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Planar 3DM Is *NP*-Complete

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A restriction of the three-dimensional matching problem 3DM, in which the associated bipartite graph is planar, is shown to remain *NP*-complete. The restriction is inspired by that of Lichtenstein's planar 3SAT (Planar formulae and their uses, *SIAM J. Comput.* **11** (1982), 329–343). Like Planar 3SAT, Planar 3DM is principally a tool for use in *NP*-completeness proofs for planar restrictions of other problems. Several examples of its applications in this respect are given. © 1986 Academic Press, Inc.

1. INTRODUCTION

In a recent paper, Lichtenstein [5] showed that a particular version of the well-known satisfiability problem 3SAT is *NP*-complete. This version has an unusual restriction, but it has proved to be a powerful tool in establishing *NP*-completeness results for planar cases of many hard problems. In his paper, Lichtenstein remarked that it would be useful to have an analogous result for exact cover by 3-sets (X3C), and conjectured that this could be done. Here we confirm a slightly stronger result, for a planar restriction of three-dimensional matching (3DM), and give some indication of its usefulness. The reader is referred to [4] for general background on *NP*-completeness.

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2. Planar Three-Dimensional Matching

Three-dimensional matching (3DM) is a "standard" NP-complete problem [4], and is used routinely in proving NP-completeness results. An instance comprises three disjoint sets R, B, Y with equal cardinality q and a set T of triples from $R \times B \times Y$. The question is to decide whether there is a subset of q triples which contains all the elements of R, B, and Y. We may associate a bipartite graph with this instance, as follows. We have a vertex for each element of R, B, and Y and each triple in T. There is an edge connecting a triple to an element if and only if the element is a member of the triple. This graph, G say, is bipartite with vertex bipartition T, $R \cup B \cup Y$. We will say that the instance is *planar* if G is planar. We will show that Planar 3DM remains NP-complete. The proof is in three parts. First we show that Planar 3SAT [5] remains NP-complete if exactly one literal in each clause is required to be true (1-3SAT). This is then used to prove NP-completeness of planar exact cover by 3-sets (X3C [4]) which is defined analogously to Planar 3DM. Finally we indicate how to modify the reduction to obtain the NP-completeness of Planar 3DM.

Lemma 2.1. *Planar* 1-3SAT *is NP-complete.*

Proof. By reduction from Planar 3SAT [5]. Suppose a typical clause in the 3SAT instance is $C_j = \{z_p, z_q, z_r\}$, where z_p, z_q, z_r are literals. In the associated graph G_1 there will be an edge from C_j to each of the variables appearing in these three literals, and this graph is planar. The 1-3SAT instance is then constructed by replacing C_j by three clauses

$$\{z_p, u_j, v_j\}, \{\bar{z}_q, u_j, w_j\}, \{\bar{z}_r, v_j, x_j\}$$

containing four variables. It is easily verified that these three clauses have a truth assignment with exactly one true literal in each clause if and only if C_j is satisfiable. This construction causes only a linear blow-up in the numbers of variables and clauses. Note that it is not necessary to assume that C_j has three *different* literals for this substitution to be valid. (It may also be observed that this reduction is not parsimonious [4], but can be made so at the expense of one extra clause and variable for each C_j.) It remains only to show that this construction preserves planarity. This is illustrated in Fig. 1. Let G_2 denote the graph constructed.

Lemma 2.2. *Planar* X3C *is NP-complete.*

Proof. By reduction from Planar 1-3SAT. The proof will be presented in diagrammatic form. In these diagrams a dot will represent an element and a small circle a 3-set. An edge joining an element to a set indicates membership (see Fig. 2). The reader should ignore the broken line appearing in

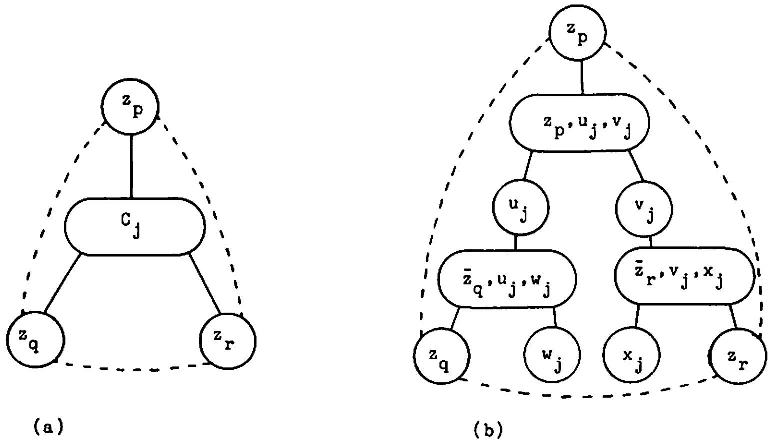

FIG. 1. (a) 3SAT instance; (b) 1-3SAT instance.

some of the diagrams, until after the proof that Planar 3DM is *NP*-complete.

A variable, v_j say, in the 1-3SAT instance will be represented by a cycle of 3-sets. If v_j occurs r times in the instance (including negations) then the cycle has $2r$ sets with each successive pair of sets sharing an element. This is illustrated, in the case $r = 3$, in Fig. 3.

It is clear that any system of which this forms a part can have an exact cover if and only if the "external" elements of this cycle are alternately covered by sets of the cycle, with alternate external elements covered by sets not in the cycle. Either alternation is possible. A successive pair of external elements will represent the appearance of v_j in a clause of the 1-3SAT instance, and the two possible alternations of these two elements being covered internally by the cycle or by sets external to the cycle will represent v_j being "true" or "false." We now augment this cycle with r additional sets and $2r$ elements by adding a 3-set to one of the external elements in each pair as illustrated in Fig. 4. The three elements now corresponding to an

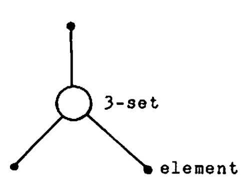

FIGURE 2

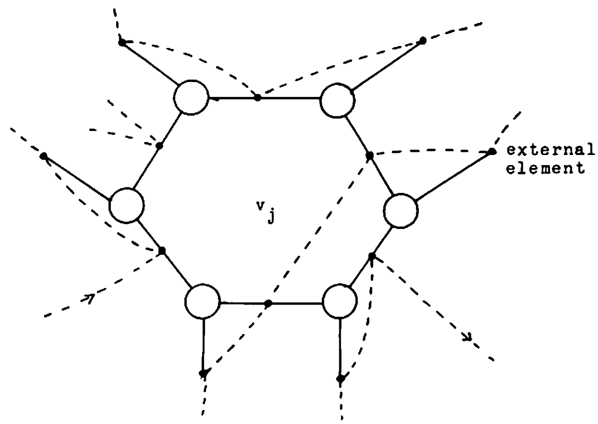

FIGURE 3

appearance of v_j will be called a "connector," and either all three, or none, of the connector elements will be covered by sets of the augmented v_j cycle as v_j is true or false. We take v_j to be true if all three connector elements are covered by the cycle when v_j appears uncomplemented in the corresponding clause.

It is easily verified that negation is handled correctly. We must now consider the clauses. Each clause C_i is represented by a copy of the configuration shown in Fig. 5. This has twelve elements and nine sets. Of the twelve elements, three are "internal" and the remaining nine are in three groups of three. Each group of three will be called a *terminal* of C_i. The construction is then completed by identifying the three connector elements for the appearance of v_j in C_i with one of the terminals of C_i. Let

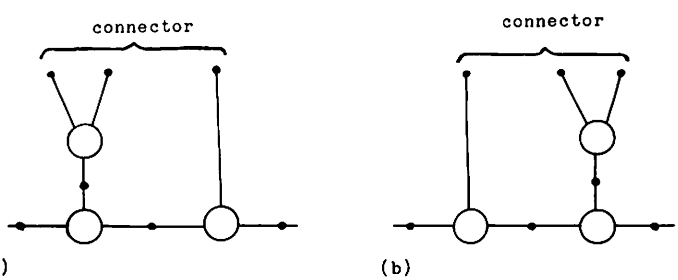

FIG. 4. (a) v_j appears uncomplemented in the corresponding clause; (b) v_j appears complemented in the corresponding clause.

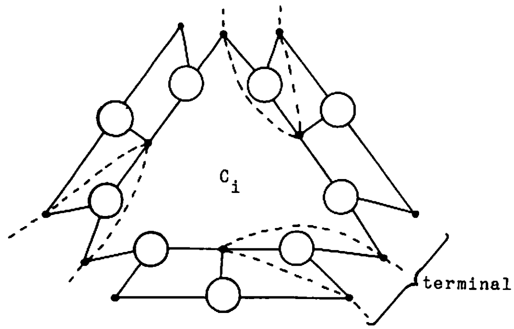

FIGURE 5

G_3 denote the graph constructed. Since we are reducing from Planar 1-3SAT and all our components are obviously planar, it is clear that we have constructed an instance of Planar X3C. We have only to show that this instance correctly simulates the 1-3SAT instance. This amounts to showing that there is an exact cover of the C_i configuration if and only if exactly one terminal is covered externally, when we restrict the covering such that either none or all three of the elements in each terminal are covered externally. Now, in this configuration, the three internal elements each appear in three of the nine sets, and no two appear in the same set. It follows that if this configuration forms part of an exact cover by 3-sets, then exactly three of its sets must be used and hence nine of its twelve elements will be covered internally. Thus exactly one terminal can be left uncovered. Now it can be verified easily from Fig. 5, using its symmetry, that if any terminal is covered externally then the remaining nine elements can be covered internally. Thus there will be an exact cover by 3-sets for this Planar X3C instance if and only if there is a satisfying truth assignment for the Planar 1-3SAT instance. This establishes the *NP*-completeness of Planar X3C.

It may also be noted that the instances of X3C constructed have the property that each element is in either two or three sets. Thus Planar X3C remains *NP*-complete under this restriction.

THEOREM 2.3. *Planar* 3DM *is NP-complete.*

Proof. An X3C instance is also a 3DM instance if the elements can be "colored" red(R), blue(B), or yellow(Y) such that each 3-set is incident with one element of each color. We will show how to modify the construction of Lemma 2.2 so that the instances admit such a coloring. First observe that the v_j cycles (as in Fig. 3) have a coloring in which all the external elements are colored B. (Simply color internal elements alternately R, Y). Thus the connector elements can be colored so that the three elements each

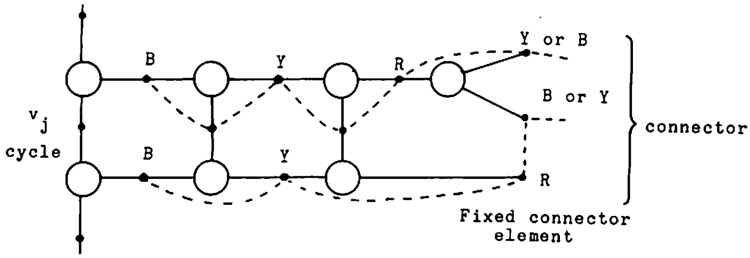

FIGURE 6

receive a different color. It is also evident that there is freedom as to which elements are colored R, Y, but the B element is fixed. We call this the fixed connector element. Now consider the clause of Fig. 5. This has a 3-coloring in which the three terminals have elements colored (in left-to-right order) RBY, BYR, and YRB. The three internal elements each receive a different color. The problem now becomes apparent. When the connector elements are identified with those of the terminals, the colors may not match because of the ordering. Now, returning to Fig. 4, it is evident that we could match the colors if we were free to choose the color of the fixed connector element. Thus we have only to show that the color of the fixed connector may be changed from B to R or Y without destroying the properties of the construction. This can be done by augmenting the variable cycles (c.f. Fig. 3 to Fig. 4) with the configuration shown in Fig. 6 if the fixed connector element needs to be colored other than B. (It will be "reflected" if v_j appears negated, similarly to Figs. 4a and b.) Let G_4 denote the graph constructed. A coloring is shown which changes the color of the fixed connector element to R. Similarly it may be changed to Y by interchanging Y, R in this diagram. Thus using this component we can arrange that the colors match at all terminals. We need only check that the configuration of Fig. 6 behaves exactly like that of Fig. 4a in its effect on the way the connector elements are covered. This can be easily verified from the diagrams. This establishes *NP*-completeness of Planar 3DM, and again we may note that every element occurs in either two or three triples.

We have proved the *NP*-completeness of Planar 3DM. However it must be stressed that this does not imply *NP*-completeness for k-dimensional matching (k DM) for $k \geq 4$ by the simple argument used in the general (i.e., nonplanar) case of this problem. This is due to the fact that this argument involves an essentially nonplanar construction. Thus we cannot conclude that Planar k DM is *NP*-complete for any $k > 3$, although we conjecture that this is the case.

Lichtenstein added the following extra condition for Planar 3SAT: a simple cycle through all the vertices representing variables can be added without destroying planarity.

Referring to Figs. 5 and 6 of [5] we see that the cycle added by Lichtenstein can be diverted to go through all clauses as well as variables without destroying planarity. Thus we can now assume that such a hamiltonian cycle H_1 has been added to G_1. Using H_1 we will show that a cycle through the vertices representing elements in Planar 3DM can be added without destroying planarity.

Consider first the transformation from G_1 to G_2. It is easy to see from Fig. 1 that the visit of H_1 to C_j can be replaced by a visit to the new variables u_j, v_j, w_j, x_j of Fig. 1b. This can be done regardless of how the cycle goes through C_j. We thus obtain a cycle H_2 through all the vertices of G_2 which represent variables.

H_2 can then be transformed into a cycle through the vertices representing elements in G_4, using the constructions indicated by the broken lines in Figs. 3, 5, and 6.

Though we make no use of this cycle in the examples here and in [3], a referee has suggested that it may be of some use elsewhere.

3. Examples

We will present five, fairly simple, applications of Theorem 2.3. We prove *NP*-completeness of some graph problems using reduction from 3DM. In each case the reduction preserves planarity and hence we can conclude that the problem remains *NP*-complete for planar graphs.

Example 1. Partitioning the vertex set of a graph into triangles. An instance of this problem is a graph $G = (V, E)$, and we ask the question: Does there exist a partition of V into $V_1, V_2, \ldots, V_p$ such that $G[V_i]$ is a triangle for $i = 1, 2, \ldots, p$?

Proof of NP-completeness for planar graphs. Garey and Johnson [4, pp. 68–69] show that this problem is *NP*-complete using a reduction from 3DM. Starting with the bipartite graph associated with the instance (as defined in Sect. 2 above), they replace the three edges incident to each triple with the configuration shown in Fig. 7. They then show that the resulting graph can be partitioned into triangles if and only if the 3DM instance contains a matching. Since the transformation is clearly planar, Theorem 2.3 establishes the *NP*-completeness of this problem even when restricted to planar graphs of degree at most 6.

In [3] similar proofs show that the following problems are hard for planar graphs: partitioning the vertices into paths, stars, trees, or connected

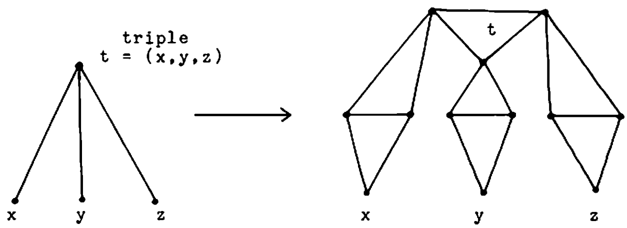

FIGURE 7

subgraphs with k (≥ 3 and fixed) vertices. See also Berman, Leighton, Shor, and Snyder [1].

EXAMPLE 2. Partitioning the edges of a graph into claws. An instance of this problem presents a graph $G = (V, E)$, and asks: Does there exist a partition of E into $E_1, E_2, \ldots, E_s$ such that each E_i induces a subgraph of G which is a claw (i.e., an isomorph of $K_{1,3}$)?

As far as we know, this result is new even for general graphs.

Proof of NP-completeness for planar graphs. Consider the bipartite graph associated with an instance of Planar 3DM. Then each triple has degree 3. We may also assume that each element has degree 2 or 3. We now modify this graph by adding a single edge to each element of degree 3, and two edges to each element of degree 2, as shown in Fig. 8. We now claim that this graph has a partition into claws if and only if the 3DM instance has a matching. Suppose there is a partition.

Since each element has at least one incident edge which is also incident to a vertex of degree 1, each element must be the center of at least one claw if the graph is partitionable. But each element now has degree 4, and hence will be the center of exactly one claw. When these claws are removed from

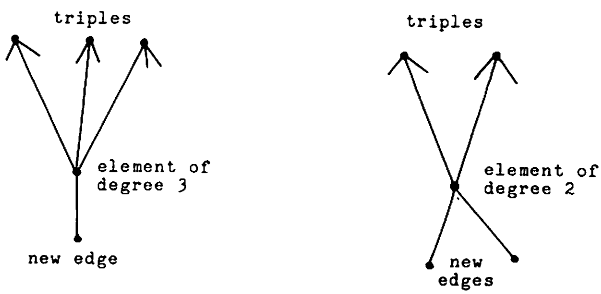

FIGURE 8

the graph, the resulting graph G' will be such that each element has degree 1 in G'. It now follows that the only way G' can be partitioned into claws is for each triple to have degree 0 or 3 in G'. The triples having degree 3 induce a matching in the 3DM instance. Conversely if the 3DM instance has a matching, this argument can be reversed to exhibit a partition of the graph into claws. The obvious planarity of the reduction now gives the desired conclusion.

It may be noted that this reduction actually proves a stronger result, that the problem is NP-complete for planar *bipartite* graphs. By considering the line graph of the bipartite graph constructed for Example 2, we obtain the simple corollary that partitioning the vertices of a line graph of a planar graph into triangles is NP-complete. In [3] similar proofs show that the following problems are hard for planar graphs: partitioning the edges into paths, trees, or connected subgraphs with k (≥ 3 and fixed) edges.

EXAMPLE 3. Dominating set. An instance of this problem comprises a graph $G = (V, E)$ and a positive integer $K \leq |V|$. The question is: Does there exist a dominating set of at most K vertices in G, i.e., is there a subset $V' \subseteq V$ with $|V'| \leq K$ such that every vertex of G is either a member of V' or adjacent to a member of V'?

Garey and Johnson [4] comment that they have shown this problem to be NP-complete for planar graphs by reduction from VERTEX COVER. However, since the VERTEX COVER problem is polynomially solvable for bipartite graphs, it would not appear that their methods could give the following result.

Proof of NP-completeness for bipartite planar graphs. Again we consider the associated bipartite graph for a Planar 3DM instance. Now we attach to each triple an independent path of two edges, as shown in Fig. 9. If the 3DM instance has $3q$ elements and t triples, we set $K = q + t$. Now the constructed graph has a dominating set of size at most K if and only if

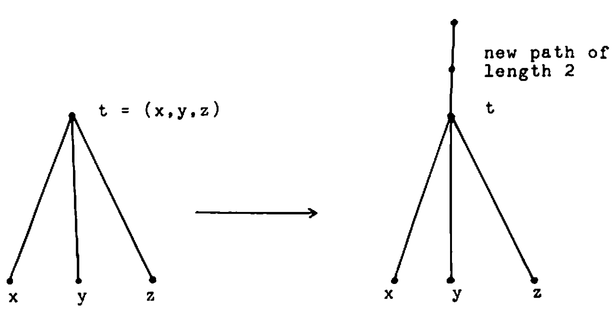

FIGURE 9

there is a matching for the 3DM instance. Suppose there is a dominating set of size at most K. For each of the added paths of length 2, one or the other of the two added vertices must be in the dominating set. There is clearly no loss in assuming that the vertex of degree 2 is in the dominating set, and the vertex of degree 1 is not. Thus all the triples are now covered and we have used t vertices in the dominating set. We now have to cover the elements. We may assume that no element is in the dominating set, since all it could now cover would be itself, and this could equally well be done by choosing any triple in which it was contained. Thus there is a dominating set of size K only if we can choose exactly q triples which cover all the elements. This will induce a matching. (There is clearly no dominating set of size less than K.) Again the argument can be reversed to construct a dominating set from a matching. Since the construction preserves both planarity and bipartiteness, Theorem 2.3 gives the conclusion.

We are indebted to Pulleyblank for pointing out the following application of Theorem 2.3.

EXAMPLE 4. Minimizing set-ups in precedence-constrained scheduling. We are given a set of n tasks $N = \{1, 2, \ldots, n\}$ of unit time length, which are to be processed on a single machine. There is an associated precedence digraph $D = (N, A)$, where $(i, j) \in A$ implies task i must precede task j. There is a fixed set-up charge if task i immediately precedes task j and $(i, j) \notin A$. The problem is to find a sequence $i_1, i_2, \ldots, i_n$ which minimizes $|\{1 \le r \le n: (i_r, i_{r+1}) \notin A\}|$.

This problem can be re-expressed in several ways:

(i) Finding the minimum number of arcs that must be added to an acyclic digraph in order to produce a hamiltonian path.

A less obvious re-formulation, for a special case, is given in Chaty and Chen [2].

(ii) If D is bipartite, with all arcs directed from one side of the vertex partition to the other, then the problem is equivalent to determining the maximum cardinality of an alternating-cycle free matching in the bipartite graph D' obtained by ignoring the orientation of the edges in D.

For the above formulation, Pulleyblank [6] has given the following reduction. Given an instance R, B, Y, T of 3DM, one replaces a triple $t = (r, b, y)$ by the configuration shown in Fig. 10. This is clearly planar, and furthermore has sufficient freedom of embedding in the plane that the connections can be made without violating planarity. (For example, the relative positions of the b, y elements can be interchanged without destroying planarity.) Thus it follows easily from Theorem 2.3 that the problem remains NP-Complete when D is restricted to be planar.

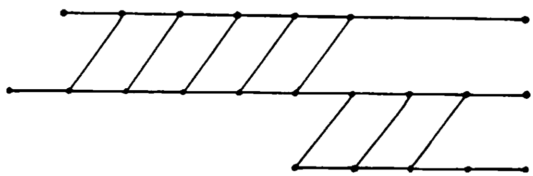

FIGURE 10

EXAMPLE 5. Elimination degree sequence. An instance presents a graph $G = (V, E)$ and a sequence $(d_1, d_2, \ldots, d_{|V|})$ of non-negative integers not exceeding $|V| - 1$. The question is: Can we number the vertices of G with the integers $1, 2, \ldots, |V|$ so that, for each i, the vertex numbered i has exactly d_i higher-numbered vertices adjacent to it?

Proof of NP-completeness for planar bipartite graphs. The construction is exactly that used for Example 2, and shown in Fig. 8. The sequence comprises $4q$ 3's, followed by $11q - 2t$ 0's. (Note that the constructed graph has $15q - 2t$ vertices.) Assume that we can number the vertices as required. Now it is clear that all degree 1 vertices must have $d_i = 0$, and hence all elements must have $d_i = 3$. This leaves q vertices to receive $d_i = 3$, and these must be q triples which are numbered before all their elements. But each element is numbered before all but one of the triples which contain it. Each such triple has $d_i = 3$. As there are only q triples with $d_i = 3$ these must induce a matching. Again the argument is reversible.

The same construction and argument gives the *NP*-completeness of the following problem: Given an undirected graph G and a set S of nonnegative integers, can we direct all the edges of G so that each vertex has an indegree in S? It remains *NP*-complete if we have to direct the edges to form an acyclic graph by the same construction, and this version is obviously very close to the elimination degree sequence problem.

We believe that our main result can be used to give easy *NP*-completeness proofs for planar cases of many other problems.

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