
SOME REMARKS ON THE GAME OF CYCLES

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Abstract

The Game of Cycles is an impartial game on a planar graph that was introduced by Francis Su. In this short note we address some questions that have been raised on the game, and raise some further questions. We assume that the reader is familiar with basic notions from combinatorial game theory.

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Subject: 91A46, 91A68.

1 The rules of the game

The Game of Cycles was invented by Francis Su [Su21]. It is an impartial game that is played on a planar graph $\Gamma \subset \mathbb{R}^2$, which is connected and simple. The complement $\mathbb{R}^2 \setminus \Gamma$ falls apart into a finite number of domains. The edges around the bounded domains are called the *cells*. Initially Γ is undirected. Alice and Bob take turns and direct one hitherto undirected edge. It is not allowed to create a sink (a vertex with all edges pointing inward) or a source (all edges pointing outward). If a player succeeds in forming a *cycle cell* in which all edges point in the same direction (either clockwise or anti-clockwise), then that player wins. Otherwise the last player to make a move wins. Given a graph, the problem is to determine whether it is winning for Alice or winning for Bob. This problem has been solved for some specific graphs in [AAG21, BBF22, Lin21, Mat22].

It is possible to remove loose ends from the graph. Suppose a vertex v is incident with only one edge e . Then e is *unmarkable* because if it is directed, then v is a source or a sink. Since e is never directed, the sink/source restriction never applies to its other vertex w . We might as well remove v and e from the graph, and declare w to be a special vertex, which is allowed to be a source or a sink (this operation of removing unmarkable

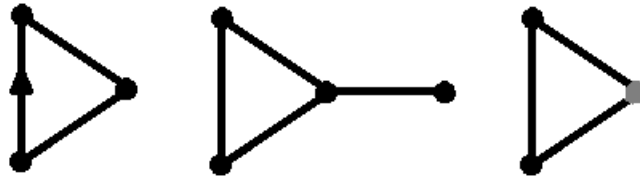


Figure 1: On the left, Alice has won the game by directing an edge of the triangle. Every move for Bob loses. The lollipop graph in the center is equivalent to the triangle with one special vertex on the right. Alice wins by the same move, but if she directs one of the other two edges, she loses. The lollipop graph has Grundy value 2.

edges is called *trimming* in [Mat22]). We only consider professional players. If a player wins by forming a cycle cell, then that can only happen if the other player had no choice on the previous move. A professional player would have resigned. Therefore, we do not really change the game if we do not allow moves that enable the closure of a cycle cell (called *death moves* in [AAG21]). This has the pleasing effect that the game is over once there are no more moves, i.e., it satisfies the normal play condition, as in a standard impartial game [ANW07].

In our (equivalent) version of the Game of Cycles, we allow special vertices and do not allow moves that enable cycle cells. It goes without saying that a vertex is special if it has degree 1. The results below are based on the work of the second author [Zan22].

2 Previous results

The Game of Cycles was analysed by Ryan Alvarado et al [AAG21], who were able to find winning strategies for graphs Γ that have certain symmetries. These winning strategies are copycat strategies. Either Bob copies the moves of Alice under the symmetry, or Alice makes a special first move on a unique edge that is fixed by the symmetry, and then she copies Bob's moves. More specifically, suppose h is a graph isomorphism of Γ such that h^2 is the identity (an *involution*) that fixes at most one edge. If none of the edges is fixed, then Bob is the copycat. If only one edge is fixed, then Alice marks it and she is the copycat. The idea behind the copycat strategy is that if e can be marked then $h(e)$ can be marked, and so the copycat always has the last move. Copycats mark *in the opposite direction*. If the opponent directs e from vertex v to vertex w , then the copycat directs $h(e)$ from $h(w)$ to $h(v)$. In this way, the copycat avoids sinks and sources. The copycat also needs to avoid death moves. If each cell is either invariant or disjoint from itself under h , then Alvarado et al [AAG21] show that the copycat's move is never a death move. Thus the copycat wins the game if such an involution h exists.

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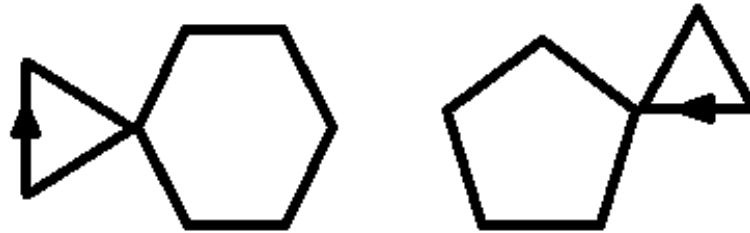


Figure 2: Winning moves for Alice for the triangle connected to an n -gon. If n is even, Alice marks the unconnected edge of the triangle and wins by a copycat strategy (this is the unique fixed edge under a reflection). If n is odd, Alice marks a connected edge. Now the unconnected edge of the triangle is unmarkable. Alice prevents the connecting vertex between n -gon and triangle from becoming a source or sink, and wins the game.

For all graphs that are solved in [AAG21], the *size* of the graph (number of edges) determines who wins the game. Alice wins if the size is odd and Bob wins if the size is even. One of the questions in [AAG21] was whether this is true for all graphs. Some care is required, as the lollipop graph has size 4 but is won for Alice. Kailee Lin [Lin21] specified the question and asked: is a graph winning for Alice if and only if its number of markable edges is odd? She established that this *parity conjecture* is true for general lollipop graphs (n -gons with loose ends attached). All games that are solved in [AAG21, BBF22, Lin21] satisfy this conjecture. However, Leah Karker and Shanise Walker [Su22] found a counterexample: a triangle connected to an n -gon is winning for Alice for all $n \geq 4$, see Fig 2. We return to this example at the end of our paper.

A tree has no cycle cells, so only the source/sink restriction applies. For tree, the game may seem simple, but it turns out to be challenging and even spider graphs (only one vertex of degree > 2) are non-trivial. Bryant Mathews [Mat22] showed that if Γ is a three-legged spider, then its Grundy value is zero if all legs of Γ are even (note that we remove loose ends, so in our game all edges in a tree are markable). The proof is long and falls apart into many different cases. For trees the parity conjecture may hold.

Conjecture 1 (parity conjecture for trees). *The Grundy value of a tree is zero if and only if its size is even.*

Alvarado et al [AAG21] left two computational challenges for the reader, as illustrated in Fig 3. They both satisfy the parity conjecture.

3 New examples

The butterfly graph can be embedded in two ways in the plane: open wings (standard) or closed wings (one triangle bounded by the other), see Fig 4. If the butterfly closes its wings, the inner cell is a part of the outer cell and therefore only the inner cell matters.

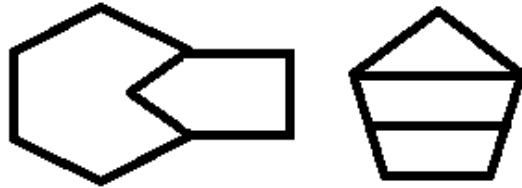


Figure 3: Some sweet graphs. The ice cream cone on the left has size ten and Grundy value zero. The layered cake on the right has size nine and Grundy value one.

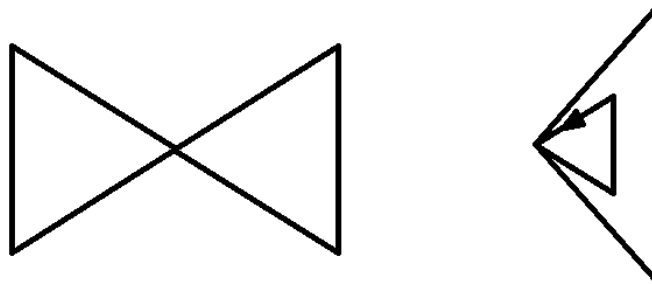


Figure 4: The butterfly graph (left) is winning for Bob, by a copycat strategy. If the butterfly closes its wings (right), then Alice wins by directing the marked edge on the inner wing, which makes the adjacent edge on that wing unplayable. Four playable edges remain and Alice wins as the second player in a game on a graph of size four.

The Grundy number of the open winged butterfly is zero. The Grundy number of the closed winged butterfly is three. A graph can be winning or losing (for Alice) depending on how it is embedded in the plane.

The wedge of n -gons in Fig 2 and the butterfly graph in Fig 4 can be disconnected by removing a single vertex. They are not 2-connected. One of the questions that we asked ourselves is: does the parity conjecture hold for 2-connected graphs? It turns out that it does not. There exists a 2-connected graph of odd size that is winning for Bob, as illustrated in Fig 5. It is invariant under reflection in the central edge. Note that the central two cells are not disjoint under this involution, so the copycat result of [AAG21] does not apply. Indeed, the first move in the copycat strategy directs the central edge. However, Bob counters by directing the left outermost edge in the opposite direction. After this move, the remaining three edges in the left hand box are unplayable. Only the edges in the right hand box remain and Bob wins as the second player on a graph of size four. Alice essentially has three other first moves, but they are all losing. The full analysis of this graph is too elaborate to reproduce here and can be found in [Zan22].

We say that a tree is *branching* if all internal vertices have degree > 2 .

Theorem 1. *A branching tree is winning for Alice if and only if its size is odd. Its Grundy number is 1 in that case.*

Proof. Essentially, we have a take-away game in which players remove chips one by one. The winning player needs to avoid sinks and sources at internal vertices. On her first move, Alice directs an edge at an inner vertex. We declare that vertex (either one, if both are inner vertices) to be the *root* of the tree. Thus we have ordered the tree and can speak about parent and child vertices. We say that a parent with an odd number of children is an *odd parent*. The parity of p , the number of odd parents, equals the parity of the size of the tree. If an edge is directed then we think of this as breaking the tie between parent and child, as if the edge is removed. Because of this the parity of p changes with each move in the game.

The winning player applies a copycat strategy. If the size of the tree is even, then Bob is the winning player. If Alice directs the edge of an even parent, then Bob counters and directs another edge of that parent in the opposite direction. The sink/source condition now is no longer relevant for this vertex. If Alice directs an edge of an odd parent, then Bob also selects an edge of an odd parent (he can, p is odd on his moves). Each internal vertex v will go through the stage of an even number of children. In that case, an edge between v and its children will first be directed by Alice, and immediately countered by Bob. Thereupon the sink/source restriction will be lifted. Therefore, Bob always has a move even if one child remains. He wins the game. If the size of the tree is odd, then Alice is the winning player by the exact same argument, because now p is odd whenever she has the move.

Note that we allowed an arbitrary first move of Alice, and used it to select a root of the tree. If the size of the tree is odd, Alice wins the game regardless of her first move. Therefore, the Grundy value is one if the size is odd. If the size is even, then Bob wins and the Grundy value is zero. \square

The parity conjecture holds for branching trees. The difficulty lies in trees with many non-branching vertices, such as spiders.

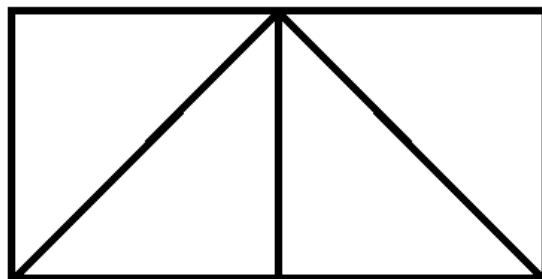


Figure 5: This graph of size nine has Grundy number zero and is a counterexample to the parity conjecture.

4 Grundy numbers

There does not seem to be a straightforward graph invariant to decide if a graph is winning for Alice or for Bob. The only possible algorithmic approach to the game seems to be by standard backward induction, which is only feasible for small graphs (size up to around ten). It is not hard to find positions in the game (that is, graphs with some directed edges) of arbitrary Grundy numbers, see [BBF22, Lin21, Mat22, Zan22], but it is a computational challenge to find a graph with a large Grundy value. We did not get very far. The maximum Grundy number that we were able to find is 3, see Fig 6, 7, and 8 below. Can anybody find a graph of Grundy number four or more?

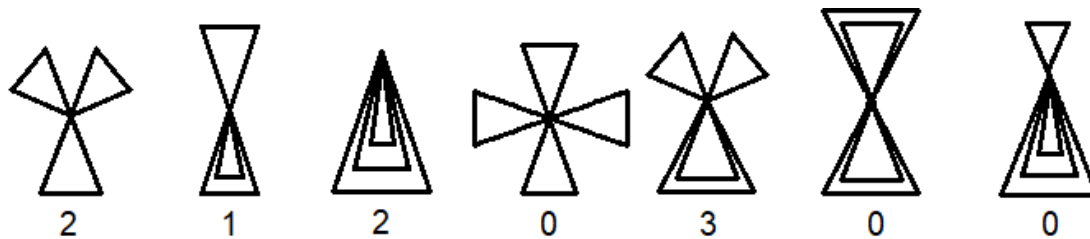


Figure 6: Windmill graphs and their Grundy numbers.

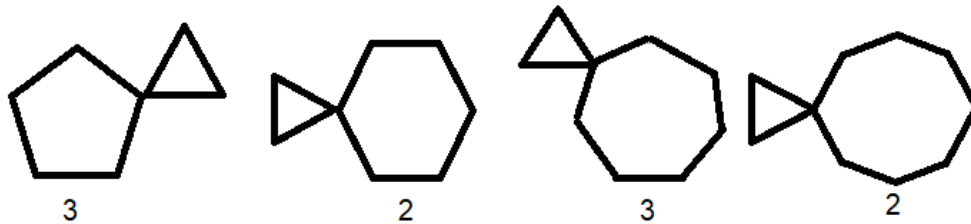


Figure 7: Fishy graphs: wedges of n -gons and a triangle.

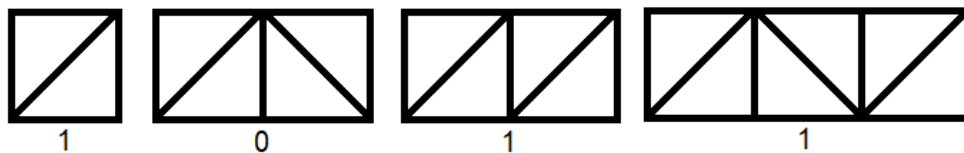


Figure 8: Box graphs and their Grundy numbers.

Leah Karker and Shanise Walker showed that the fishy graphs in Fig 7 are all winning for Alice. Our computational results suggest that their Grundy numbers are 2 if n is even and 3 if n is odd. We prove that these are the Grundy numbers if the connecting vertex is special.

Theorem 2. *The Grundy number of an n -gon with a special vertex is zero if n is even and one if n is odd, except for the triangle which has Grundy number two.*

Proof. Place the n -gon in the plane such that it is invariant under reflection in the x -axis and such that the special vertex is on this axis. This reflection is an involution and the copycat strategy works. If the opponent marks one edge of the special vertex, then the copycat responds by marking its other edge, so the special vertex is not really special for the copycat. Bob wins if n is even and Alice wins if n is odd. We only have to determine by ‘how much’ Alice wins.

If $n = 3$ then we have the graph of Fig 1. Alice has a winning move and a losing move, which implies that the Grundy number is two. The case of odd $n > 3$ remains. We prove that it is losing for Alice if we add the option of a *pass*. This option is available once and only once. If one of the players passes, then the pass is off the table. The reflection in the x -axis fixes one edge denoted f . We modify the reflection by defining $h(f)$ to be the pass (and vice versa). If Alice marks e then Bob marks $h(e)$ in the opposite direction. If Alice passes, then Bob marks f in any possible direction. Observe that $h(e)$ is unmarked and that an adjacent edge of $h(e)$ is directed if and only if its corresponding adjacent edge at e is directed. Also note that the adjacent edges of f are copies under h . If Bob needs to direct f , either both of its adjacent edges are unmarked or they point in the same direction. Therefore Bob can direct f if Alice passes. However, there is a catch. Directing $h(e)$ as prescribed may be a death move. If Alice keeps directing edges other than f clockwise, then Bob follows suit until only f remains. Alice then wins by closing the cycle. In this case, Bob’s final move is a death move, which is unprofessional. We need to modify copycat into a professional strategy: if directing $h(e)$ as prescribed is a death move, then Bob takes a pass. Is this possible? If $h(e)$ is a death move, then $n - 2$ edges are marked when it is Bob’s move. If the pass is unavailable, then the game has gone through $n - 1$ moves, which is an even number. However, if Bob moves, then the game has gone through an odd number of moves, so this cannot happen. If $n - 2$ edges are marked and Bob has the move, then he may pass. Now two edges remain for Alice,

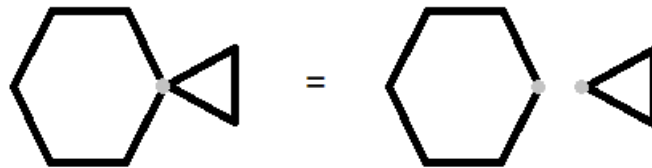


Figure 9: The game on a wedge of two n -gons at a special vertex is a sum of two games.

one of which is f . It has no special vertex and at least one of its adjacent edges has to be marked. It can only be directed in one way and that is a death move. So Alice cannot mark f . The other remaining edge is $h(e)$. If it has no special vertex, it can only be directed in one way for the same reason as f . This is a death move and not allowed. If $h(e)$ has a special vertex, then it is adjacent to e at this vertex. The other adjacent

edge has already been marked (here we need $n > 3$ to rule out that f is adjacent to $h(e)$). Again, $h(e)$ only admits a death move. After Bob passes, Alice is out of moves. It follows that this game with one pass is winning for Bob. Since one pass is equivalent to a game of Grundy number 1, we conclude that odd n -gons with $n > 3$ have Grundy number 1. \square

5 Acknowledgement

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