith block of s_1 or s_2.

Now, assume that none of the symbols of $s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))$ belongs to the ith block of s. Then, if v_i does not belong to the ith block of s, we can obtain a better solution adding v_i to the ith block of s.□

Hence, a feasible solution s of 2-ELCS$(1, \leq 1)$ over s_1 and s_2 consists of $f_1x_1\ldots f_ix_i\ldots f_nx_n$, where each block f_i is either v_i or a subsequence of $s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))$.

Theorem 3. *The* 2-ELCS$(1, \leq 1)$ *problem is **APX**-hard even when each symbol appears at most twice in each input sequence.*

Proof. Consider the symbols of a common subsequence s contained in $b_1(v_i)$ and $b_2(v_i)$. The common subsequence s contains the symbol x_i and either v_i or some symbols in $e_1(v_i)e_2(v_i)e_3(v_i)$. Observe that each edge symbol is mandatory, which means that it must appear exactly once in a common subsequence. Moreover, an edge symbol encoding edge (v_i, v_j) appears in blocks $b_1(v_i)$ and $b_1(v_j)$ of s_1 and in blocks $b_2(v_i)$ and $b_2(v_j)$ of s_2. Thus, a common subsequence takes such an edge symbol either from $b_1(v_i)$ and $b_2(v_i)$ or from $b_1(v_j)$ and $b_2(v_j)$.

Let I be the set of vertices appearing in s; we will show that I is an independent set of G. Assume that symbols v_i, $v_j \in I$. Then, (v_i, v_j) is not an edge of G; otherwise, s in f_i and f_j contains symbols v_i and v_j, respectively. An immediate consequence is that the edge symbol associated with $e = (v_i, v_j)$, which can appear only in f_i and f_j, is not contained in s. Since each edge symbol is mandatory, it must appear in any feasible solution of 2-ELCS$(1, \leq 1)$, which is a contradiction. Observe that the length of a feasible solution s of 2-ELCS$(1, \leq 1)$ over s_1 and s_2 is $|V| + |E| + |I|$, where I is an independent set of G. Indeed, s will contain some symbols associated with an independent set I and one occurrence of each mandatory symbol. Notice that the set of mandatory symbols has size $|V| + |E|$.

On the other side, let I be an independent set of G; we can compute a feasible solution of 2-ELCS$(1, \leq 1)$ over s_1 and s_2 of size $|V| + |E| + |I|$, retaining in the exemplar common subsequence only the symbols associated with vertices in I. Since I is an independent set, for each edge $e = (v_i, v_j)$, at least one of v_i and v_j is not in I; hence, each symbol associated with e can be retained once in a feasible solution of 2-ELCS$(1, \leq 1)$ over s_1 and s_2. □

3.2 Complexity of 2-ELCS$(\geq 1, \leq 1)$

Next, we show that 2-ELCS$(\geq 1, \leq 1)$ is also **APX**-hard with a reduction similar to the previous one. Let $G = (V, E)$ be a cubic graph; for each vertex $v_i \in V$, we introduce four optional symbols, $v_i^a v_i^b v_i^c v_i^d$, and the blocks $b_1(v_i)$ and $b_2(v_i)$

associated with v_i in sequences s_1 and s_2, respectively, are defined as follows:

$$b_1(v_i) = v_i^a v_i^b v_i^c v_i^d s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))x_i$$

and

$$b_2(v_i) = s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))v_i^a v_i^b v_i^c v_i^d x_i.$$

Notice that x_i, $s(e_1(v_i))$, $s(e_2(v_i))$, and $s(e_3(v_i))$ are all mandatory symbols.

Since the symbols x_i are mandatory and there is only one occurrence of each x_i in s_1 and s_2, it follows that Lemma 1 also holds for this problem. Each symbol x_i appears in blocks $b_1(v_i)$ and $b_2(v_i)$ of s_1 and s_2, respectively, and any symbol x_i in an exemplar common subsequence must be taken from the blocks of s_1, s_2 associated with v_i, that is, $b_1(v_i)$ and $b_2(v_i)$. Since each mandatory edge symbol appears twice in each input sequence, it must appear once or twice in a common subsequence.

Lemma 4. *The ith block of an exemplar common subsequence s contains either sequence $v_i^a v_i^b v_i^c v_i^d$ or some symbols in $s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))$.*

Proof. It is easy to see that if any symbol of the sequence $v_i^a v_i^b v_i^c v_i^d$ is in a feasible solution of 2-ELCS$(\geq 1, \leq 1)$ over s_1 and s_2, then this solution does not contain any occurrences of symbols of sequence $s(e_1(v_1))s(e_2(v_1))s(e_3(v_1))$ in $b_1(v_i)$ and $b_2(v_i)$. This means that a feasible solution s of 2-ELCS$(\geq 1, \leq 1)$ over s_1 and s_2 consists of $g_1x_1 \ldots g_ix_i \ldots g_nx_n$, where each g_i is either a subsequence of $v_i^a v_i^b v_i^c v_i^d$ or a subsequence of $s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))$.

Now, assume that none of the symbols of $s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))$ belongs to the ith block of s. Then, if some of the symbols $v_i^a v_i^b v_i^c v_i^d$ do not belong to the ith block of s, we can obtain a better solution by adding it to the ith block of s. Conversely, if none of the symbols of $v_i^a v_i^b v_i^c v_i^d$ belongs to the ith block of s, then having the sequence $v_i^a v_i^b v_i^c v_i^d$ in the ith block of s does not shorten s. □

Observe that each edge symbol is mandatory, which means that it must appear exactly once in an exemplar common subsequence. Thus, an exemplar common subsequence takes each edge symbol from one of the two blocks where it appears.

Theorem 5. *The* 2-ELCS$(\geq 1, \leq 1)$ *problem is **APX**-hard even when each symbol appears at most twice in each input sequence.*

Proof. Let I be an independent set of G, then $s = g_1x_1 \ldots g_ix_i \ldots g_nx_n$, where each $g_i = v_i^a v_i^b v_i^c v_i^d$ if $v_i \in I$ and $g_i = s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))$ otherwise. It is immediate to note that s is a common subsequence of s_1 and s_2 of length $|V| + 3(|V| - |I|) + 4|I| = 4|V| + |I|$ and that all mandatory symbols encoding an edge are included in s. Without loss of generality, assume to the contrary that a symbol encoding edge (v_1, v_2) is not included in s. This fact implies that $g_1 = v_1^a v_1^b v_1^c v_1^d$ and $g_2 = v_2^a v_2^b v_2^c v_2^d$; hence, $v_1, v_2 \in I$, contradicting the assumption that I is an independent set of G.

Assume now that there exists a feasible solution s of 2-ELCS$(\geq 1, \leq 1)$ over s_1 and s_2 with length $4|V| + |I|$. We can assume that, for each block, either $v_i^a v_i^b v_i^c v_i^d$ or $s(e_1(v_i))s(e_2(v_i))s(e_3(v_i))$ appears as a substring of s. Let Y be the set of blocks for which $v_i^a v_i^b v_i^c v_i^d$ is part of s. Hence, the vertices corresponding to Y are an independent set of G. By a trivial counting argument, it is easy to show that, for $|I|$ blocks, s includes $v_i^a v_i^b v_i^c v_i^d$. We claim that such blocks encode an independent set. Without loss of generality, assume that $v_1^a v_1^b v_1^c v_1^d$ and $v_2^a v_2^b v_2^c v_2^d$ are included in s. Then, there is no edge (v_1, v_2) in G; otherwise, the mandatory symbol encoding such an edge would not be in s. □

4 Existence of a Feasible Solution

Given an instance of 2-ELCS, a problem related to 2-ELCS is that of determining if a feasible solution exists. In what follows, we will consider a general version of the 2-ELCS problem, where the instance consists of two sequences, s_1 and s_2, over alphabet $A_o \cup A_m$ and we want to compute if there exists a subsequence of s_1 and s_2 containing all of the mandatory symbols in A_m. Observe that computing if a feasible solution of 2-ELCS exists implies computing if a feasible solution exists for each of the problems 2-ELCS$(1, \leq 1)$, 2-ELCS(1), 2-ELCS$(\geq 1, \leq 1)$, and 2-ELCS(≥ 1). Notice that both reductions described in the previous section hold for instances that are known to admit a feasible solution; therefore, they are not sufficient to deal with the problem.

A simple observation allows us to simplify the complexity of the problem; in fact, only mandatory symbols are relevant as removing all optional symbols does not change the fact of whether a feasible solution exists or not. Therefore, in what follows, we can assume that both input sequences are made only of mandatory symbols. Clearly, in order to have a feasible solution, each mandatory symbol must appear in both input sequences s_1 and s_2. It is trivial to verify in polynomial time such a property; hence, in what follows, we assume that all mandatory symbols appear in both input sequences.

The number of occurrences of each mandatory symbol in the instance is a fundamental parameter when studying the complexity of the 2-ELCS problem. Indeed, we will show that finding a feasible solution can be done in polynomial time for small values of such parameter but becomes intractable when each symbol occurs three times in each input sequence.

4.1 A Polynomial-Time Algorithm

First, we investigate the case where each mandatory symbol appears in total at most three times in the input sequences. We will present a polynomial-time algorithm for this case via a reduction to 2SAT (the restriction of SATISFIABILITY to instances where each clause contains at most two literals). It is well-known that 2SAT can be solved in linear time [2].

For each symbol s, let $o_1(s)$ (respectively, $o_2(s)$) be the set of positions of the input sequence s_1 (respectively, s_2) where s appears. Clearly, both $o_1(s)$ and $o_2(s)$ are not empty and $|o_1(s)| + |o_2(s)| \leq 3$. It follows that, for each symbol s, there exists one of s_1 and s_2 containing exactly

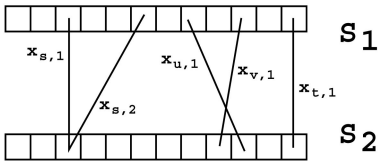


Fig. 2. Reducing 2-ELCS to 2SAT.

one occurrence of s, whereas, in the other sequence, there are one or two occurrences of s. It follows that, for each symbol s, there are at most two pairs in $o_1(s) \times o_2(s)$; otherwise, $|o_1(s)| + |o_2(s)| > 3$. Let us associate with each such pair a variable $x_{s,i}$, where $i \in \{1, 2\}$ if there are two pairs in $o_1(s) \times o_2(s)$ and $i = 1$ if there is only one pair in $o_1(s) \times o_2(s)$. Graphically, the possible variables are represented in Fig. 2 with a line connecting two identical symbols belonging to different sequences. The case $|occ_1(s)| + |occ_2(s)| = 3$ is represented by the two leftmost lines and the variables $x_{s,1}$ and $x_{s,2}$, whereas the case $|occ_1(s)| + |occ_2(s)| = 2$ is represented by the rightmost line and the variable $x_{t,1}$. Each truth assignment to the variables can be viewed as picking the lines corresponding to true variables.

Let C be the set of clauses of the instance of 2SAT that we are constructing. For each pair $x_{s,1}, x_{s,2}$ of variables, the clauses $\neg x_{s,1} \vee \neg x_{s,2}$ and $x_{s,1} \vee x_{s,2}$ are added to C. Moreover, for each symbol s such that there is only one pair in $o_1(s) \times o_2(s)$, add the clause $x_{s,1}$ to C (this corresponds to forcing the variable $x_{s,1}$ to be true). Two lines (or two variables) are called *crossing* if they cross in the drawing built as in Fig. 2.

If there exists a solution S of 2SAT that satisfies all of the clauses in C, then S picks exactly one of the lines associated with each symbol. More formally, notice that each variable $x_{s,i}$ is associated with an occurrence of symbol s in sequence s_1 (denoted as $s_1(s,i)$) and an occurrence of symbol s in sequence s_2 (denoted as $s_2(s,i)$). A pair $x_{s,i}, x_{t,j}$ of variables is crossing if, in s_1, the symbol $s_1(s,i)$ precedes $s_1(t,j)$ and, in s_2, the symbol $s_2(s,i)$ does not precede $s_2(t,j)$ or, symmetrically, if, in s_1, the symbol $s_1(s,i)$ does not precede $s_1(t,j)$ and, in s_2, the symbol $s_2(s,i)$ precedes $s_2(t,j)$. For each pair $x_{s,i}, x_{t,j}$ of crossing variables, the clause $\neg x_{s,i} \vee \neg x_{t,j}$ is added to C.

Theorem 6. *The problem of determining if a feasible solution exists for an instance of 2-ELCS where each mandatory symbol appears in total at most three times in the input sequences can be solved in polynomial time.*

Proof. We prove that the original instance of 2-ELCS has a feasible solution iff the corresponding instance of 2SAT is satisfiable, that is, there is a truth assignment for all variables such that all clauses in C are evaluated to be true. Assume that there is a feasible solution z of the instance of 2-ELCS, then, for each symbol s, we pick the lines connecting the symbols retained in z. By definition of common subsequence, there cannot be two crossing lines and exactly one of the lines associated with each symbol must be picked as z in an exemplar common subsequence; thus, all the symbols must belong to s. Therefore, we have constructed a feasible solution of 2SAT.

Conversely, given a truth assignment A for variables that satisfies all clauses in C, it follows that there are no two crossing variables in A. Indeed, for each pair of crossing variables $x_{s,i}, x_{t,j}$, a clause $\neg x_{s,i} \vee \neg x_{t,j}$ is in C and this clause can be true iff at least one of $x_{s,i}$ and $x_{t,j}$ is false. Moreover, the two clauses $\neg x_{s,1} \vee \neg x_{s,2}$ and $x_{s,1} \vee x_{s,2}$ are true iff there is exactly one of the variables $x_{s,1}$ and $x_{s,2}$ true in A and one of the variables $x_{s,1}$ and $x_{s,2}$ false in A. Hence, there is exactly one line for each symbol; therefore, it is immediate to construct a feasible solution of 2-ELCS that contains all symbols. □

The overall complexity of the algorithm is quadratic since we build a clause for each pair $x_{s,i}, x_{t,j}$ of crossing variables.

Notice that the above result holds for all of the restrictions of the 2-ELCS considered here as no symbol appears twice in both input sequences; therefore, it can appear at most once in any solution.

4.2 NP-Hardness

In what follows, we will show that slightly relaxing the constraint on the number of occurrences of each symbol makes the problem **NP**-hard.

Theorem 7. *The problem of determining if a feasible solution exists for an instance of 2-ELCS where each mandatory symbol appears at most three times in each input sequence is **NP**-hard.*

Proof. We will prove the theorem reducing 3SAT to 2-ELCS, with a reduction very similar to the one shown before. Let $C = \{C_1, \ldots, C_k\}$ be a set of clauses, each one consisting of at most three (possibly negated) literals. We construct an instance of 2-ELCS associating a block with each variable. The block of s_1 associated with variable x_i is defined as the symbol x_i, followed by the sequence of clauses containing x_i, and then by the sequence of clauses containing $\neg x_i$, where, in each sequence, the clauses are ordered according to the index in $\{C_1, \ldots, C_k\}$. In s_2, the block associated with variable x_i is defined as the symbol x_i, followed by the sequence of clauses containing $\neg x_i$, and then by the sequence of clauses containing x_i (again the clauses are ordered according to the index in $\{C_1, \ldots, C_k\}$). For example, if x_1 appears negated in C_1 and not negated in C_2 and C_3, then the corresponding blocks are $x_1C_2C_3C_1$ (in s_1) and $x_1C_1C_2C_3$ (in s_2). Both sequences s_1 and s_2 consist of the sequence of all blocks associated with the variables of the original instance of 3SAT. All symbols are mandatory; also, notice that each symbol appears at most three times in each sequence as each clause contains at most three literals.

Each symbol x_i appears exactly once in each sequence; hence, there is no ambiguity on which occurrence is retained in any exemplar common subsequence. Consequently, each symbol retained must correspond to occurrences taken from the same block. Inside the block associated with x_i, retaining the clauses where x_i appears as a positive literal is mutually exclusive with retaining the clauses where x_i appears as a negative literal, by definition of exemplar common subsequence. The first case (that is, retaining the clauses where x_i appears as a positive literal) corresponds to setting x_i to be true, whereas the second case corresponds to setting x_i to be

false. In both cases, the clauses retained are satisfied by the assignment of variables x_i.

Any feasible solution of 2-ELCS over sequences s_1 and s_2 must contain all symbols associated with the clauses; therefore, we have computed a truth assignment of the variables that satisfies all clauses in C, completing the proof. □

The above results have a definitive consequence on the approximability of the 2-ELCS problem where each mandatory symbol appears at most three times in both input sequences as they rule out any polynomial-time approximation algorithm (regardless of the approximation factor).

5 Instances of More Than Two Sequences

Since the problem can be extended to instances consisting of a set of sequences, it is interesting to know if the above results can be made stronger. In fact, the well-known inapproximability results in [6] for the LCS problem immediately also apply to the ELCS(≥ 1) problem since ELCS(≥ 1) is more general than LCS. A closer inspection of their proofs shows that their results also apply to all versions of ELCS as the optimal solutions in their reductions contain at most one occurrence of each symbol, excluding any $O(n^{1-\epsilon})$ ratio polynomial-time approximation algorithm unless $\mathbf{P} = \mathbf{NP}$, even if no mandatory symbol is allowed and all symbols appear at most twice in each sequence.

6 Instances Containing No Mandatory Symbol

Consider the restrictions of problems 2-ELCS$(1, \leq 1)$ and 2-ELCS$(\geq 1, \leq 1)$ where $A_m = \emptyset$. Observe that the two problems are equivalent since each feasible solution of the two problems consists only of optional symbols and each optional symbol can occur at most once. Denote by 2-ELCS$(^*, \leq 1)$ the restriction above. Next, we will show that the 2-ELCS$(^*, \leq 1)$ is **NP**-hard by modifying the reduction in Section 3.1, replacing all of the mandatory symbols by optional symbols.

First, each mandatory symbol x_i can be replaced by a sufficiently long sequence w_j of new optional symbols. Let $|w_j| = 10n$, where n represents the number of vertices of the cubic graph G, that is, $n = |V|$. It follows that, for each x_i, either all or no symbols of w_j are included in the solution. Indeed, if a set of symbols of w_j appears in a solution, it follows that we could add all of the remaining symbols of w_i without shortening the resulting exemplar common subsequence. Furthermore, since $|w_i| = 10n$, all sequences w_i must be included in an exemplar common subsequence; otherwise, the resulting solution is too short. Notice that each x_i appears exactly once in the reduction.

It remains to replace the mandatory symbols associated with edges, each with a sequence of unique symbols. Replace each edge symbol $s(e_{ij})$ with a sequence $z(e_{ij})$ of new optional symbols such that $|z(e_{ij})| = n$. Again, either all or no edge symbols are included in the solution.

Now, if edge e_{ij} is incident to vertices v_i and v_j, $z(e_{ij})$ will appear in blocks i and j of s_1 and s_2. It follows that one of the two occurrences of $z(e_{ij})$ might be taken. Since all symbols of w_i are taken, either the occurrences of $z(e_{ij})$ in block i of both s_1 and s_2 or the occurrences of $z(e_{ij})$ in block j of both s_1 and s_2 are taken, that is, the threading scheme of $z(e_{ij})$ cannot cross the threading scheme of w_i. Observe that, at most one occurrence of $z(e_{ij})$ can be taken in a solution of 2-ELCS$(^*, \leq 1)$. Still, at least one symbol of both occurrences of $z(e_{ij})$ must be taken; otherwise, the resulting sequence is too short and it is always possible to take only the symbols of one of the occurrences of $z(e_{ij})$ without shortening the resulting exemplar common subsequence.

7 Fixed-Parameter Algorithms

In this section, we propose some fixed-parameter algorithms for the resolution of the 2-ELCS(1) and 2-ELCS(≥ 1) problems, where the parameter is the number of mandatory symbols. First, we describe a naive approach and, then, we present two dynamic programming algorithms. In what follows, we denote by s_1 and s_2 the two input sequences by $A_m = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, the set of mandatory symbols, and by n, the maximum of $|s_1|$ and $|s_2|$.

7.1 Naive Approach

We present a naive algorithm for 2-ELCS(1) that is based on two phases: The first step consists of guessing the exact ordering of all mandatory symbols in the optimal solution and the second step basically fills in the gaps between each pair of mandatory symbols. Since each mandatory symbol appears exactly once in a feasible solution, the correct ordering of the mandatory symbol is a permutation of A_m, which can be computed in $O(m!)$ time.

Assume that s is an optimal permutation of mandatory symbols, the second phase consists of computing an LCS s^* of $\{s_1, s_2\}$. Notice that each optional symbol can appear an unrestricted number of times in any solution. Let us denote by $s[i]$ the ith character of the sequence s and by $s[i \ldots j]$ the substring of s starting with $s[i]$ and ending with $s[j]$. The recurrence equation for $\mathrm{EL}[i, j, k]$, that is, the length of an optimal solution over $s_1[1 \ldots i]$ and $s_2[1 \ldots j]$, which are both supersequences of the sequence $s[1] \cdots s[k]$, is

$$\mathrm{EL}[i,j,k] = \max \begin{cases} \mathrm{EL}[i-1,j-1,k]+1 & \text{if } s_1[i] = s_2[j], s_1[i] \in A_o \\ \mathrm{EL}[i-1,j-1,k-1]+1 & \text{if } s_1[i] = s_2[j] = s[k] \\ \mathrm{EL}[i-1,j,k], \mathrm{EL}[i,j-1,k] & \text{always.} \end{cases}$$

The boundary conditions are $\mathrm{EL}[0, j, 0] = 0$ and $\mathrm{EL}[i, 0, 0] = 0$ for $0 \leq i \leq |s_1|$ and $0 \leq j \leq |s_2|$. The value of an optimal solution can be read in $\mathrm{EL}[|s_1|, |s_2|, |s|]$. Once the matrix EL has been completely filled in, the actual optimal subsequence can be constructed with standard backtracking techniques [3]. The recurrence equation described above can be easily modified for the 2-ELCS(≥ 1) by removing the requirement $s_1[i] \in A_o$ in the first condition of the equation.

7.2 Dynamic Programming Algorithms

The algorithm described above computes the maximum length of an exemplar common subsequence by computing

all of the possible permutations of mandatory symbols. Observe that, if the number of mandatory symbols is m, then the number of permutations is $m!$ and the above algorithm has time complexity $O(m!n^2)$. Next, we present dynamic programming algorithms to compute the maximum length of an exemplar common subsequence of time complexity $O(m2^m n^2)$.

First, we describe a dynamic programming algorithm to compute the existence of a feasible solution of 2-ELCS. Denote by $\mathrm{ES}[i,j]$, where $1 \leq i \leq |s_1|$ and $1 \leq j \leq |s_2|$, a Boolean function that is $true$ iff there exists a feasible solution of 2-ELCS with input sequences $s_1[1 \dots i]$ and $s_2[1 \dots j]$; otherwise, $\mathrm{ES}[i,j]$ is $false$. Let z be a feasible solution of 2-ELCS; we call it the *restriction* of z and denote it by z_r, the subsequence of z consisting only of the rightmost occurrence of each mandatory symbol.

Lemma 8. *Let z_r be the restriction of a feasible solution z of 2-ELCS and let $\alpha \in A_m$ be the rightmost mandatory symbol of z_r. Then, there exist two occurrences j_1 and j_2 of α in s_1 and s_2, respectively, such that $z_r[1 \dots m-1]$ is a restriction of an exemplar common subsequence of $s_1[1 \dots j_1 - 1]$ and $s_2[1 \dots j_2 - 1]$ with the set of mandatory symbols $A_m - \{\alpha\}$.*

Proof. In order to obtain a feasible solution, we have to guarantee that each mandatory symbol has at least one occurrence. Since α is the rightmost symbol in z_r, it follows that $z_r[1 \dots m-1]$ must contain all mandatory symbols in $A_m - \{\alpha\}$. Now, assume that $z_r[m]$ is taken from two occurrences j_1 and j_2 of α in s_1 and s_2, respectively. It follows that all of the mandatory symbols in $z_r[1 \dots m-1]$, that is, in $A_m - \{\alpha\}$, must be taken from $s_1[1 \dots j_1 - 1]$ and $s_2[1 \dots j_2 - 1]$; thus, $z_r[1 \dots m-1]$ is a restriction of an exemplar common subsequence of $s_1[1 \dots j_1 - 1]$ and $s_2[1 \dots j_2 - 1]$ with the set of mandatory symbols $A_m - \{\alpha\}$. □

Observe that there must be a mandatory symbol $\alpha \in A_m$ that is the rightmost mandatory symbol in a feasible solution. Thus, function $\mathrm{ES}[n,m]$ is $true$ iff there exists a feasible solution $\mathrm{ES}[r(o_1(\alpha)) - 1, r(o_2(\alpha)) - 1]$ over the sets of mandatory symbols in $A_m - \{\alpha\}$, where $r(o_1(\alpha))$ (respectively, $r(o_2(\alpha))$) represents the rightmost occurrence of α in s_1 (respectively, s_2) with $r(o_1(\alpha)), r(o_2(\alpha)) \leq n$.

Denote by $\mathrm{ES}[j_1, j_2, A']$, where $A' \subseteq A_m$ is a subset of the mandatory symbols, a Boolean function that is $true$ iff there exists a feasible solution of 2-ELCS with input sequences $s_1[1 \cdots j_1], s_2[1 \cdots j_2]$ containing all of the mandatory symbols in A'; otherwise, it is $false$:

$$\mathrm{ES}[i,j,A'] = \bigvee_{\alpha \in A'} \begin{cases} \mathrm{ES}[i-1, j-1, A' - \{\alpha\}] & \text{if } s_1[i] = s_2[j], s_1[i] \in A' \\ \mathrm{ES}[i-1, j-1, A] & \text{if } s_1[i] = s_2[j], s_1[i] \notin A' \\ \mathrm{ES}[i, j-1, A'], \mathrm{ES}[i-1, j, A'] & \text{always.} \end{cases} \tag{1}$$

The boundary conditions are $\mathrm{ES}[i,j,\emptyset] = true$ for all $0 \leq i \leq |s_1|$ and $0 \leq j \leq |s_2|$, $\mathrm{ES}[0,j,A'] = false$, and $\mathrm{ES}[i,0,A'] = false$ for $0 \leq i \leq |s_1|$ and $0 \leq j \leq |s_2|$ and, for all subsets $A' \subseteq A_m$, $A' \neq \emptyset$. The existence of a feasible solution of 2-ELCS can be read in $\mathrm{ES}[|s_1|, |s_2|, A_m]$.

The time complexity of the above algorithm is $O(m2^m n^2)$. Indeed, each partial solution is computed by evaluating at most $O(m)$ equations since we have to choose a mandatory symbol $\alpha \in A'$, $|A'| \leq m$. The number of partial solutions is $O(2^m n^2)$ since the possible subsets $A' \subseteq A_m$ are $O(2^m)$, whereas indices i and j range over $[1, |s_1|]$ and $[1, |s_2|]$, respectively.

Now, we extend the approach to compute a feasible solution in order to design an algorithm that computes an ELCS, that is, a solution of the optimization problem. Informally, since (1) computes the rightmost occurrence of a mandatory symbol of set A' in a (possible) feasible solution, we have to add to the solution some symbols between a pair of consecutive mandatory symbols.

First, we discuss the case where the solution must contain exactly one occurrence of each mandatory symbol, whereas the occurrences of each optional symbol are unrestricted. Denote by $\mathrm{EL}[j_1, j_2, A']$, where $A' \subseteq A_m$ is a subset of the mandatory symbols, a function that represents the length of the longest exemplar common subsequence with input sequences $s_1[1 \cdots j_1]$ and $s_2[1 \cdots j_2]$ containing one occurrence of each mandatory symbol in A'. Indeed, the occurrences of mandatory symbols in $A' - \{\alpha\}$ occur at the left of i_1 and i_2 since α is the rightmost mandatory symbol by hypothesis, whereas symbols in $A_m - A' - \{\alpha\}$ already have an occurrence in the exemplar subsequence. The following is the recurrence to compute $\mathrm{EL}[j_1, j_2; A']$:

$$\mathrm{EL}[i,j,A'] = \max_{\alpha \in A'} \begin{cases} \mathrm{EL}[i-1, j-1, A' - \{\alpha\}] & \text{if } s_1[i] = s_2[j] = \alpha, \ \alpha \in A' \\ \mathrm{EL}[i-1, j-1, A'] & \text{if } s_1[i] = s_2[j], \ s_1[i] \in A_o \\ \mathrm{EL}[i, j-1, A'], \mathrm{EL}[i-1, j, A'] & \text{always.} \end{cases} \tag{2}$$

Denote by $\mathrm{LSO}[j_1, j_2]$ the size of an LCS with input sequences $s_1[1 \cdots j_1], s_2[1 \cdots j_2]$, where all mandatory symbols in A_m are removed from intervals $[1, j_1]$ and $[1, j_2]$. The boundary conditions are $\mathrm{EL}[i,j,\emptyset] = \mathrm{LSO}[i,j]$ for $0 \leq i \leq |s_1|$ and $0 \leq j \leq |s_2|$, $\mathrm{EL}[0,j,A'] = -\infty$, and $\mathrm{EL}[i,0,A'] = -\infty$ for $0 \leq i \leq |s_1|$ and $0 \leq j \leq |s_2|$ and for each subset $A' \subseteq A_m$, $A' \neq \emptyset$. The value of the optimal solution can be read in $\mathrm{EL}[|s_1|, |s_2|, A_m]$.

The time complexity of the algorithm is $O(m2^m n^2)$. Indeed, each partial solution is computed by evaluating at most $4m$ equations. The number of partial solutions is $O(2^m n^2)$ since the possible subsets $A' \subseteq A_m$ are $O(2^m)$, whereas indices i and j range over $[1, |s_1|]$ and $[1, |s_2|]$, respectively.

Next, we consider the case of 2-ELCS when a solution contains at least one occurrence of each mandatory symbol, whereas the occurrences of each optional symbol are unrestricted. Once again, we assume that α is the rightmost mandatory symbol of a longest exemplar common subsequence of length $\mathrm{EL}[j_1, j_2, A']$. With respect to (2), observe that we can also add to a solution mandatory symbols that are not in A' since each mandatory symbol can appear more than once in a solution:

$$\mathrm{EL}[i,j,A'] = \max_{\alpha \in A'} \begin{cases} \mathrm{EL}[i-1,j-1,A'-\{\alpha\}] & \text{if } s_1[i] = s_2[j] = \alpha, \\ & \alpha \in A' \\ \mathrm{EL}[i-1,j-1,A'] & \text{if } s_1[i] = s_2[j], \\ & s_1[i] \in A_o \cup A_m - A' \\ \mathrm{EL}[i,j-1,A'], \mathrm{EL}[i-1,j,A'] & \text{always.} \end{cases} \quad (3)$$

Denote by $\mathrm{LSM}[j_1, j_2]$ the size of an LCS with input sequences $s_1[1 \cdots j_1], s_2[1 \cdots j_2]$. The boundary conditions are $\mathrm{EL}[i,j,\emptyset] = \mathrm{LSM}[i,j]$ for $0 \le i \le |s_1|$ and $0 \le j \le |s_2|$, $\mathrm{EL}[0,j,A'] = -\infty$ and $\mathrm{EL}[i,0,A'] = -\infty$ for $0 \le i \le |s_1|$ and $0 \le j \le |s_2|$ and, for each subset $A' \subseteq A_m$, $A' \neq \emptyset$. The value of the optimal solution can be read in $\mathrm{EL}[|s_1|, |s_2|, A_m]$.

The time complexity of the algorithm is $O(m2^m n^2)$. Indeed, each partial solution is computed by evaluating at most $4m4$ equations. As before, the number of partial solutions is $O(2^m n^2)$ since the possible subsets $A' \subseteq A_m$ are $O(2^m)$, whereas indices i and j range over $[1, |s_1|]$ and $[1, |s_2|]$, respectively.

8 Implementation

The algorithm described in (2) has been implemented and tested on randomly generated data. More precisely, we have tested the algorithm with two input sequences of length 200 and with an alphabet of mandatory symbols A_m of size 10. The algorithm produces the output in a few seconds. However, the space complexity of the algorithm, which grows exponentially with the size of A_m, makes the algorithm not applicable when the size of A_m is 20 or more.

We have implemented and tested a different dynamic programming algorithm to deal with the problem. This second algorithm uses a different approach and it preprocesses subsequences of the input sequences consisting only of optional symbols. However, the first approach turns out to be much more efficient both in time and space than the latter one. Both implementations are freely available at http://www.algo.disco.unimib.it/ and licensed under the GNU General Public License.

9 Open Problems

In this paper, we have investigated the computational and approximation complexity of several versions of the ELCS problem. Some interesting cases concerning the computational complexity of the ELCS problem still need to be addressed. More precisely, we have shown that the 2-ELCS problem when each mandatory symbol appears in total at most three times in the input sequences admits a polynomial-time algorithm. Such an algorithm determines if a feasible solution exists, but different feasible solutions can lead to exemplar common subsequences of different lengths. Indeed, the computational complexity of the general problem of computing an ELCS when each mandatory symbol appears in total at most three times in the input sequences is still not known. Furthermore, we have shown that the 2-ELCS problem is **NP**-hard when each mandatory symbol appears at least three times in both input sequences. Hence, we do not know the computational complexity of the 2-ELCS problem when each mandatory symbol appears less than three times in at least one sequence, whereas it appears in total more than three times in the two input sequences.

We have proposed some fixed-parameter algorithms to compute an ELCS. Observe that both the time and space complexity of these algorithms are exponential on the size of the set of mandatory symbols A_m. In particular, the space complexity makes the algorithm not applicable when the size of A_m is 20 or more. Hence, an interesting issue concerning the implementation of these algorithms is the reduction of the space complexity of such algorithms.

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