



Faster space-efficient STR-IC-LCS computation [☆]

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ABSTRACT

One of the most fundamental method for comparing two given strings A and B is the *longest common subsequence* (LCS), where the task is to find (the length) of an LCS of A and B . In this paper, we deal with the *STR-IC-LCS*¹ problem which is one of the constrained LCS problems proposed by Chen and Chao [J. Comb. Optim, 2011]. A string Z is said to be an STR-IC-LCS of three given strings A , B , and P , if Z is a longest string satisfying that (1) Z includes P as a substring and (2) Z is a common subsequence of A and B . We present three efficient algorithms for this problem: First, we begin with a space-efficient solution which computes the length of an STR-IC-LCS in $O(n^2)$ time and $O((\ell + 1)(n - \ell + 1))$ space, where ℓ is the length of an LCS of A and B of length n . When $\ell = O(1)$ or $n - \ell = O(1)$, then this algorithm uses only linear $O(n)$ space. Second, we present a faster algorithm that works in $O(nr/\log r + n(n - \ell + 1))$ time, where r is the length of P , while retaining the $O((\ell + 1)(n - \ell + 1))$ space efficiency. Third, we give an alternative algorithm that runs in $O(nr/\log r + n(n - \ell' + 1))$ time with $O((\ell' + 1)(n - \ell' + 1))$ space, where ℓ' denotes the STR-IC-LCS length for input strings A , B , and P .

1. Introduction

Comparison of two given strings (sequences) has been a central task in Theoretical Computer Science, since it has many applications including alignments of biological sequences, spelling corrections, and similarity searches.

One of the most fundamental method for comparing two given strings A and B is the *longest common subsequence* LCS, where the task is to find (the length of) a common subsequence L that can be obtained by removing zero or more characters from both A and B , and no such common subsequence longer than L exists. A classical dynamic programming (DP) algorithm is able to compute an LCS of A and B in quadratic $O(n^2)$ time with $O(n^2)$ working space, where n is the length of the input strings [2]. In the word RAM model with ω machine word size, the so-called “Four-Russian” method allows one to compute the length of an LCS of two given strings in $O(n^2/k + n)$ time, for any $k \leq \omega$, in the case of constant-size alphabets [3]. Under a common assumption that $\omega = \log_2 n$, this method leads to weakly sub-quadratic $O(n^2/\log n)$ time solution for constant alphabets. In the case of general alphabets, the state-of-the-art algorithm computes the length of an LCS in $O(n^2 \log^2 k/k^2 + n)$ time [4], which is weakly sub-quadratic $O(n^2(\log \log n)^2/\log^2 n)$ time

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¹ STR-IC-LCS stands for “subSTRing InCluding LCS” [1].

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Table 1

Time and space complexities of algorithms for STR-IC-LCS, for input strings A and B of length n and constraint string P of length r , where $r \leq n$. ℓ denotes the LCS length of A and B , and ℓ' denotes the STR-IC-LCS length of A , B , and P .

Algorithm	Time Complexity	Space Complexity
Deorowicz's Algorithm [9]	$O(n^2)$	$O(n^2)$
Algorithm I	$O(n^2)$	$O((\ell + 1)(n - \ell + 1))$
Algorithm II [our work]	$O(nr/\log r + n(n - \ell + 1))$	$O((\ell' + 1)(n - \ell' + 1))$
Algorithm III	$O(nr/\log r + n(n - \ell' + 1))$	$O((\ell' + 1)(n - \ell' + 1))$

for $k \leq \omega = \log_2 n$. It is widely believed that such “log-shaving” improvements would be the best possible one can hope, since an $O(n^{2-\epsilon})$ -time LCS computation for any constant $\epsilon > 0$ refutes the famous strong exponential time hypothesis (SETH) [5].

Recall however that this conditional lower-bound under the SETH does not enforce us to use (strongly) quadratic space in LCS computation. Indeed, a simple modification to the DP method permits us to compute the length of an LCS in $O(n^2)$ time with $O(n)$ working space. There also exists an algorithm that computes an LCS string in $O(n^2)$ time with only $O(n)$ working space [6]. The aforementioned log-shaving methods [3,4] use only $O(2^k + n)$ space, which is $O(n)$ for $k \leq \omega = \log_2 n$.

In this paper, we follow a line of research called the *Constrained* LCS problems, in which a pattern P that represents a-priori knowledge of a user is given as a third input, and the task is to compute the longest common subsequence of A and B that meets the condition w.r.t. P [7,8,1,9–11]. The variant we consider here is the *STR-IC-LCS* problem of computing a longest string Z which satisfies that (1) Z includes P as a substring and (2) Z is a common subsequence of A and B . We present new solutions to the STR-IC-LCS problem, named Algorithm I, Algorithm II, and Algorithm III.

Algorithm I is the first space-efficient algorithm for the STR-IC-LCS problem running in $O(n^2)$ time with $O((\ell + 1)(n - \ell + 1))$ working space, where $\ell = \text{lcs}(A, B)$ denotes the length of an LCS of A and B . This solution improves on the state-of-the-art STR-IC-LCS algorithm of Deorowicz [9] that uses $\Theta(n^2)$ time and $\Theta(n^2)$ working space, since $O((\ell + 1)(n - \ell + 1)) \subseteq O(n^2)$ always holds. This method requires only sub-quadratic $o(n^2)$ space whenever $\ell = o(n)$. In particular, when $\ell = O(1)$ or $n - \ell = O(1)$, which can happen when we compare very different strings or very similar strings, respectively, then our algorithm uses only linear $O(n)$ space. Algorithm I, is built on a non-trivial extension of the LCS computation algorithm by Nakatsu et al. [12] that runs in $O(n(n - \ell + 1))$ time with $O((\ell + 1)(n - \ell + 1))$ working space (Section 3).

Algorithm II is a faster version of Algorithm I, which works in faster $O(nr/\log r + n(n - \ell + 1))$ time with the same $O((\ell + 1)(n - \ell + 1))$ working space, where $r = |P|$ (Section 4). Recall that $r \leq n$ holds.

Algorithm III is an alternative version to Algorithm II, that works in $O(nr/\log r + n(n - \ell' + 1))$ time with $O((\ell' + 1)(n - \ell' + 1))$ space, where ℓ' is the length of an STR-IC-LCS of A , B , and P . Since $\ell' \leq \ell$, $n - \ell' + 1 \geq n - \ell + 1$ holds, implying that Algorithm III takes at least as much time as Algorithm II. Still, we show that Algorithm III uses less space than Algorithm II for some strings, by presenting strings for which $\ell' = O(1)$ and $\ell = \Theta(n)$ (Section 5).

Table 1 summarizes the complexities of the existing and proposed algorithms for STR-IC-LCS.

We remark that the $O(n^{2-\epsilon})$ -time conditional lower-bound for LCS also applies to our case since STR-IC-LCS with the pattern P being the empty string is equal to LCS, and thus, our solution is almost time optimal.

A preliminary version of this work appeared in [13], in which Algorithm I was proposed. The new materials in this full version are our second and third solutions, Algorithm II and Algorithm III, described in Sections 4 and 5.

1.1. Related work

There exist four variants of the *Constrained* LCS problems, STR-IC-LCS/SEQ-IC-LCS/STR-EC-LCS/SEQ-EC-LCS, each of which is to compute a longest string Z such that (1) Z includes/excludes the constraint pattern P as a substring/subsequence and (2) Z is a common subsequence of the two target strings A and B [7,8,1,9–11]. Yamada et al. [14] proposed an $O(n\sigma + (\ell'' + 1)(n - \ell'' + 1)r)$ -time and space algorithm for the STR-EC-LCS problem, which is also based on the method by Nakatsu et al. [12], where σ is the alphabet size, ℓ'' is the length of an STR-EC-LCS and r is the length of P . However, the design of our solution to STR-IC-LCS is quite different from that of Yamada et al.'s solution to STR-EC-LCS.

2. Preliminaries

2.1. Strings

Let Σ be an *alphabet*. An element of Σ^* is called a *string*. The length of a string S is denoted by $|S|$. The empty string ϵ is a string of length 0. For a string $S = uvw$, u , v and w are called a *prefix*, *substring*, and *suffix* of S , respectively.

The i -th character of a string S is denoted by $S[i]$, where $1 \leq i \leq |S|$. For a string S and two integers $1 \leq i \leq j \leq |S|$, let $S[i..j]$ denote the substring of S that begins at position i and ends at position j , namely, $S[i..j] = S[i] \cdots S[j]$. For convenience, let $S[i..j] = \epsilon$ when $i > j$. S^R denotes the reversed string of S , i.e., $S^R = S[|S|] \cdots S[1]$. A non-empty string Z is called a *subsequence* of another string S if there exist increasing positions $1 \leq i_1 < \cdots < i_{|Z|} \leq |S|$ in S such that $Z = S[i_1] \cdots S[i_{|Z|}]$. The empty string ϵ is a subsequence of any string. A string that is a subsequence of two strings A and B is called a *common subsequence* of A and B .

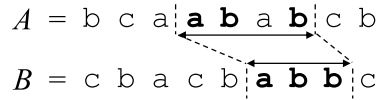


Fig. 1. Let $A = \text{bcdababcb}$, $B = \text{cbacbaaba}$, and $P = \text{abb}$. The length of an STR-IC-LCS of these strings is 6. One of such strings can be obtained by minimal intervals $[4..7]$ over A and $[6..8]$ over B because $\text{lcs}(\text{bca}, \text{cbacb}) = 2$, $|P| = 3$, and $\text{lcs}(\text{cb}, \text{c}) = 1$.

2.2. STR-IC-LCS

Let A, B , and P be strings. A string Z is said to be an STR-IC-LCS of two target strings A and B including the pattern P if Z is a longest string such that (1) P is a substring of Z and (2) Z is a common subsequence of A and B .

For ease of exposition, we assume that $n = |A| = |B|$, but our algorithm to follow can deal with the general case where $|A| \neq |B|$. We can also assume that $|P| \leq n$, since otherwise there clearly is no solution. In this paper, we present a space-efficient algorithm that computes an STR-IC-LCS in $O(n^2)$ time and $O((\ell + 1)(n - \ell + 1))$ space, where $\ell = \text{lcs}(A, B)$ is the longest common subsequence length of A and B . In case where there is no solution, we use a convention that $Z = \perp$ and its length $|\perp|$ is -1 . We remark that $\ell \geq |Z|$ always holds.

3. Space-efficient solution (Algorithm I) for STR-IC-LCS problem

In this section, we propose a space-efficient solution for the STR-IC-LCS problem.

Problem 1 (STR-IC-LCS problem). For any given strings A, B of length n and P , compute an STR-IC-LCS of A, B , and P .

Theorem 1. The STR-IC-LCS problem can be solved in $O(n^2)$ time and $O((\ell + 1)(n - \ell + 1))$ space where ℓ is the length of LCS of A and B .

In Section 3.1, we explain an overview of our algorithm. In Section 3.2, we show a central technique for our space-efficient solution and Section 3.3 concludes with the detailed algorithm.

3.1. Overview of our solution

Our algorithm is built on the previous algorithm for the STR-IC-LCS problem which was proposed by Deorowicz [9]. Firstly, we explain an outline of his algorithm. An interval $[i..j]$ over the string A is said to be a *minimal occurrence* of string P , if P is a subsequence of $A[i..j]$ and P is not a subsequence of either $A[i + 1..j]$ or $A[i..j - 1]$. In what follows, we will simply call such $[i..j]$ as a *minimal interval* for an arbitrarily fixed P . Let I_A be the set of minimal intervals over A , whose size is defined to be the number of intervals in it. Remark that I_A is of size linear in the length of A since each interval in I_A cannot contain any other intervals in I_A . There exists a pair of minimal intervals $[b_A..e_A]$ over A and $[b_B..e_B]$ over B such that the length of an STR-IC-LCS is equal to the sum of the three values $\text{lcs}(A[1..b_A - 1], B[1..b_B - 1])$, $|P|$, and $\text{lcs}(A[e_A + 1..n], B[e_B + 1..n])$ (see also Fig. 1 for an example). First, the algorithm computes I_A and I_B and computes the sum of three values for any pair of intervals. If we have an LCS table d of size $n \times n$ such that $d(i, j)$ stores $\text{lcs}(A[1..i], B[1..j])$ for any integers $i, j \in [1..n]$, we can check any LCS value between prefixes of A and B in constant time. It is known that this table can be computed in $O(n^2)$ time by using a simple dynamic programming. Since the LCS tables for prefixes and suffixes require $O(n^2)$ space, the algorithm also requires $O(n^2)$ space.

Our algorithm uses a space-efficient LCS table by Nakatsu et al. [12] instead of the table d for computing LCSs of prefixes (suffixes) of A and B . The algorithm by Nakatsu et al. also computes a table by dynamic programming, but the LCS values of $\text{lcs}(A[1..i], B[1..j])$ for some pairs (i, j) are missing in their table. In what follows, we show how we can resolve this issue.

3.2. Space-efficient prefix LCS

First, we explain a dynamic programming solution by Nakatsu et al. for computing an LCS of given strings A and B . We give a slightly modified description in order to describe our algorithm. For any integers $i, s \in [1..n]$, let $f_A(s, i)$ be the length of the shortest prefix $B[1..f_A(s, i)]$ of B such that the length of the longest common subsequence of $A[1..i]$ and $B[1..f_A(s, i)]$ is s . For convenience, $f_A(s, i) = \infty$ if no such prefix exists. The values $f_A(s, i)$ will be computed using dynamic programming as follows:

$$f_A(s, i) = \min\{f_A(s, i - 1), j_{s,i}\},$$

where $j_{s,i}$ is the index of the leftmost occurrence of $A[i]$ in $B[f_A(s - 1, i - 1) + 1..n]$. Let s' be the largest value such that $f_A(s', i) < \infty$ for some i , i.e., the s' -th row is the lowest row which has an integer value in the table f_A . We can see that the length of the longest common subsequence of A and B is s' (i.e., $\ell = \text{lcs}(A, B) = s'$). See Fig. 2 for an instance of f_A . Due to the algorithm, we do not need to compute all the values in the table f_A for obtaining the length of an LCS. Let F_A be the sub-table of f_A such that $F_A(s, i)$ stores a value $f_A(s, i)$ if $f_A(s, i)$ is computed in the algorithm of Nakatsu et al. Intuitively, F_A stores the first $n - l + 1$ diagonals of length at most l . Let $\langle i \rangle$ be the set of pairs in the i -th diagonal line ($1 \leq i \leq n$) of the table f_A :

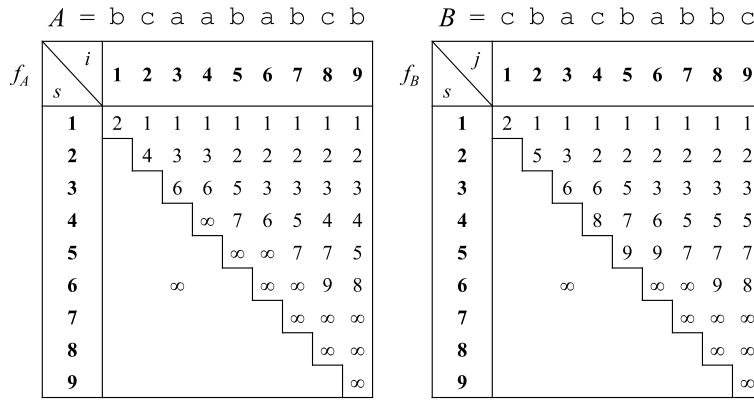


Fig. 2. The LCS-table f_A which is defined by Nakatsu et al. of $A = bcdababcb$. This figure also illustrates the table f_B of $B = cbacbaaba$.

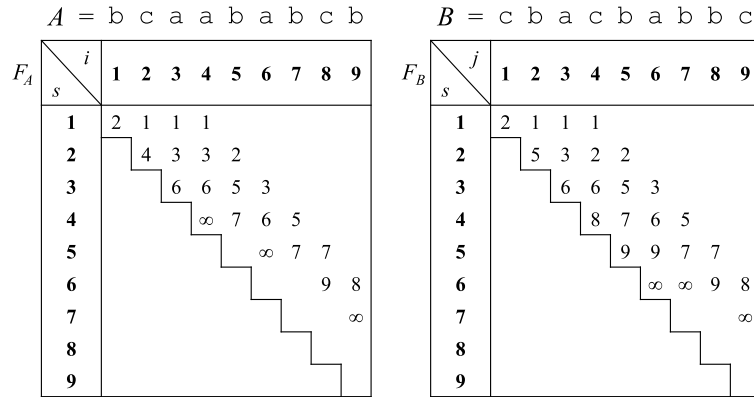


Fig. 3. A sparse table F_A of f_A for $A = bcdababcb$ and $B = cbacbaaba$ does not give $\text{lcs}(A[1..i], B[1..j])$ for some (i, j) .

$$\langle i \rangle = \{(s, i + s - 1) \mid 1 \leq s \leq n - i + 1\}.$$

Formally, $F_A(s, i) = \text{undefined}$ if

1. $s > i$,
2. $(s, i) \in \langle j \rangle$ ($j > n - \ell + 1$), or
3. $F_A(s - 1, i - 1) \in \{\infty, \text{undefined}\}$.

Any other $F_A(s, i)$ stores the value $f_A(s, i)$. Since the lowest row number of each diagonal line $\langle j \rangle$ ($j > n - \ell + 1$) is less than ℓ , we do not need to compute values which is described by the second item. Actually, we do not need to compute the values in $\langle n - \ell + 1 \rangle$ for computing the LCS since the maximum row number in the last diagonal line is also ℓ . However, we need the values on the last line in our algorithm. Hence the table F_A uses $O((\ell + 1)(n - \ell + 1))$ space (subtable which need to compute is parallelogram-shaped of height ℓ and base $n - \ell$). See Fig. 3 for an instance of F_A .

Now we describe a main part of our algorithm. Recall that a basic idea is to compute $\text{lcs}(A[1..i], B[1..j])$ from F_A . If we have all the values on the table f_A , we can check the length $\text{lcs}(A[1..i], B[1..j])$ as follows.

Observation 1. The length of an LCS of $A[1..i]$ and $B[1..j]$ for any $i, j \in [1..n]$ is the largest s such that $f_A(s, i) \leq j$. If no such s exists, $A[1..i]$ and $B[1..j]$ have no common subsequence of length s .

However, F_A does not store several integer values with respect to the second condition of undefined for some i and j . See also Fig. 3 for an example of this fact. In this example, we can see that $\text{lcs}(A[1..7], B[1..4]) = f_A(3, 7) = 3$ from the table f_A , but $F_A(3, 7) = \text{undefined}$ in F_A . In order to resolve this problem, we also define F_B (and f_B). Formally, for any integers $j, s \in [1..n]$, let $f_B(s, j)$ be the length of the shortest prefix $A[1..f_B(s, j)]$ of A such that the length of the longest common subsequence of $B[1..j]$ and $A[1..f_B(s, j)]$ is s . Our algorithm accesses the length of an LCS of $A[1..i]$ and $B[1..j]$ for any given i and j by using two tables F_A and F_B . The following lemma shows a key property for the solution.

Lemma 2. Let s be the length of an LCS of $A[1..i]$ and $B[1..j]$. If $F_A(s, i) = \text{undefined}$, then $F_B(s, j) \neq \text{undefined}$.

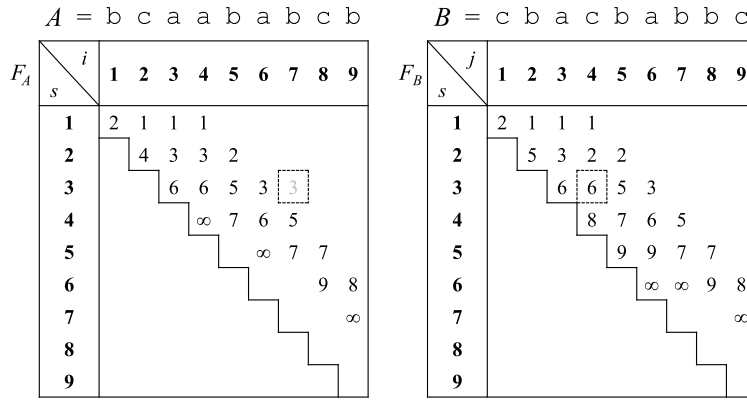


Fig. 4. Due to Observation 1, $f_A(3,7)$ gives the fact that $\text{lcs}(A[1..7], B[1..4]) = 3$. However, $F_A(3,7) = \text{undefined}$. Then we can obtain the fact that $\text{lcs}(A[1..7], B[1..4]) = 3$ by using F_B . Namely, $F_B(3,4)$ gives the LCS value.

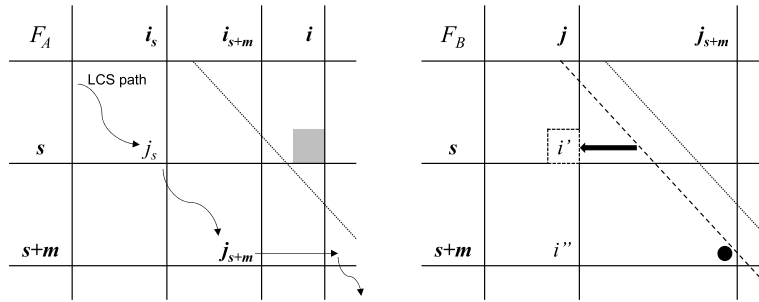


Fig. 5. This figure shows an illustration for the proof of Lemma 2 (and Lemma 3). The length s of an LCS of $A[1..i]$ and $B[1..j]$ cannot be obtained over F_A because $F_A(s, i) = \text{undefined}$ (the highlighted cell). However, the length can be obtained by $F_B(s, j)$ over F_B . The existence of $F_B(s+m, j_{s+m})$ from an LCS path guarantees the fact that $F_B(s, j) \neq \text{undefined}$.

This lemma implies that the length of an LCS of $A[1..i]$ and $B[1..j]$ can be obtained if we have the two sparse tables (see also Fig. 4).

Before proving Lemma 2, we show Lemma 3 below:

Lemma 3. Let U_{F_A} be the set of pairs (s, i) of integers such that $F_B(s, j) \neq \text{undefined}$, where $F_A(s, i) = j$. For any $1 \leq s \leq \text{lcs}(A, B)$, there exists i such that $(s, i) \in U_{F_A}$.

Proof. Let $\ell = \text{lcs}(A, B)$. We consider a sequence $i_1, \dots, i_s, \dots, i_\ell$ of positions in A such that $A[i_1] \dots A[i_s] \dots A[i_\ell]$ is an LCS of A and B which can be obtained by backtracking over F_A . Suppose that $F_B(s, F_A(s, i_s)) = \text{undefined}$ for some $s \in [1.. \ell]$. $F_B(s', F_A(s', i_{s'})) = \text{undefined}$ for any $s' \in [s.. \ell]$, since $F_A(1, i_1) < \dots < F_A(\ell, i_\ell)$ and s' increases as one goes either to the next diagonal cell or even further right. However, $F_B(\ell, F_A(\ell, i_\ell))$ is not undefined. Therefore, $F_B(s, F_A(s, i_s)) \neq \text{undefined}$ for any $s \in [1.. \ell]$. Thus that the lemma holds. \square

Now we are ready to prove Lemma 2 as follows.

Proof of Lemma 2. Let $\ell = \text{lcs}(A, B)$ and $X = A[i_1] \dots A[i_s] \dots A[i_\ell]$ be an LCS of A and B which can be obtained from F_A , as in the proof for Lemma 3. We also consider a sequence $j_1, \dots, j_s, \dots, j_\ell$ of positions in B such that $B[j_1] \dots B[j_s] \dots B[j_\ell]$ is an LCS of A and B , which satisfies that $j_s = F_A(k, i_s)$ for each $s \in [1.. \ell]$.

Assume that $F_A(s, i) = \text{undefined}$. Let m be the largest integer such that $i_{s+m} \leq i$ holds and $F_A(s+m, i_{s+m}) \neq \text{undefined}$. If no such m exists, namely $i < i_1$, the statement holds since $F_A(s, i) \neq \text{undefined}$. Due to Observation 1, $F_A(s+m, i) > j$. Thus $j < j_{s+m}$ holds. On the other hand, we consider the table F_B (and f_B). Let $i' = f_B(s, j)$ and $i'' = f_B(s+m, j)$. Due to Observation 1, $i' \leq i < i''$ holds. By Lemma 3, $F_B(s+m, j_{s+m}) \neq \text{undefined}$. This implies that $F_B(s+m, j) (= i'')$ is not undefined. By the definition of X , $j_{s+m} - j \geq m - 1$. Notice that (s, j) is in $(j - s + 1)$ -th diagonal line. These facts imply that $F_B(s, j) \neq \text{undefined}$. See also Fig. 5 for an illustration. \square

3.3. Algorithm 1

A pseudo-code of our first space-efficient algorithm is given in Algorithm 1.

Algorithm 1 Algorithm I for computing the length of STR-IC-LCS.**Input:** A, B, P ($|A| = n, |B| = n, |P| = r$)**Output:** l, C

```

1: compute  $I_A$  and  $I_B$ 
2: compute  $F_A, F_B, F_{A^R},$  and  $F_{B^R}$ 
3:  $\ell \leftarrow \text{lcs}(A, B)$ 
4:  $l \leftarrow 0$ 
5: for  $i = 1$  to  $|I_A|$  do
6:    $k_1^A \leftarrow 1, k_1^B \leftarrow 1, k_2^A \leftarrow \ell, k_2^B \leftarrow \ell$ 
7:   for  $j = 1$  to  $|I_B|$  do
8:      $k_1 \leftarrow 0, k_2 \leftarrow 0$ 
9:     compute  $\text{lcs}(A[1..b_A(i) - 1], B[1..b_B(j) - 1])$  // as  $k_1$  by Algorithm 2
10:    compute  $\text{lcs}(A[e_A(i) + 1..n], B[e_B(j) + 1..n])$  // as  $k_2$  by Algorithm 3
11:    if  $k_1 + k_2 + r > l$  then
12:       $l \leftarrow k_1 + k_2 + r$ 
13:    end if
14:  end for
15: end for
16: return  $l$ 

```

Algorithm 2 Computing $\text{lcs}(A[1..b_A(i) - 1], B[1..b_B(j) - 1])$ in Algorithm I.

```

1: for  $k \leftarrow k_1^A$  to  $\ell$  do
2:   if  $F_A(b_A(i) - 1, k) \leq b_B(j) - 1$  then
3:     if  $F_A(b_A(i) - 1, k + 1) > b_B(j) - 1$  then
4:        $k_1 \leftarrow k$ 
5:        $k_1^A \leftarrow k$ 
6:       break
7:     end if
8:   else if  $F_A(b_A(i) - 1, k) > b_B(j) - 1$  then
9:     if  $F_A(b_A(i) - 1, k - 1) = \text{undefined}$  then
10:       $k_1^A \leftarrow k$ 
11:      for  $k' = k_1^B$  to  $\ell$  do
12:        if  $F_B(b_B(j) - 1, k') > b_A(i) - 1$  then
13:           $k_1 \leftarrow 0$ 
14:           $k_1^B \leftarrow k'$ 
15:          break
16:        else if  $F_B(b_B(j) - 1, k' + 1) > b_A(i) - 1$  then
17:           $k_1 \leftarrow k'$ 
18:           $k_1^B \leftarrow k'$ 
19:          break
20:        end if
21:      end for
22:    else
23:       $k_1 \leftarrow 0$ 
24:       $k_1^A \leftarrow k$ 
25:      break
26:    end if
27:  end if
28: end for

```

First, our algorithm computes the sets of minimal intervals I_A and I_B (similar to the algorithm by Deorowicz [9]). Second, compute the tables F_A and F_B for computing LCSs of prefixes, and the tables F_{A^R} and F_{B^R} for computing LCSs of suffixes (similar to the algorithm by Nakatsu et al. [12]). Third, for any pairs of intervals in I_A and I_B , compute the length of an LCS of corresponding prefixes/suffixes and obtain a candidate of the length of an STR-IC-LCS. As stated above, the first and the second steps are similar to the previous work. Here, we describe our method to compute the length of an LCS of prefixes on F_A and F_B in the third step. We can also compute the length of an LCS of suffixes on F_{A^R} and F_{B^R} in a similar way.

We assume that I_A and I_B are sorted in increasing order of the beginning positions. Let $[b_A(x)..e_A(x)]$ and $[b_B(y)..e_B(y)]$ denote the x -th interval in I_A and the y -th interval in I_B , respectively. We process $O(n^2)$ -queries in increasing order of the beginning position of the intervals in I_A . For each interval $[b_A(x)..e_A(x)]$ in I_A , we want to obtain the length of an LCS of $A[1..b_A(x) - 1]$ and $B[1..b_B(y) - 1]$. For convenience, let $i_x = b_A(x) - 1$ and $j_y = b_B(y) - 1$. In the rest of this section, we use a pair (x, y) of integers to denote a prefix-LCS query (computing $\text{lcs}(A[1..i_x], B[1..j_y])$). We will find the LCS by using Observation 1. Here, we describe how to compute prefix-LCS queries $(i_x, j_1), \dots, (i_x, j_{|I_B|})$ in this order for a fixed i_x .

Lemma 4. All required prefix-LCS values for an interval $[b_A(x)..e_A(x)]$ in I_A and all intervals in I_B can be computed in $O(n)$ time.

Proof. There exist two cases for each i_x . Formally, (1) $F_A(1, i_x) \neq \text{undefined}$ or (2) $F_A(1, i_x) = \text{undefined}$.

Algorithm 3 Computing $\text{lcs}(A[e_A(i) + 1..n], B[e_B(j) + 1..n])$ in Algorithm I.

```

1: for  $k = k_2^A$  to 1 do
2:   if  $F_{A^R}(n - e_A(i), k) \leq n - e_B(j)$  then
3:      $k_2 \leftarrow k$ 
4:      $k_2^A \leftarrow k$ 
5:     break
6:   else if  $F_{A^R}(n - e_A(i), k) > n - e_B(j)$  then
7:     if  $F_{A^R}(n - e_A(i), k - 1) = \text{undefined}$  then
8:        $k_2^A \leftarrow k$ 
9:       for  $k' = k_2^B$  to 1 do
10:        if  $F_{B^R}(n - e_B(j), k') \leq n - e_A(i)$  then
11:           $k_2 \leftarrow k'$ 
12:           $k_2^B \leftarrow k'$ 
13:          break
14:        else if  $F_{B^R}(n - e_B(j), k' - 1) = \text{undefined}$  then
15:           $k_2 \leftarrow 0$ 
16:           $k_2^B \leftarrow k'$ 
17:          break
18:        end if
19:      end for
20:    end if
21:  end if
22: end for

```

In the first case, we scan the i_x -th column of F_A from the top to the bottom in order to find the maximum value which is less than or equal to j_1 . If such a value exists in the column, then the row number s_1 is the length of an LCS. After that, we are given the next prefix-LCS query (i_x, j_2) . It is easy to see that $s_0 = \text{lcs}(A[1..i_x], B[1..j_1]) \leq \text{lcs}(A[1..i_x], B[1..j_2])$ since $j_1 < j_2$. This implies that the next LCS value is equal to s_0 or that is placed in a lower row in the column. This means that we can start to scan the column from the s_0 -th row. Thus we can answer all prefix-LCSs for a fixed i_x in $O(n)$ time (that is linear in the size of I_B).

In the second case, we start to scan the column from the top $F_A(i_x - n - \ell + 1, i_x)$ (the first $i_x - n - \ell$ rows are undefined). If $F_A(i_x - n - \ell + 1, i_x) \leq j_1$, then the length of an LCS for the first query (i_x, j_1) can be found in the table (similar to the first case) and any other queries $(i_x, j_2), \dots, (i_x, j_{|I_B|})$ can be also answered in a similar way. Otherwise (if $F_A(i_x - n - \ell + 1, i_x) > j_1$), the length which we want may be in the “undefined” domain. Then we use the other table F_B . We scan the j_1 -th column in F_B from the top to the bottom in order to find the maximum value which is less than or equal to i_x . By Lemma 2, such a value must exist in the column (if $\text{lcs}(A[1..i_x], B[1..j_1]) > 0$ holds) and the row number s' is the length of an LCS. After that, we are given the next query (i_x, j_2) . If $F_A(i_x - n - \ell + 1, i_x) \leq j_2$, then the length can be found in the table (similar to the first case). Otherwise (if $F_A(i_x - n - \ell + 1, i_x) > j_2$), the length must be also in the “undefined” domain. Since such a value must exist in the j_2 -th column in F_B by Lemma 2, we scan the column in F_B . It is easy to see that $s' = \text{lcs}(A[1..i_x], B[1..j_1]) \leq \text{lcs}(A[1..i_x], B[1..j_2])$. This implies that the length of an LCS that we want to find is in lower row. Thus it is enough to scan the j_2 -th column from the s' -th row to the bottom. Then we can answer the second query (i_x, j_2) . Hence we can compute all LCSs for a fixed i_x in $O(n + \ell)$ time (that is linear in the size of I_B or the number of rows in the table F_B).

Therefore we can compute all prefix-LCSs for each interval in I_A in $O(n)$ time (since $n \geq \ell$). \square

On the other hand, we can compute all required suffix-LCS values with computing prefix-LCS values. We want a suffix-LCS value of $A[e_A(x) + 1..n]$ and $B[e_B(y) + 1..n]$ ($1 \leq y \leq |I_B|$) when we compute the length of an LCS of $A[1..b_A(x) - 1]$ and $B[1..b_B(y) - 1]$. Recall that we process all intervals of I_B in increasing order of the beginning positions when computing prefix-LCS values with a fixed interval of I_A . This means that we need to process all intervals of I_B in “decreasing order” when computing suffix-LCS values with a fixed interval of I_A . We can do that by using an almost similar way on F_{A^R} and F_{B^R} . The most significant difference is that we scan the $|A[e_A(x) + 1..n]|$ -th column of F_{A^R} from the ℓ -th row to the first row.

Overall, we can obtain the length of an STR-IC-LCS in $O(n^2)$ time in total. Also this algorithm requires space for string all minimal intervals and tables, namely, requiring $O(n + (\ell + 1)(n - \ell + 1)) \subseteq O((\ell + 1)(n - \ell + 1))$ space in the worst case. Finally, we can obtain Theorem 1.

In addition, we can also compute an STR-IC-LCS (as a string), if we store a pair of minimal intervals which produce the length of an STR-IC-LCS. Namely, we can find a cell which gives the prefix-LCS value over F_A or F_B . Then we can obtain a prefix-LCS string by a simple backtracking (a suffix-LCS can be also obtained by backtracking on F_{A^R} or F_{B^R}). On the other hand, we can also use an algorithm that computes an LCS string in $O(n^2)$ time and $O(n)$ space by Hirschberg [6].

4. Faster solution (Algorithm II) for STR-IC-LCS problem

In this section, we propose a faster solution (Algorithm II) for the STR-IC-LCS problem that achieves the following:

Theorem 5. *The STR-IC-LCS problem can be solved in $O(nr/\log r + n(n - \ell + 1))$ time and $O((\ell + 1)(n - \ell + 1))$ space where ℓ is the length of an LCS of A and B .*

	1	2	3	4	5	6	7	8	9
$\theta_A =$	2	[2, 3]	5	5	[5, 7]	8	8	[8, 9]	-
	1	2	3	4	5	6	7	8	9
$\theta_B =$	2	[2, 4]	5	5	[5, 6]	7	[7, 9]	-	-

Fig. 6. θ_A of the set $I_A = \{[2,3],[5,7],[8,9]\}$ of minimal intervals on string $A = \text{cabcadbab}$ and pattern ab , and θ_B of the set $I_B = \{[2,4],[5,6],[7,9]\}$ of minimal intervals on string $B = \text{cacbabadb}$ and pattern ab .

Algorithm II with $O(nr/\log r + n(n - \ell + 1))$ running time is built on the previous solution Algorithm I with $O(n^2)$ running time. Thus, in this section we discuss only the differences between Algorithm I and Algorithm II.

An overview of Algorithm II is as follows: First, we compute the sets of minimal intervals I_A and I_B by using Das et al.'s algorithm [15] that works in $O(nr/\log r)$ time and $O(n)$ space. Second, we compute two arrays θ_A and θ_B from I_A and I_B in $O(n)$ time and space, respectively. Third, following the approach of Algorithm I, we compute the tables F_A and F_B for computing LCSs of the prefixes, and the tables F_{AR} and F_{BR} for computing LCSs of the suffixes, in faster $O(n(n - \ell + 1))$ time with $O((\ell + 1)(n - \ell + 1))$ space. Finally, for some pairs of intervals in θ_A and θ_B which respectively have intervals from I_A and I_B , we compute the length of an LCS of corresponding prefixes/suffixes and obtain a candidate of the length of an STR-IC-LCS by using the faster method that works in $O((\ell + 1)(n - \ell + 1))$ time. The main idea of this step is to omit rescanning of the same element in F_A and F_B .

In the following subsections, we explain the details of Algorithm II. Section 4.1 shows how to compute θ_A and θ_B . Section 4.2 shows the faster method of computing candidates of STR-IC-LCS.

4.1. Computing θ_A and θ_B

As stated before, I_A and I_B are the sets of minimal intervals over A and B which have P as a subsequence, respectively. We define two arrays θ_A and θ_B : For each $1 \leq i \leq n$,

- If there is an interval $[i..j]$ in I_A that begins with i , then $\theta_A[i]$ stores this interval $[i..j]$. These minimal intervals stored in θ_A are called *interval elements* of θ_A .
- If there is no interval in I_A that begins with i , then $\theta_A[i]$ stores the smallest index s such that $s > i$ and $\theta_A[s]$ stores an interval element. In case where there is no such interval to the right of i , then $\theta_A[i]$ stores null.

For example, Fig. 6 shows θ_A of the set I_A of minimal intervals for string $A = \text{cabcadbab}$ and pattern ab . The array θ_B is defined analogously for string B . These arrays are used for our efficient algorithm in Section 4.2 for computing candidates.

4.2. Computing candidates

Recall that Algorithm I computes candidates by scanning each column of F_A and some columns F_B from the top to the bottom. The total time complexity of Algorithm I is $O(n^2)$ since it depends on the number of minimal intervals which is $O(n^2)$, in other words, the number of candidates considered by Algorithm I is $O(n^2)$. To achieve a faster solution, Algorithm II excludes some “redundant” candidates that are considered by Algorithm I but cannot actually be a solution. Below, we explain such candidates.

Here, let us recall the process in Algorithm I in which prefix-LCS queries are performed in this order $(i_x, j_k), \dots, (i_x, j_{|I_B|})$ for a fixed i_x ($1 \leq k < |I_B|$). Assume that $F_A(s_k, i_x)$ satisfying $F_A(s_k, i_x) \geq j_k$ exists. We scan the i_x -th column of F_A in order to find the value $F_A(s_k, i_x)$. After this scanning, we are given the next prefix-LCS query (i_x, j_{k+1}) and restart to scan the column from $F_A(s_1, i_x)$. In this situation, if $s_k = \text{lcs}(A[1..i_x], B[1..j_k]) = \text{lcs}(A[1..i_x], B[1..j_{k+1}])$, we scan only $F_A(s_k, i_x)$. However, the candidate obtained by such scanning cannot be a solution of STR-IC-LCS since $\text{lcs}(A[1..i_x], B[1..j_k]) + |P| + \text{lcs}(A[i'_x..n], B[j'_k..n]) \leq \text{lcs}(A[1..i_x], B[1..j_{k+1}]) + |P| + \text{lcs}(A[i'_x..n], B[j'_{k+1}..n])$ holds, where i'_x, j'_k and j'_{k+1} satisfy $(i_x + 1, i'_x - 1) \in I_A$, $(j_k + 1, j'_k - 1) \in I_B$ and $(j_{k+1} + 1, j'_{k+1} - 1) \in I_B$, respectively. This implies that the rescanning of the same value in F_A is useless in computing a solution, and the symmetric arguments hold when we compute F_B . We will prove Theorem 5 based on this observation.

Algorithm II performs prefix-LCS and suffixes LCS-queries which are given from θ_A and θ_B , respectively. Thus, we describe how to perform prefix-LCS and suffix-LCS queries by scanning the elements $\theta_A[i]$ and $\theta_B[1], \dots, \theta_B[n]$ in this order for a fixed i ($1 \leq i \leq n$).

Lemma 6. We can compute all prefix/suffix LCSs with respect to θ_A and θ_B in a total of $O((\ell + 1)(n - \ell + 1))$ time and space.

Proof. We use an array W_{AR} of size n such that, for each $1 \leq i \leq n$, $W_{AR}[i]$ initially stores the largest row index of the i -th column of F_{AR} that stores a non- ∞ value. We also use an array W_{BR} of size n that is built on F_{BR} in an analogous manner. See Fig. 7 for examples of initialized W_{AR} and W_{BR} . In our faster algorithm to follow, we update these two arrays W_{AR} and W_{BR} while keeping the invariants that $W_{AR}[i]$ and $W_{BR}[i]$ respectively store the row indices of the i -th columns of F_{AR} and F_{BR} from which we restart the scanning, for every $1 \leq i \leq n$.

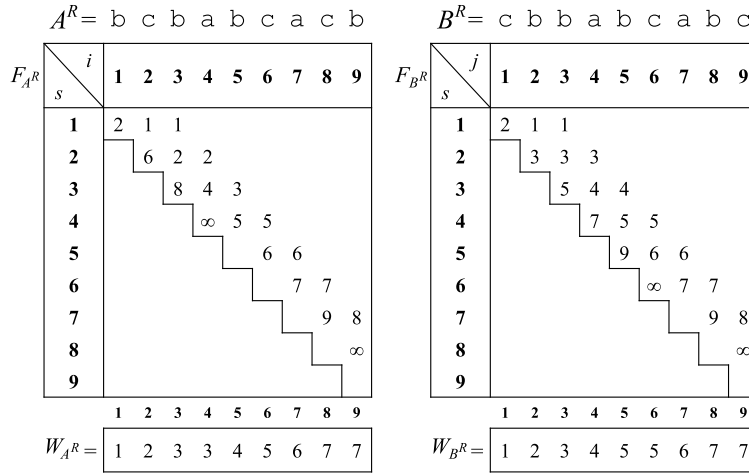


Fig. 7. Sparse tables F_A^R and F_B^R for $A = bcacbabcb$ and $B = cbacbabbc$, and integer arrays W_A^R and W_B^R in their initial states, whose i -th elements are initialized with the largest row indices of the i -th columns of F_A^R and F_B^R storing non- ∞ values, respectively.

For each $1 \leq i \leq n$ such that $\theta_A[i]$ stores an interval element, let us denote this interval by $[i..e_i]$. For a given i , if $\theta_A[i]$ does not store an interval element, then we access the next interval element in θ_A by updating the index i with $i \leftarrow \theta_A[i]$. Let j be the least index in θ_B such that $\theta_B[j]$ stores an interval element. We then proceed as described above.

There exist two cases for each i such that $\theta_A[i]$ stores an interval element:

- (1) When $F_A(1, i - 1) \neq \text{undefined}$: In this case, first, we compute the prefix-LCS length of $A[1..i - 1]$ and $B[1..j - 1]$. We scan the $(i - 1)$ -th column of F_A from the top to the bottom in order to find the maximum value which is less than or equal to $j - 1$. If such a value exists in the column, then the row number s is the length of a prefix-LCS.

Next, we compute the suffix-LCS length of $A[e_i + 1..n]$ and $B[e_j + 1..n]$. If $F_{A^R}(1, n - e_i) \neq \text{undefined}$, we scan the $(n - e_i)$ -th column of F_{A^R} by the same methods as Algorithm I. If $F_{A^R}(1, n - e_i) = \text{undefined}$, we scan the $(n - e_i)$ -th column of F_{A^R} by the same method as Algorithm I if $F_A(e_i - \ell, n - e_i) \leq n - e_j$, and otherwise we can scan the $(n - e_j)$ -th column of F_{B^R} from the $(W_{B^R}[n - e_j])$ -th row to the top and we get the row number s' since $n - e_i$ is decreasing order and $\text{lcs}(A[e_i + 1 + k..n], B[e_j + 1..n]) \geq \text{lcs}(A[e_i + 1..n], B[e_j + 1..n])$ for $1 \leq k \leq n - (e_i + 1)$. Then, in the case that we scan F_{B^R} , we store s' at $W_{B^R}[n - e_j]$.

After that, we update j with the beginning position of the next interval element, which is either directly stored at $\theta_B[F_A(s + 1, i - 1) + 1]$ or pointed by a pointer stored at $\theta_B[F_A(s + 1, i - 1) + 1]$. This operation allows us to skip all interval elements which can cause rescanning $F_A(s, i - 1)$. This way, we can start scanning from $F_A(s + 1, i - 1)$ in order to avoid rescanning $F_A(s, i - 1)$. Thus we can answer all prefix-LCSs for a fixed i in time linear in the size of the $i - 1$ -th row of F_A , meaning that we can compute the prefix-LCSs for all i in time linear in the size of F_A , which is $O((\ell + 1)(n - \ell + 1))$. We remark that, in other words, the number of candidates has been reduced to the size of F_A .

In addition, for obtaining the next suffix-LCS length of $A[e_i + 1..n]$ and $B[e_j + 1..n]$, we can use the same method as in the case of scanning only F_{A^R} with Algorithm I. In this case, we scan the $(n - e_j)$ -th column of F_{B^R} from the $(\min\{s', W_{B^R}[n - e_j]\})$ -th row to the top and we obtain the row number t . Then, we store t at $W_{B^R}[n - e_j]$.

Overall, since the total number of rescanned cells in F_{A^R} and F_{B^R} is linear in the size of F_A , and since the total number of scanned cells except for the aforementioned rescanned cells is linear in the total size of F_{A^R} and F_{B^R} , we can answer the suffix-LCSs for all i in time linear in the total size of F_{A^R} , F_{B^R} and F_A , which is $O((\ell + 1)(n - \ell + 1))$.

- (2) When $F_A(1, i - 1) = \text{undefined}$: In this second case, we compute candidates (i.e. the sums of the lengths of pairs of a prefix-LCS and suffix-LCS) as follows: We start scanning the column from the top $F_A(i - n - \ell, i - 1)$. If $F_A(i - n - \ell, i - 1) \leq j - 1$, then the length of a prefix-LCS for the first query $(i - 1, j - 1)$ can be found in the table (similar to Case (1)) and next queries about interval elements can also be answered in a similar way. Otherwise (if $F_A(i - n - \ell, i - 1) > j - 1$), the length which we want may be in the “undefined” domain. Then we use the other table F_B . We scan the $(j - 1)$ -th column in F_B from the top to the bottom in order to find the maximum value which is less than or equal to $i - 1$. Then, the row number s is the length of a prefix-LCS of $A[1..i - 1]$ and $B[1..j - 1]$. In addition, we compute the suffix-LCS length of $A[e_i + 1..n]$ and $B[e_j + 1..n]$ by the same method as Case (1).

After that, we compute the prefix-LCS of $A[1..i' - 1]$ and $B[1..j - 1]$, and the suffix-LCS of $A[e_{i'} + 1..n]$ and $B[e_j + 1..n]$ for the fixed j and the next index i' ($i < i' \leq n$) in a similar way, where i' is the index of the next interval element that is stored after i in θ_A . We obtain the next interval element $[i'..e_{i'}]$ or an index to that element from $\theta_A[F_B(s + 1, j - 1) + 1]$. Here again, we can start scanning from $F_B(s + 1, i - 1)$ in order to avoid rescanning $F_B(s, i - 1)$ for computing the prefix-LCS of $A[i' - 1..n]$ and $B[1..j - 1]$. In addition, we compute the suffix-LCS of $A[e_{i'} + 1..n]$ and $B[e_j + 1..n]$ in a similar way to Case (1). The difference is that we scan F_{A^R} and we use W_{A^R} instead of W_{B^R} . We repeat this method up to the end of θ_A .

We then delete the interval element $\theta_B[j] = [j, e_j]$ and $\theta_B[1]$ is assigned to $j + 1$, since we have completed computation for the interval $[j, e_j]$. With these updates done, we are able to access the interval elements by accessing at most two elements on θ_B . Next, we search $\theta_B[j + 1]$ for the next interval and repeat the above operations.

Overall, we can answer all prefix-LCSs in $O((\ell + 1)(n - \ell + 1))$ for all i (that is linear in the sum of the sizes of F_A and F_B). In other words, the number of candidates has been reduced to the total size of F_A and F_B . Since the total number of rescanned cells in F_{AR} and F_{BR} is linear in the total size of F_A and F_B , and since the total number of scanned cells except for the aforementioned rescanned cells is linear in the total size of F_{AR} and F_{BR} , we conclude that the suffix-LCSs for all i can be computed in time linear in the total sizes of F_A , F_B , F_{AR} , and F_{BR} , which is $O((\ell + 1)(n - \ell + 1))$.

Finally, we can compute all prefix/suffix-LCSs in $O((\ell + 1)(n - \ell + 1))$ total time. \square

Below we present some concrete examples on how our Algorithm II operates efficiently.

Example 1. Here we show an example of scanning F_{AR} , F_{BR} and updating W_{BR} in prefix/suffix-LCS computation in Algorithm II. See in Fig. 8 for illustrations.

Consider searching for s with $n - e_i = 8$ and $n - e_j = 5$ by scanning from F_{AR} and F_{BR} . Here we cannot obtain s from F_{AR} . Thus, we start scanning from $F_{BR}(W_{BR}[5], 5) = F_{BR}(5, 5)$ toward the top of the column, and we obtain $s = 4$ from $F_{BR}(4, 5) = 5 \leq 8$. Then, we update $W_{BR}[5] \leftarrow s = 4$.

Next, we search for s with $n - e_i = 7$ and $n - e_j = 5$ (since $n - e_i$ is in decreasing order). Likewise, we cannot obtain s from F_{AR} . Then we can restart scanning from $F_{BR}(W_{BR}[5], 5) = F_{BR}(4, 5)$ toward the top of the column for s . This permits us to avoid rescanning $F_{BR}[5, 5]$.

Overall, we can obtain the length of an STR-IC-LCS in $O(nr/\log r + n(n - \ell + 1))$ time in total. Also this algorithm requires space for storing all minimal intervals, tables, and arrays, namely, requiring $O(n + (\ell + 1)(n - \ell + 1)) = O((\ell + 1)(n - \ell + 1))$ space in the worst case. We have thus proven Theorem 5.

In addition, we can also compute an STR-IC-LCS (as a string) by storing a pair of minimal intervals that correspond to an STR-IC-LCS, and then using the same method as Algorithm I.

5. Alternative solution (Algorithm III) for STR-IC-LCS problem

In this section, we present our third solution Algorithm III to STR-IC-LCS. In the sequel, ℓ' denotes the solution length of the STR-IC-LCS for input strings A , B , and P .

We first compute the minimal intervals in A and B where P occurs as subsequence, using $O(nr/\log r)$ time as in Section 4. Let $\max b_A$ and $\max b_B$ respectively denote the beginning positions of the *rightmost* minimal intervals in A and B where P occurs as subsequence.

The following observation and lemma are a key to our third solution Algorithm III.

Observation 2. Prefix-LCS queries are never performed in Algorithm II for any pair (i, j) of positions such that $\max b_A \leq i \leq n$ and $\max b_B \leq j \leq n$.

Lemma 7. For any $1 \leq i < \max b_A$ and $1 \leq j < \max b_B$, $\text{lcs}(A[1..i], B[1..j]) \leq \ell'$.

Proof. Assume on the contrary that there exists a pair (i, j) of positions in A and B such that $1 \leq i < \max b_A$, $1 \leq j < \max b_B$, and $l = \text{lcs}(A[1..i], B[1..j]) > \ell'$. Then, there must exist an STR-IC-LCS of A , B , and P of length at least $l + r > \ell' + r \geq \ell'$. However, this contradicts that ℓ' is the STR-IC-LCS of A , B , and P . \square

Due to Observation 2, it suffices for us to work on the prefixes $\hat{A} = A[1..\max b_A - 1]$ and $\hat{B} = B[1..\max b_B - 1]$, namely, we build tables $F_{\hat{A}}$ and $F_{\hat{B}}$ and the other auxiliary tables for \hat{A} and \hat{B} . Then, it follows from Lemma 7 that the sparse tables $F_{\hat{A}}$ and $F_{\hat{B}}$ (and the other auxiliary tables) require $O((\ell' + 1)(n - \ell' + 1))$ space, and can be computed in $O(n(n - \ell' + 1))$ time, by applying the method described in Section 4 to A' and B' . We can use the same techniques for suffix-LCS queries.

Overall, we obtain the following:

Theorem 8. The STR-IC-LCS problem can be solved in $O(nr/\log r + n(n - \ell' + 1))$ time and $O((\ell' + 1)(n - \ell' + 1))$ space, where ℓ' is the length of an STR-IC-LCS of A , B , and P .

The merit of Algorithm III when compared to Algorithm II is summarized in the following lemma:

Lemma 9. There exist strings of length n for which Algorithm III uses only $O(n)$ space, while Algorithm II needs $\Theta(n^2)$ space.

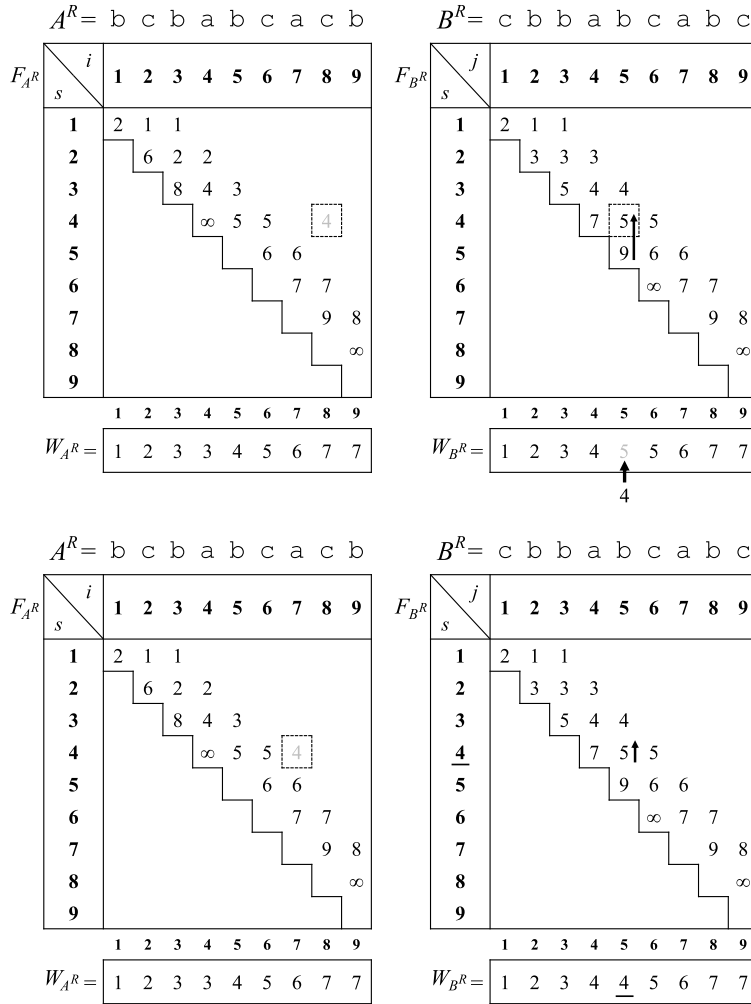


Fig. 8. An example of scanning F_{A^R}, F_{B^R} and updating W_{B^R} in prefix/suffix-LCS computation in Algorithm II. After the computation in the upper table, we can skip scanning $F_{B^R}(5,5)$ in the lower table.

Proof. Consider two strings $A = a^{n-1}$ and $B = a^{\frac{n}{2}} b^{\frac{n}{2}-1}$ of length $n - 1$ each such that $\ell = \text{lcs}(A, B) = n/2$. Then, consider strings $A' = a^i c a^{n-i-1}$ and $B' = a^{\frac{n-i}{2}} c a^i b^{\frac{n-i}{2}-1}$ of length n each, where $c \in \Sigma \setminus \{a, b\}$, and i is a constant. Then, the solution length ℓ' of the STR-IC-LCS of A', B' , and $P = c$ is $2i + 1 = O(1)$. Thus, Algorithm III uses $O((\ell' + 1)(n - \ell' + 1)) = O(n)$ space, while Algorithm II uses $O((\ell + 1)(n - \ell + 1)) = \Theta(n^2)$ space for A', B' , and P . \square

6. Conclusions

This paper proposed three space-efficient algorithms that find an STR-IC-LCS of two given strings A and B of length n with constrained pattern P of length r .

Our first solution, Algorithm I, works in $O(n^2)$ time with $O((\ell + 1)(n - \ell + 1))$ working space, where ℓ is the length of an LCS of A and B . This method improves on the space requirements of the algorithm by Deorowicz [9] that uses $\Theta(n^2)$ space, irrespective of the value of ℓ .

Our second solution, Algorithm II, runs in faster $O(nr/\log r + n(n - \ell + 1))$ time with the same $O((\ell + 1)(n - \ell + 1))$ working space. We have achieved this improved $O(nr/\log r + n(n - \ell + 1))$ -time complexity by carefully avoiding redundant scans in the dynamic programming tables.

Our third solution, Algorithm III, runs in $O(nr/\log r + n(n - \ell' + 1))$ time with $O((\ell' + 1)(n - \ell' + 1))$ working space, where ℓ' is the STR-IC-LCS length.

We note that all of our proposed algorithms are based on Nakatsu et al.'s algorithm [12] for finding (standard) LCS that runs in $O(n(n - \ell + 1))$ time with $O((\ell + 1)(n - \ell + 1))$ working space. The only overhead in our faster solution Algorithm II is the $nr/\log r$ additive factor for finding minimal intervals where P occur by using Das et al.'s method [15]. Bille et al. [16] showed that there is no strongly sub-quadratic $O((nr)^{1-\epsilon})$ time algorithm for finding such minimal intervals unless the famous SETH fails. Thus, as

long as computing such minimal intervals is involved, it might be difficult to drastically improve our $O(nr/\log r + n(n - \ell + 1))$ time complexity for STR-IC-LCS.

CRedit authorship contribution statement

Yuki Yonemoto: Writing – original draft, Methodology. **Yuto Nakashima:** Writing – original draft. **Shunsuke Inenaga:** Writing – review & editing, Writing – original draft, Conceptualization. **Hideo Bannai:** Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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