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Periodicity in rectangular arrays

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ABSTRACT

We discuss several two-dimensional generalizations of the familiar Lyndon–Schützenberger periodicity theorem for words. We consider the notion of primitive array (as one that cannot be expressed as the repetition of smaller arrays). We count the number of $m \times n$ arrays that are primitive. Finally, we show that one can test primitivity and compute the primitive root of an array in linear time.

1. Introduction

Let Σ be a finite alphabet. One very general version of the famous Lyndon–Schützenberger theorem [18] can be stated as follows:

Theorem 1. *Let $x, y \in \Sigma^+$. Then the following five conditions are equivalent:*

(1) $xy = yx$;

(2) There exist $z \in \Sigma^+$ and integers $k, \ell > 0$ such that $x = z^k$ and $y = z^\ell$;

(3) There exist integers $i, j > 0$ such that $x^i = y^j$;

(4) There exist integers $r, s > 0$ such that $x^r y^s = y^s x^r$;

(5) $x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset$.

Proof. For a proof of the equivalence of (1), (2), and (3), see, for example [23, Theorem 2.3.3].

Condition (5) is essentially the “defect theorem”; see, for example, [17, Cor. 1.2.6].

For completeness, we now demonstrate the equivalence of (4) and (5) to each other and to conditions (1)–(3):

(3) $\Longrightarrow$ (4): If $x^i = y^j$, then we immediately have $x^r y^s = y^s x^r$ with $r = i$ and $s = j$.

(4) $\Longrightarrow$ (5): Let $z = x^r y^s$. Then by (4) we have $z = y^s x^r$. So $z = xx^{r-1}y^s$ and $z = yy^{s-1}x^r$. Thus $z \in x\{x, y\}^*$ and $z \in y\{x, y\}^*$. So $x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset$.

(5) $\Longrightarrow$ (1): By induction on the length of $|xy|$. The base case is $|xy| = 2$. More generally, if $|x| = |y|$ then clearly (5) implies $x = y$ and so (1) holds. Otherwise without loss of generality $|x| < |y|$. Suppose $z \in x\{x, y\}^*$ and $z \in y\{x, y\}^*$. Then x is a proper prefix of y, so write $y = xw$ for a nonempty word w. Then z has prefix xx and also prefix xw. Thus $x^{-1}z \in x\{x, w\}^*$ and $x^{-1}z \in w\{x, w\}^*$, where by $x^{-1}z$ we mean remove the prefix x from z. So $x\{x, w\}^* \cap w\{x, w\}^* \neq \emptyset$, so by induction (1) holds for x and w, so $xw = wx$. Then $yx = (xw)x = x(wx) = xy$. $\square$

A nonempty word z is *primitive* if it cannot be written in the form $z = w^e$ for a word w and an integer $e \geq 2$. We will need the following fact (e.g., [17, Prop. 1.3.1] or [23, Thm. 2.3.4]):

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Fact 2. Given a nonempty word x, the shortest word z such that $x = z^i$ for some integer $i \geq 1$ is primitive. It is called the *primitive root* of x, and is unique.

In this paper we consider generalizations of the Lyndon–Schützenberger theorem and the notion of primitivity to two-dimensional rectangular arrays (sometimes called *pictures* in the literature). For more about basic operations on these arrays, see, for example, [11].

2. Rectangular arrays

By $\Sigma^{m\times n}$ we mean the set of all $m \times n$ rectangular arrays A of elements chosen from the alphabet Σ. Our arrays are indexed starting at position 0, so that $A[0,0]$ is the element in the upper left corner of the array A. We use the notation $A[i..j,k..\ell]$ to denote the rectangular subarray with rows i through j and columns k through ℓ. If $A \in \Sigma^{m\times n}$, then $|A| = mn$ is the number of entries in A.

We also generalize the notion of powers as follows. If $A \in \Sigma^{m\times n}$ then by $A^{p\times q}$ we mean the array constructed by repeating A pq times, in p rows and q columns. More formally $A^{p\times q}$ is the $pm \times qn$ array B satisfying $B[i,j] = A[i \bmod m, j \bmod n]$ for $0 \leq i < pm$ and $0 \leq j < qn$. For example, if

$$A = \begin{bmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} \\ \mathtt{d} & \mathtt{e} & \mathtt{f} \end{bmatrix},$$

then

$$A^{2\times 3} = \begin{bmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{b} & \mathtt{c} \\ \mathtt{d} & \mathtt{e} & \mathtt{f} & \mathtt{d} & \mathtt{e} & \mathtt{f} & \mathtt{d} & \mathtt{e} & \mathtt{f} \\ \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{b} & \mathtt{c} \\ \mathtt{d} & \mathtt{e} & \mathtt{f} & \mathtt{d} & \mathtt{e} & \mathtt{f} & \mathtt{d} & \mathtt{e} & \mathtt{f} \end{bmatrix}.$$

We can also generalize the notation of concatenation of arrays, but now there are two annoyances: first, we need to decide if we are concatenating horizontally or vertically, and second, to obtain a rectangular array, we need to insist on a matching of dimensions.

If A is an $m \times n_1$ array and B is an $m \times n_2$ array, then by $A \obar B$ we mean the $m \times (n_1 + n_2)$ array obtained by placing B to the right of A.

If A is an $m_1 \times n$ array and B is an $m_2 \times n$ array, then by $A \ominus B$ we mean the $(m_1 + m_2) \times n$ array obtained by placing B underneath A.

3. Generalizing the Lyndon–Schützenberger theorem

We now state our first generalization of the Lyndon–Schützenberger theorem to two-dimensional arrays, which generalizes claims (2), (3), and (4) of Theorem 1.

Theorem 3. *Let A and B be nonempty arrays. Then the following three conditions are equivalent:*

(a) There exist positive integers p_1, p_2, q_1, q_2 such that $A^{p_1\times q_1} = B^{p_2\times q_2}$.

(b) There exist a nonempty array C and positive integers r_1, r_2, s_1, s_2 such that $A = C^{r_1\times s_1}$ and $B = C^{r_2\times s_2}$.

(c) There exist positive integers t_1, t_2, u_1, u_2 such that $A^{t_1\times u_1} \circ B^{t_2\times u_2} = B^{t_2\times u_2} \circ A^{t_1\times u_1}$ where $\circ$ can be either $\obar$ or $\ominus$.

Proof.

(a) $\Longrightarrow$ (b). Let A be an array in $\Sigma^{m_1\times n_1}$ and B be an array in $\Sigma^{m_2\times n_2}$ such that $A^{p_1\times q_1} = B^{p_2\times q_2}$. By dimensional considerations we have $m_1p_1 = m_2p_2$ and $n_1q_1 = n_2q_2$. Define $P = A^{p_1\times 1}$ and $Q = B^{p_2\times 1}$. We have $P^{1\times q_1} = Q^{1\times q_2}$. Viewing P and Q as words over $\Sigma^{m_1p_1\times 1}$ and considering horizontal concatenation, this can be written $P^{q_1} = Q^{q_2}$. By Theorem 1 there exist a word R over $\Sigma^{m_1p_1\times 1}$ and integers s_1, s_2 such that $P = R^{1\times s_1}$ and $Q = R^{1\times s_2}$. Let r denote the number of columns of R and let $S = A[0\ldots m_1 - 1, 0\ldots r-1]$ and $T = B[0\ldots m_2 - 1, 0\ldots r-1]$. Observe $A = S^{1\times s_1}$ and $B = T^{1\times s_2}$. Considering the r first columns of P and Q, we have $S^{p_1\times 1} = T^{p_2\times 1}$. Viewing S and T as words over $\Sigma^{1\times r}$ and considering vertical concatenation, we can rewrite $S^{p_1} = T^{p_2}$. By Theorem 1 again, there exist a word C over $\Sigma^{1\times r}$ and integers r_1, r_2 such that $S = C^{r_1\times 1}$ and $T = C^{r_2\times 1}$. Therefore, $A = C^{r_1\times s_1}$ and $B = C^{r_2\times s_2}$.

(b) $\Longrightarrow$ (c). Without loss of generality, assume that the concatenation operation is $\obar$. Let us recall that $A = C^{r_1\times s_1}$ and $B = C^{r_2\times s_2}$. Take $t_1 = r_2$ and $t_2 = r_1$ and $u_1 = s_2$ and $u_2 = s_1$. Then we have

$$\begin{aligned} &A^{t_1\times u_1} \obar B^{t_2\times u_2} \\ &= C^{r_1t_1\times s_1u_1} \obar C^{r_2t_2\times s_2u_2} \\ &= C^{r_1t_1\times(s_1u_1+s_2u_2)} \qquad \text{(Observe that } r_1t_1 = r_2t_2\text{)} \\ &= C^{r_2t_2\times s_2u_2} \obar C^{r_1t_1\times s_1u_1} \\ &= B^{t_2\times u_2} \obar A^{t_1\times u_1}. \end{aligned}$$

(c) $\Longrightarrow$ (a). Without loss of generality, assume that the concatenation operation is $\obar$. Assume the existence of positive integers t_1, t_2, u_1, u_2 such that

$$A^{t_1\times u_1} \obar B^{t_2\times u_2} = B^{t_2\times u_2} \obar A^{t_1\times u_1}.$$

An immediate induction allows to prove that for all positive integers i and j,

$$A^{t_1\times iu_1} \obar B^{t_2\times ju_2} = B^{t_2\times ju_2} \obar A^{t_1\times iu_1}. \tag{1}$$

Assume that A is in $\Sigma^{m_1\times n_1}$ and B is in $\Sigma^{m_2\times n_2}$. For $i = n_2u_2$ and $j = n_1u_1$, we get $iu_1n_1 = ju_2n_2$. Then, by considering the first iu_1n_1 columns of the array defined in (1), we get $A^{t_1\times iu_1} = B^{t_2\times ju_2}$. $\square$

Note that generalizing condition (1) of Theorem 1 requires considering arrays with the same number of rows or same number of columns. Hence the next result is a direct consequence of the previous theorem.

Corollary 4. *Let A, B be nonempty rectangular arrays. Then*
(a) if A and B have the same number of rows, $A \obar B = B \obar A$ if and only if there exist a nonempty array C and integers $e, f \geq 1$ such that $A = C^{1\times e}$ and $B = C^{1\times f}$;
(b) if A and B have the same number of columns, $A \ominus B = B \ominus A$ if and only if there exist a nonempty array C and integers $e, f \geq 1$ such that $A = C^{e\times 1}$ and $B = C^{f\times 1}$.

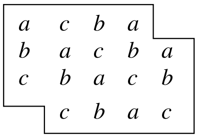

Fig. 1. A typical plane figure (from [13,14]).

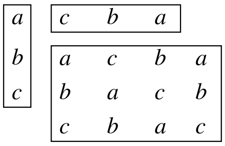

Fig. 2. Tiling of Fig. 1.

4. Labeled plane figures

We can generalize condition (5) of Theorem 1. We begin with the following lemma. As in the case of Corollary 4, we need conditions on the dimensions.

Lemma 5. *Let X and Y be rectangular arrays having same number of rows or same numbers of columns. In the former case set $\circ = \obar$. In the latter case set $\circ = \ominus$. If*

$$X \circ W_1 \circ W_2 \circ \cdots \circ W_i = Y \circ Z_1 \circ Z_2 \circ \cdots \circ Z_j \tag{2}$$

holds, where $W_1, W_2, \ldots, W_i, Z_1, Z_2, \ldots, Z_j \in \{X, Y\}$ for $i, j \geq 0$, then X and Y are powers of a third array T.

Proof. Without loss of generality we can assume that X and Y have the same number r of rows. Then the lemma is just a rephrasing of part (5) $\Longrightarrow$ (2) in Theorem 1, considering X and Y as words over $\Sigma^{r\times 1}$. $\square$

Now we can give our maximal generalization of (5) $\Longrightarrow$ (3) in Theorem 1. To do so, we need the concept of labeled plane figure (also called "labeled polyomino").

A *labeled plane figure* is a finite union of labeled cells in the plane lattice, that is, a map from a finite subset of $\mathbb{Z} \times \mathbb{Z}$ to a finite alphabet Σ. A sample plane figure is depicted in Fig. 1. Notice that such a figure does not need to be connected or convex.

Let S denote a finite set of rectangular arrays. A *tiling* of a labeled plane figure F is an arrangement of translates of the arrays in S so that the label of every cell of F is covered by an identical entry of an element of S, and no cell of F is covered by more than one entry of an element of S. For example, Fig. 2 depicts a tiling of the labeled plane figure in Fig. 1 by the arrays $[c\ b\ a]$, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and

$$\begin{bmatrix} a & c & b & a \\ b & a & c & b \\ c & b & a & c \end{bmatrix}.$$

Theorem 6. *Let F be a labeled plane figure, and suppose F has two different tilings U and V by two nonempty rectangular arrays A and B. Then both A and B are powers of a third array C.*

Proof. Assume that F has two different tilings by rectangular arrays, but A and B are not powers of a third array C. Without loss of generality also assume that F is the smallest such figure (with the fewest cells) and also that A and B are arrays with the fewest total entries that tile F, but are not powers of a third array.

Consider the leftmost cell L in the top row of F. If this cell is covered by the same array, in the same orientation, in both tilings U and V, remove the array from U and V, obtaining a smaller plane figure F' with the same property. This is a contradiction, since F was assumed minimal. So F must have a different array in U and V at this cell. Assume U has A in its tiling and V has B.

Without loss of generality, assume that the number of rows of A is equal to or larger than r, the number of rows of B. Truncate A at the first r rows and call it A'. Consider the topmost row of F. Since it is topmost and contains L at the left, there must be nothing above L. Hence the topmost row of F must be tiled with the topmost rows of A and B from left to right, aligned at this topmost row, until either the right end of the figure or an unlabeled cell is reached. Restricting our attention to the r rows underneath this topmost row, we get a rectangular tiling of these r rows by arrays A' and B in both cases, but the tiling of U begins with A' and the tiling of V begins with B.

Now apply Lemma 5 to these r rows (with $\circ = \obar$). We get that A' and B are both expressible as powers of some third array T. Then we can write A as a concatenation of some copies of T and the remaining rows of A (call the remaining rows C). Thus we get two tilings of F in terms of T and C. Since A and B were assumed to be the smallest nonempty tiles that could tile F, and $|T| \leq |B|$ and $|C| < |A|$, the only remaining possibility is that $T = B$ and C is empty. But then $A = A'$ and so both A and B are expressible as powers of T. $\square$

Remark 7. The papers [21,22] claim a proof of Theorem 6, but the partial proof provided is incorrect in some details and missing others.

Remark 8. As shown by Huova [13,14], Theorem 6 is not true for three rectangular arrays. For example, the plane figure in Fig. 1 has the tiling in Fig. 2 and also another one.

5. Primitive arrays

In analogy with the case of ordinary words, we can define the notion of primitive array. An array M is said to be *primitive* if the equation $M = A^{p\times q}$ for $p, q > 0$ implies that $p = q = 1$. For example, the array

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

is primitive, but

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{and} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

are not, as they can be written in the form $[1]^{2\times 2}$ or $[1\ 2]^{2\times 1}$ respectively.

Table 1

	1	2	3	4	5	6	7
1	2	2	6	12	30	54	126
2	2	10	54	228	990	3966	16254
3	6	54	498	4020	32730	261522	2097018
4	12	228	4020	65040	1047540	16768860	268419060
5	30	990	32730	1047540	33554370	1073708010	34359738210
6	54	3966	261522	16768860	1073708010	68718945018	4398044397642
7	126	16254	2097018	268419060	34359738210	4398044397642	562949953421058

As a consequence of Theorem 3 we get another proof of Lemma 3.3 in [10].

Corollary 9. *Let A be a nonempty array. Then there exist a unique primitive array C and positive integers i, j such that $A = C^{i\times j}$.*

Proof. Choose i as large as possible such that there exist an integer k and an array D such that $A = D^{i\times k}$. Now choose j as large as possible such that there exists an integer j and an array C such that $A = C^{i\times j}$. We claim that C is primitive. For if not, then there exists an array B such that $C = B^{i'\times j'}$ for positive integers i, j, not both 1. Then $A = C^{i\times j} = B^{ii'\times jj'}$, contradicting either the maximality of i or the maximality of j.

For uniqueness, assume $A = C^{i_1\times j_1} = D^{i_2\times j_2}$ where C and D are both primitive. Then by Theorem 3 there exists an array E such that $C = E^{p_1\times q_1}$ and $D = E^{p_2\times q_2}$. Since C and D are primitive, we must have $p_1 = q_1 = 1$ and $p_2 = q_2 = 1$. Hence $C = D$. □

Remark 10. In contrast, as Bacquey [4] has recently shown, two-dimensional biperiodic *infinite* arrays can have two distinct primitive roots.

6. Counting the number of primitive arrays

There is a well-known formula for the number of primitive words of length n over a k-letter alphabet (see e.g. [17, p. 9]):

$$\psi_k(n) = \sum_{d\mid n} \mu(d) k^{n/d}, \tag{3}$$

where μ is the well-known Möbius function, defined as follows:

$$\mu(n) = \begin{cases} (-1)^t, & \text{if } n \text{ is squarefree and the product} \\ & \text{of } t \text{ distinct primes;} \\ 0, & \text{if } n \text{ is divisible by a square} > 1. \end{cases}$$

We recall the following well-known property of the sum of the Möbius function $\mu(d)$ (see, e.g., [12, Thm. 263]):

Lemma 11.

$$\sum_{d\mid n} \mu(d) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1. \end{cases}$$

In this section we generalize Eq. (3) to two-dimensional primitive arrays:

Theorem 12. *There are*

$$\psi_k(m,n) = \sum_{d_1\mid m} \sum_{d_2\mid n} \mu(d_1)\mu(d_2) k^{mn/(d_1 d_2)}$$

primitive arrays of dimension $m \times n$ over a k-letter alphabet.

Proof. We will use Lemma 11 to prove our generalized formula, which we obtain via Möbius inversion.

Define $g(m,n) := k^{mn}$; this counts the number of $m \times n$ arrays over a k-letter alphabet. Each such array has, by Corollary 9, a unique primitive root of dimension $d_1 \times d_2$, where evidently $d_1 \mid m$ and $d_2 \mid n$. So $g(m,n) = \sum_{\substack{d_1\mid m \\ d_2\mid n}} \psi_k(d_1,d_2)$. Then

$$\begin{aligned} &\sum_{\substack{d_1\mid m \\ d_2\mid n}} \mu(d_1)\mu(d_2)\, g\left(\frac{m}{d_1}, \frac{n}{d_2}\right) \\ &\quad = \sum_{d_1\mid m} \mu(d_1) \sum_{d_2\mid n} \mu(d_2)\, g\left(\frac{m}{d_1}, \frac{n}{d_2}\right) \\ &\quad = \sum_{d_1\mid m} \mu(d_1) \sum_{d_2\mid n} \mu(d_2) \sum_{\substack{c_1\mid m/d_1 \\ c_2\mid n/d_2}} \psi_k(c_1,c_2) \\ &\quad = \sum_{c_1 d_1\mid m} \mu(d_1) \sum_{c_2 d_2\mid n} \mu(d_2)\, \psi_k(c_1,c_2) \\ &\quad = \sum_{c_1\mid m} \sum_{c_2\mid n} \psi_k(c_1,c_2) \sum_{\substack{d_1\mid m/c_1 \\ d_2\mid n/c_2}} \mu(d_1)\mu(d_2). \end{aligned}$$

Let $r = m/c_1$ and $s = n/c_2$. By Lemma 11, the last sum in the above expression is 1 if $r = 1$ and $s = 1$; that is, if $c_1 = m$ and $c_2 = n$. Otherwise, the last sum is 0. Thus, the sum reduces to $\psi_k(m,n)$ as required. □

Table 1 gives the first few values of the function $\psi_2(m,n)$:

Remark 13. As a curiosity, we note that $\psi_2(2,n)$ also counts the number of *pedal triangles* with period exactly n. See [24,15].

7. Checking primitivity in linear time

In this section we give an algorithm to test primitivity of two-dimensional arrays. We start with a useful lemma.

Lemma 14. *Let A be an $m \times n$ array. Let the primitive root of row i of A be r_i and the primitive root of column j of A be c_j.*

Then the primitive root of A has dimension $p \times q$, where $q = \operatorname{lcm}(|r_0|, |r_1|, \ldots, |r_{m-1}|)$ and $p = \operatorname{lcm}(|c_0|, |c_1|, \ldots, |c_{n-1}|)$.

Proof. Let P be the primitive root of the array A, of dimension $m' \times n'$. Then the row $A[i, 0..n-1]$ is periodic with period n'. But since the primitive root of $A[i, 0..n-1]$ is of length r_i, we know that $|r_i|$ divides n'. It follows that $q \mid n'$, where $q = \operatorname{lcm}(|r_0|, |r_1|, \ldots, |r_{m-1}|)$. Now suppose $n' \neq q$. Then since $q \mid n'$ we must have $n'/q > 1$. Define $Q := P[0..m'-1, 0..q-1]$. Then $Q^{1\times(n'/q)} = P$, contradicting our hypothesis that P is primitive. It follows that $n' = q$, as claimed.

Applying the same argument to the columns proves the claim about p. □

Now we state the main result of this section.

Theorem 15. *We can check primitivity of an $m \times n$ array and compute the primitive root in $O(mn)$ time, for fixed alphabet size.*

Proof. As is well known, a word u is primitive if and only if u is not an interior factor of its square uu [7]; that is, u is not a factor of the word $u_F u_L$, where u_F is u with the first letter removed and u_L is u with the last letter removed. We can test whether u is a factor of $u_F u_L$ using a linear-time string matching algorithm, such as the Knuth–Morris–Pratt algorithm [16]. If the algorithm returns no match, then u is indeed primitive. Furthermore, if u is not primitive, the length of its primitive root is given by the index (starting with position 1) of the first match of u in $u_F u_L$. We assume that there exists an algorithm 1DPrimitiveRoot to obtain the primitive root of a given word in this manner.

We use Lemma 14 as our basis for the following algorithm to compute the primitive root of a rectangular array. This algorithm takes as input an array A of dimension $m \times n$ and produces as output the primitive root C of A and its dimensions.

Algorithm 1 Computing the primitive root of A.

procedure 2DPrimitiveRoot(A, m, n)
  for $0 \le i < m$ **do** ▷ compute primitive root of each row
    $r_i \leftarrow$ 1DPrimitiveRoot($A[i,\ 0..n-1]$)
  $q \leftarrow \operatorname{lcm}(|r_0|, |r_1|, \ldots, |r_{m-1}|)$ ▷ compute lcm of lengths of primitive roots of rows
  for $0 \le j < n$ **do** ▷ compute primitive root of each column
    $c_j \leftarrow$ 1DPrimitiveRoot($A[0..m-1, j]$)
  $p \leftarrow \operatorname{lcm}(|c_0|, |c_1|, \ldots, |c_{n-1}|)$ ▷ compute lcm of lengths of primitive roots of columns
  for $0 \le i < p$ **do**
    for $0 \le j < q$ **do**
      $C[i, j] \leftarrow A[i, j]$
  return (C, p, q)

The correctness follows immediately from Lemma 14, and the running time is evidently $O(mn)$. □

Remark 16. The literature features a good deal of previous work on pattern matching in two-dimensional arrays. The problem of finding every occurrence of a fixed rectangular pattern in a rectangular array was first solved independently by Bird [6] and by Baker [5]. Amir and Benson later introduced the notion of two-dimensional periodicity in a series of papers [2,1,3]. Mignosi, Restivo, and Silva [20] considered two-dimensional generalizations of the Fine–Wilf theorem. A survey of algorithms for two-dimensional pattern matching may be found in Chapter 12 of Crochemore and Rytter's text [9]. Marcus and Sokol [19] considered two-dimensional Lyndon words. Crochemore, Iliopoulos, and Korda [8] and, more recently, Gamard and Richomme [10], considered quasiperiodicity in two dimensions. However, with the exception of this latter paper, where Corollary 9 can be found, none of this work is directly related to the problems we consider in this paper.

Remark 17. One might suspect that it is easy to reduce 2-dimensional primitivity to 1-dimensional primitivity by considering the array A as a 1-dimensional word, and taking the elements in row-major or column-major order. However, the natural conjectures that A is primitive if and only if (a) either its corresponding row-majorized or column-majorized word is primitive, or (b) both its row-majorized or column-majorized words are primitive, both fail. For example, assertion (a) fails because

$$\begin{bmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{b} \end{bmatrix}$$

is not primitive, while its row-majorized word aabb is primitive. Assertion (b) fails because

$$\begin{bmatrix} \mathtt{a} & \mathtt{b} & \mathtt{a} \\ \mathtt{b} & \mathtt{a} & \mathtt{b} \end{bmatrix}$$

is 2-dimensional primitive, but its row-majorized word ababab is not.

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