sciendo FCT Fundação para a Ciência e a Tecnologia CIUHCT Ludus

Domino Antimagic Squares and Rectangles

Alison M. Marr[1], *Christina Napier*[2], *Lauren M. Nelsen*[3], *Luke L. Nelsen*[4], *and Violeta Vasilevska*[5]

[1] *Southwestern University* , marra@southwestern.edu
[2] *Cedarville University* , christinanapier@cedarville.edu
[3] *University of Colorado-Colorado Springs* , lnelsen@uccs.edu
[4] *Colorado Springs, Colorado* , luke.l.nelsen@gmail.com
[5] *Utah Valley University* , Violeta.Vasilevska@uvu.edu

Abstract

A *domino antimagic square of order* n is an $n \times n$ array formed from a subset of the standard set of 28 dominoes such that the sums of the rows, columns, and two main diagonals form a set of $2n+2$ distinct, consecutive integers while an $m \times n$ *domino antimagic rectangle* is an $m \times n$ rectangular array formed from a subset of the standard set of 28 dominoes such that the sums of the rows and columns form a set of $m+n$ distinct, consecutive integers. This paper outlines what the possible dimensions are for $m \times n$ domino antimagic rectangles and provides many examples of both domino antimagic rectangles and squares. Many open questions are given at the end of the paper for future exploration.

Keywords: antimagic square, dominoes, domino magic square, domino antimagic square.

1 Introduction

The idea of creating magic squares using dominoes has been around for centuries. Martin Gardner describes the history of dominoes and some of the fun puzzles (including domino magic squares) people have attempted with dominoes in [Gar69]. The original idea was to use all the dominoes to form a 7×7 square with a column of zeros (thus, in actuality a 7×8 rectangle). As mathematicians studied these domino squares further, they formed squares of even order (i.e., 2×2, 4×4, and 6×6) and created odd domino magic squares by leaving a hole in the center [ZaS06]. See Figure 1 for two examples. Note that for each square the rows, columns, and main diagonals have equal sums and neither square has a repeated domino. Domino magic rectangles have also been studied, as have counting the number of different domino magic squares of a certain order [SpG08, Fir11].

DOI 10.2478/rmm-2024-0010

0	0	3	2
0	3	0	2
3	1	1	0
2	1	1	1

3	1	0	2	2
1	1	3	2	1
4	2		0	2
0	4	3	1	0
0	0	2	3	3

Figure 1: Domino magic squares.

Antimagic squares (as opposed to magic squares) were first defined by J.A. Lindon in 1962 in *Recreational Mathematics Magazine* [Lin62]. An antimagic square of order n is a square array of the first n^2 positive integers such that no two of the rows, columns, or main diagonals have the same sum and the list of sums formed is a set of consecutive integers. No antimagic square of order 2 exists as there would need to be six different sums and with the numbers 1 through 4 only five different sums of two numbers are possible. It has also been shown that no antimagic square of order 3 exists using a case by case analysis of the possible placements of the even numbers in the order 3 square [Tri69]. Hence, the smallest possible antimagic squares are of order 4. An example of an antimagic square of order 4 is shown in Figure 2 proving that order 4 is the smallest possible order for an antimagic square.

1	13	3	12
15	9	4	10
7	2	16	8
14	6	11	5

Figure 2: An antimagic square of order 4.

Given the existing work on antimagic squares and domino magic squares, it is natural to combine the two concepts. In this article, we uncover the world of domino antimagic squares and rectangles and provide some open questions for further exploration.

2 Definitions

A domino is a 1×2 rectangle with its two 1×1 squares labeled by the natural numbers. Here we consider the familiar double-6 set of 28 dominoes in which both labels (or pip counts) are from 0 to 6, inclusive.

Definition 2.1. *A* domino antimagic square of order n *is an $n \times n$ array tiled by a subset of the standard set of 28 dominoes such that the sums of the rows, columns, and two main diagonals form a set of $2n+2$ distinct, consecutive integers.*

Unlike with antimagic squares, using dominoes means that some numbers can be repeated as entries in the square (so long as the dominoes themselves are distinct). In addition, only the numbers 0 through 6 can be used. By noting there are only 28 dominoes and hence 56 possible entries covered by the dominoes, we can only possibly construct domino antimagic squares of order 2, 4, and 6 (odd ordered squares are not possible since dominoes are of size 1×2). However, there are several natural variations and generalizations of domino antimagic squares. For example, we can consider rectangular arrays instead of just squares.

Definition 2.2. *An $m \times n$* domino antimagic rectangle *is an $m \times n$ rectangular array formed from a subset of the standard set of 28 dominoes such that the sums of the rows and columns form a set of $m+n$ distinct, consecutive integers.*

It is important to recognize that not all $n \times n$ domino antimagic rectangles are domino antimagic squares of order n, as the diagonal sums of a domino antimagic rectangle do not have to be part of the set of consecutive integers formed by the sums. Likewise, not every domino antimagic square of order n is an $n \times n$ domino antimagic rectangle. If the diagonal sums happen to be in the middle of the list of ordered consecutive integers formed by the sums, the set of row and column sums would not be a set of consecutive integers.

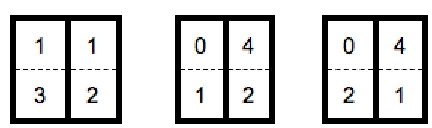

Figure 3: Examples of a 2×2 domino antimagic rectangle, a domino antimagic square, and one that is both.

When constructing domino antimagic squares/rectangles, close attention should be paid to both the antimagic sum properties and the existence of a legal tiling (see Figure 4). The counting of all possible domino antimagic squares or rectangles will be done up to symmetry where two domino antimagic squares or rectangles are the same if they can be obtained from each other by rotations and reflections.

Two straightforward methods allow us to generate new domino antimagic squares from a given one, assuming the resulting pip counts are between 0 and 6: adding a constant c to every cell (since all sums increase by nc) and subtracting every cell from a constant b (since each sum s becomes $nb - s$).

0	0	0	2
0	0	2	2
0	1	2	2
1	2	2	2

Figure 4: A 4×4 square that satisfies the rectangle antimagic sum property, but it is not a domino antimagic square.

Definition 2.3. *Two domino antimagic squares of order n are* equivalent *if one can be obtained from the other by adding the same integer value to all of the pip counts.*

Definition 2.4. *The* complement *of a domino antimagic square of order n is another domino antimagic square of order n created by replacing each entry i of the original domino antimagic square with entry $6 - i$.*

The concepts of equivalence and complements of squares cannot generally be extended to $m \times n$ domino antimagic rectangles as these operations might not shift the row sums in the same way as the column sums since m may not equal n. In this paper, we seek to find distinct domino antimagic squares up to symmetry, equivalence, and complements.

3 Domino antimagic rectangles

We first investigate domino antimagic rectangles, which despite having fewer conditions and therefore a greater flexibility, do not allow for as many possible sizes as one might expect.

3.1 Initial necessary conditions

Let's consider an $m \times n$ rectangle, $m \leqslant n$, that is tiled with a subset of the standard double-6 dominoes set without repetition.

- There are a finite number of possible rectangular arrays.
- The number of entries mn is at most 56 and must be an even number (since each domino covers exactly two entries). Hence, $n \leqslant \frac{56}{m}$ and m, n cannot be both odd numbers at the same time.

3.2 Additional parity conditions

Let Σ_R denote the sum of the row sums, Σ_C denote the sum of the column sums, and Σ_E denote the sum of all entries. Then $\Sigma_R = \Sigma_C = \Sigma_E$, which means that the sum of all row and column sums $\Sigma_R + \Sigma_C = 2\Sigma_E$ is an even integer. If the row and column sums

DOI 10.2478/rmm-2024-0010

form a set of distinct, consecutive integers, then we can write the least of these sums as k and we have

$$\begin{aligned}\Sigma_R + \Sigma_C &= \sum_{i=k}^{k+m+n-1} i = \sum_{i=0}^{m+n-1} (k+i) \\ &= (m+n)k + \frac{(m+n-1)(m+n)}{2} \qquad (1) \\ &= (m+n)\left[k + \frac{m+n-1}{2}\right] \qquad (2)\end{aligned}$$

Equation (1) shows that the case $m+n \equiv 2 \mod 4$ is impossible since $\Sigma_R + \Sigma_C$ would be odd. If $m+n$ is odd, then by equation (2) the values k and $\frac{m+n-1}{2}$ must have the same parity. We will now see that most of the remaining cases are also impossible.

3.3 Antimagic rectangle sizes must be somewhat balanced

By considering that the integer entries of the $m \times n$ rectangle are nonnegative, we can extract a relationship between m and n by using information about the least sum k. Since Σ_C is at least the sum of the smallest n possible sums, we have

$$\Sigma_C \geqslant \sum_{i=k}^{k+n-1} i = \frac{1}{2}(n^2 + 2kn - n)$$

Similarly, since Σ_R is at most the sum of the greatest m possible sums we have

$$\Sigma_R \leqslant \sum_{i=k+n}^{k+n+m-1} i = \frac{1}{2}(m^2 + 2km + 2nm - m)$$

Since $\Sigma_C = \Sigma_R$, this gives us the inequality

$$\begin{aligned} n^2 + n(2k - 2m - 1) + (m - 2km - m^2) &\leqslant 0 \qquad (3) \\ n^2 + n(-2m - 1) + (m - m^2) &\leqslant 0 \end{aligned}$$

where the last step uses the assumptions that k is nonnegative and $m \leqslant n$. Solving this last inequality for n yields

$$n \leqslant \frac{1}{2}\left(1 + 2m + \sqrt{8m^2+1}\right) \qquad (4)$$

So we have an upper bound for the number of columns in terms of the number of rows. When $1 \leqslant m \leqslant 4$, this upper bound is better than the area condition $n \leqslant \frac{56}{m}$. In Tables 1 and 3 the "□" symbol indicates that the sizes $m \times n$ meet the necessary conditions for a domino antimagic rectangle and to simplify the presentation, we assume $m \leqslant n$. In addition, an "E" symbol means that k must be even, while "O" means that k is odd.

$m\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
1		$\boxed{\text{O}}$													
2		$\square$	$\boxed{\text{E}}$		$\boxed{\text{O}}$										
3				$\boxed{\text{O}}$		$\boxed{\text{E}}$									
4				$\square$	$\boxed{\text{E}}$		$\boxed{\text{O}}$	$\square$	$\boxed{\text{E}}$						
5						$\boxed{\text{O}}$		$\boxed{\text{E}}$		$\boxed{\text{O}}$					
6						$\square$	$\boxed{\text{E}}$		$\boxed{\text{O}}$						
7								$\boxed{\text{O}}$							

Table 1: Table of possible sizes for domino antimagic rectangles.

An interesting note: By applying the Triangle Inequality in (4) we can show that $n \leqslant m + m\sqrt{2} + 1$. So regardless of whether dominoes are being used, we have the following "balance" for any $m \times n$ rectangular array of nonnegative integers which has distinct row and column sums forming a set of consecutive integers: the ratio $\frac{n}{m}$ cannot exceed $1 + \sqrt{2} + \frac{1}{m}$.

3.4 Tighter bounds using domino sums

While the previously mentioned balance condition for antimagic rectangles is perhaps of independent interest when entries can be any nonnegative integers, there is a stricter relationship between m and n in a domino antimagic rectangle. Because entries in a domino antimagic rectangle must be placed from the standard set of dominoes, the least sum k can take on only certain values. By using inequality (3) and these values of k, we can use the additional information from the dominoes to identify more impossible cases.

As an example of how the sum of domino pips can provide much more additional information, consider the 7×8 case. Since all 28 dominoes must be used in the rectangle, $\Sigma_E = 8 \sum_{i=0}^{6} i = 168$. From equation (1) we obtain

$$336 = 2\Sigma_E = \Sigma_R + \Sigma_C = 15k + 105$$

which implies $k = 15.4$, which is impossible. Hence, no 7×8 domino antimagic rectangle can be constructed.

In general, we can use equation (1) and $\Sigma_R + \Sigma_C = 2\Sigma_E$ to solve for k:

$$k = \frac{2\Sigma_E - \frac{1}{2}(m+n-1)(m+n)}{m+n} \tag{5}$$

However, in our remaining cases we have $mn < 56$ and not all 28 dominoes would be used in the rectangular array. Although we cannot exactly calculate a single value for Σ_E, we can still calculate exact bounds for Σ_E. Let $s(N)$ and $S(N)$ denote the minimum and maximum values, respectively, that can be obtained by adding the pips from N distinct dominoes from the standard set. (For ease of reference, we have included the values of

DOI 10.2478/rmm-2024-0010

N	0	1	2	3	4	5	6	7	8	9	$\cdots$	26	27	28
$s(N)$	0	0	1	3	5	8	11	15	19	23	$\cdots$	145	156	168
$S(N)$	0	12	23	33	43	52	61	69	77	85	$\cdots$	167	168	168

Table 2: Values of s and S for number of dominoes N. Note that $s(N)$ is the sequence of partial sums of OEIS A55086 for $N \leqslant 19$.

the functions s and S in Table 2.) Replacing Σ_E in equation (5) the following lower and upper bounds for k are obtained:

$$\frac{2s\left(\frac{mn}{2}\right) - \frac{1}{2}(m+n-1)(m+n)}{m+n} \leqslant k \leqslant \frac{2S\left(\frac{mn}{2}\right) - \frac{1}{2}(m+n-1)(m+n)}{m+n}$$

Every remaining triple m, n, k is now checked and those which do not satisfy inequality (3) are removed. Finally, $k \leqslant 6 \cdot \max(m,n) - (m+n-1) = 5n - m + 1$ so that the required sums do not get too large. The possible values of k for each of the remaining cases (that obey the necessary bounds and parity conditions) are listed inside each box in Table 3.

$m\backslash n$	2	3	4	5	6	7
1	1					
2	$k \leqslant 9$	0,2,4				
3			1,3,5,7,9		2	
4			$2 \leq k \leq 15$	4,6,8,10,12,14,16		
5					7,9,11,13,15,17	
6					$8 \leq k \leq 17$	10,12,14,16

Table 3: Table of possible sizes and k values for domino antimagic rectangles.

With this table in hand, a natural question is whether there exist such domino antimagic rectangles for these ten remaining possible sizes and their k values. We provide examples of all 61 cases in Figures 5, 6, and 7.

We also make some remarks in some of the cases:

- Exactly one 1×2 domino antimagic rectangle exists (see Figure 5).
- The 2×2 case is small enough that we have performed an exhaustive search with computer assistance: there are 70 such rectangles (up to symmetry) which take on k-values 0 through 9.
- The 2×3 case is small enough that we have performed an exhaustive search with computer assistance: there are 331 such rectangles (up to symmetry) which take on k-values 0, 2, and 4.
- The examples of 4×4 domino antimagic rectangles shown in Figure 6 are actually domino antimagic squares of order 4 with the property that their diagonals have sums $k-2$ and $k-1$. This, along with the complements of the examples with $k = 9$ and $k = 10$ (which have least sums 15 and 14, respectively), shows the existence of domino antimagic squares with all possible values for the least sum (0 through 15).

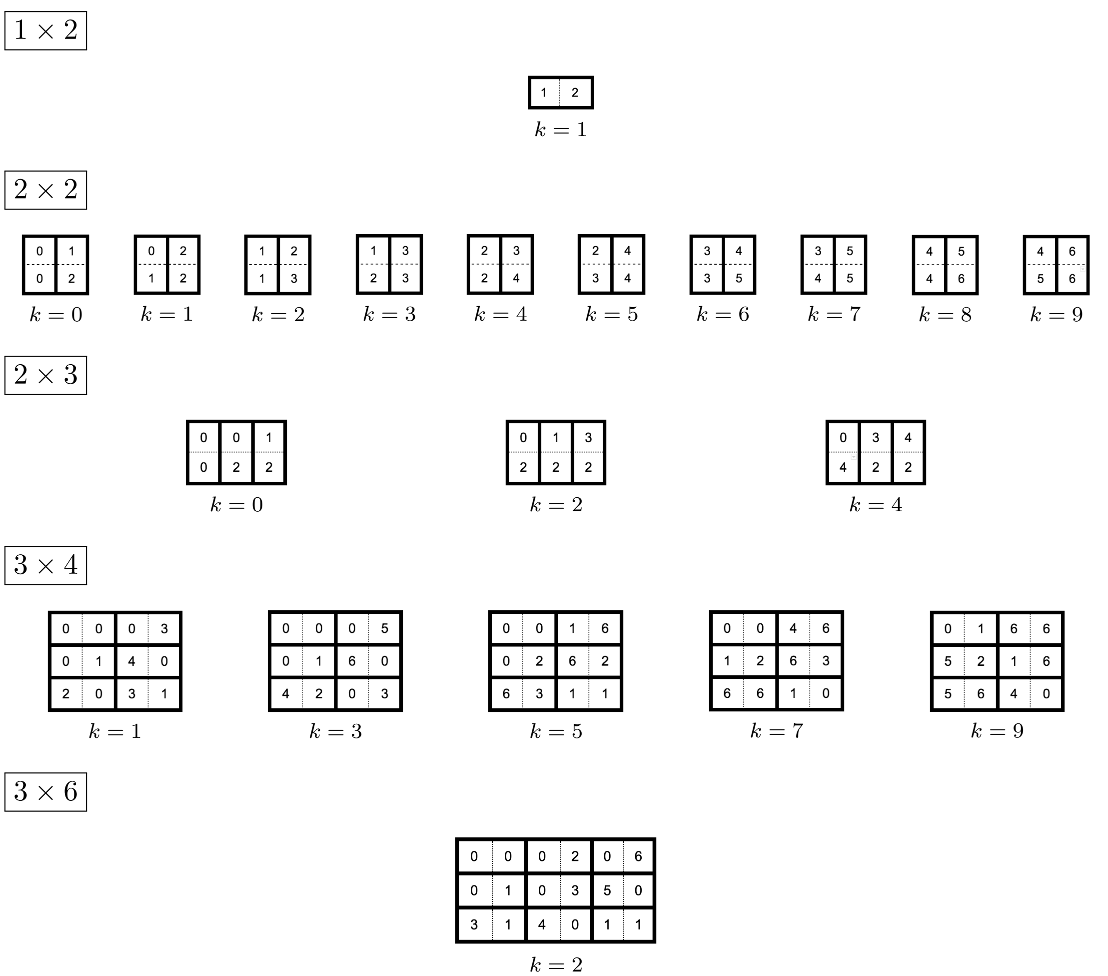

Figure 5: Examples of domino antimagic rectangles of dimensions up to 3×6 and their possible least sums k.

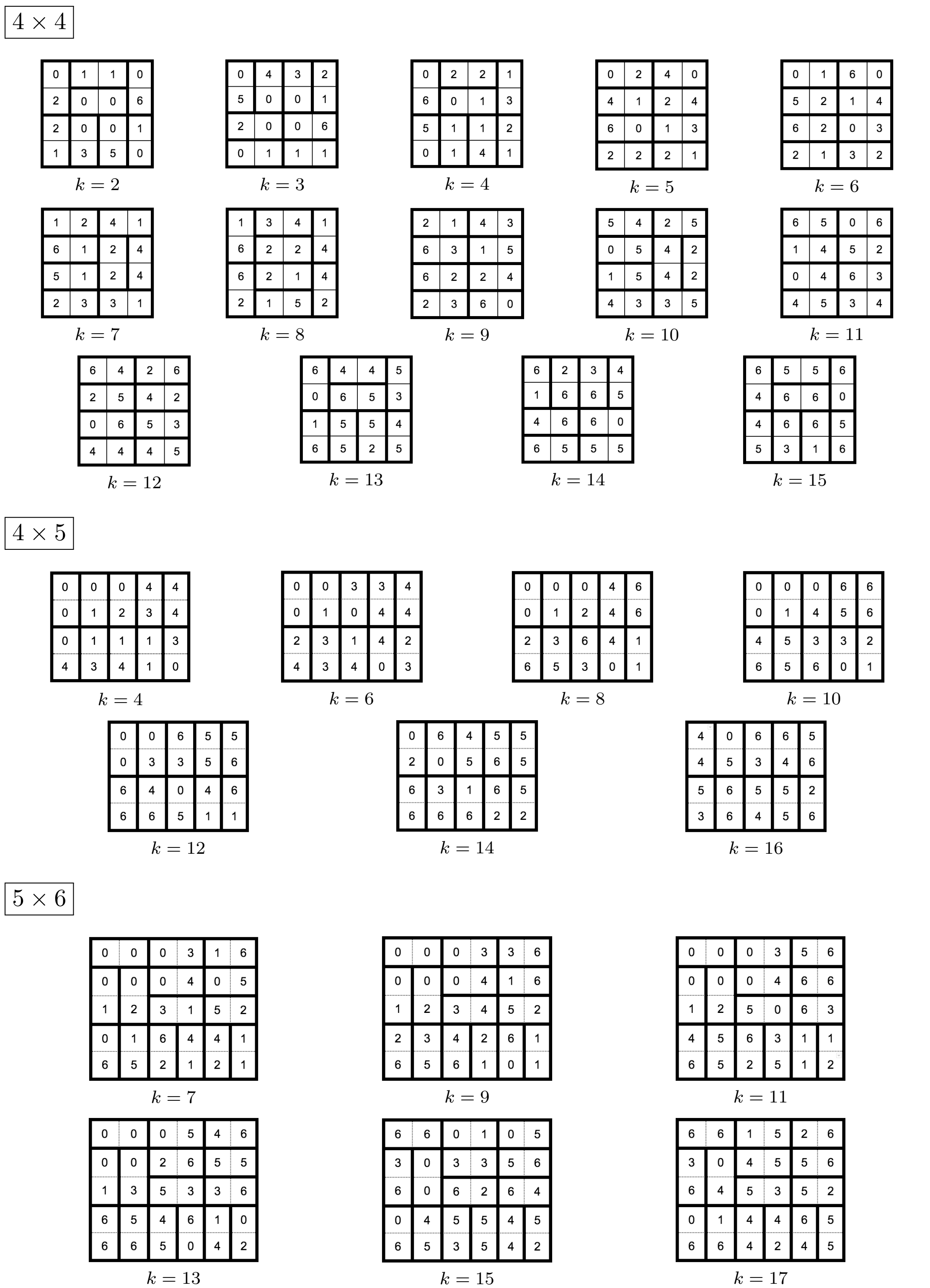

Figure 6: Examples of domino antimagic rectangles of dimensions 4×4, 4×5, and 5×6 and their possible least sums k.

$\boxed{6 \times 6}$

2	1	0	0	0	5
3	6	0	1	0	3
2	6	0	4	0	6
3	3	3	1	5	1
0	2	4	2	6	1
4	1	2	2	1	1

$k = 8$

2	1	0	0	0	6
3	6	0	1	0	3
2	6	0	4	0	5
3	3	1	1	1	3
0	2	5	6	6	1
4	1	4	3	4	2

$k = 9$

2	1	0	0	1	6
3	6	0	1	0	3
2	6	0	4	0	5
3	3	0	6	4	2
0	2	5	1	5	2
4	1	6	4	2	3

$k = 10$

2	1	0	0	3	5
3	6	0	1	0	3
2	6	0	4	0	5
3	3	0	6	1	3
0	2	6	5	5	4
4	1	6	4	6	1

$k = 11$

2	1	0	0	4	5
3	6	0	1	0	3
2	6	0	4	0	5
3	3	6	0	5	1
0	2	4	6	5	6
4	1	5	5	6	1

$k = 12$

4	5	6	6	2	1
3	0	6	5	6	3
4	0	6	2	6	1
3	3	0	6	1	5
6	4	2	0	1	0
2	5	1	1	0	5

$k = 13$

4	5	6	6	3	1
3	0	6	5	6	3
4	0	6	2	6	1
3	3	6	0	5	3
6	4	0	1	1	2
2	5	0	2	0	5

$k = 14$

4	5	6	6	5	0
3	0	6	5	6	3
4	0	6	2	6	1
3	3	6	0	2	4
6	4	1	5	1	4
2	5	0	2	4	3

$k = 15$

4	5	6	6	6	0
3	0	6	5	6	3
4	0	6	2	6	1
3	3	5	5	5	3
0	4	1	0	0	5
2	5	2	3	2	4

$k = 16$

4	5	6	6	6	1
3	0	6	5	6	3
4	0	6	2	6	0
3	3	3	5	1	5
6	4	2	4	0	5
2	5	4	4	5	5

$k = 17$

$\boxed{6 \times 7}$

2	3	0	0	0	1	5
1	3	0	3	6	3	1
3	0	0	0	1	5	5
6	2	1	4	1	2	3
2	4	4	0	6	3	2
6	1	5	5	1	2	2

$k = 10$

2	3	0	0	0	5	6
1	3	0	3	5	2	4
3	0	0	0	6	3	5
6	2	1	4	0	5	5
2	4	5	1	1	5	1
6	1	6	6	3	1	1

$k = 12$

2	3	6	0	0	1	5
1	3	6	1	4	4	4
3	0	5	0	0	6	5
6	2	5	3	5	0	3
2	4	0	5	1	5	4
6	2	0	6	6	2	4

$k = 14$

2	3	6	0	0	3	4
1	3	6	2	6	4	4
3	2	5	0	1	5	3
6	2	5	5	6	2	2
2	4	0	4	5	5	5
6	2	0	6	3	4	6

$k = 16$

Figure 7: Examples of domino antimagic rectangles of dimensions 6×6 and 6×7 and their possible least sums k.

DOI 10.2478/rmm-2024-0010

4 Domino antimagic squares

Our investigation into domino antimagic rectangles sheds some light on domino antimagic squares, but the latter will require separate attention. As mentioned before, because of the additional condition of the diagonal sums, not all domino antimagic squares are domino antimagic rectangles.

4.1 Domino antimagic squares of order 2

First, note that pip counts cannot be repeated as this would create two rows/ columns/ diagonals with the same sum. Hence, a subset of size four from the set $\{0, 1, 2, 3, 4, 5, 6\}$ is needed to create the four distinct entries of an antimagic square of order 2. In addition, from the four distinct integers we need the six possible pairs to have distinct sums that form a set of consecutive integers.

Let $\{a, a+i, a+j, a+z\}$, where $1 \leqslant i < j < z$, be our subset of size four and let k be the least of the six sums (i.e., $k = 2a + i$). First note that $z \geqslant 3$ and also that each entry appears in exactly three of the six sums. So we have

$$3\Big(a + (a+i) + (a+j) + (a+z)\Big) = k + (k+1) + \cdots + (k+5) = 6k + 15$$

which gives $z = 5 + i - j$. So $z > 3$ since $(i, j, z) = (1, 2, 3)$ is not a solution to this equation, and $z < 5$ since $i - j < 0$. Therefore, we must have $z = 4$ and $j - i = 1$. The list below contains all possible subsets of integers that can be used to create domino antimagic squares of order 2:

$$\{0, 1, 2, 4\} \;\; \{0, 2, 3, 4\} \;\; \{1, 2, 3, 5\} \;\; \{1, 3, 4, 5\} \;\; \{2, 3, 4, 6\} \;\; \{2, 4, 5, 6\}$$

This list also shows that all sets of entries from domino antimagic squares of order 2 can be obtained from the set $\{0, 1, 2, 4\}$ via equivalence and complements. Considering placement of dominoes, each set of entries from the list above can generate six domino antimagic squares of order 2 up to symmetry (see Figure 8).

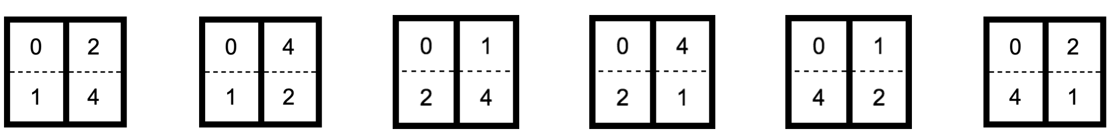

Figure 8: The six domino antimagic squares of order 2 formed from the entries $\{0, 1, 2, 4\}$ (up to symmetry).

4.2 Domino antimagic squares of larger order

The example in Figure 9 completes the list of possible orders of domino antimagic squares using the standard double-6 set of dominoes.

DOI 10.2478/rmm-2024-0010

2	1	0	0	4	5
3	6	0	1	0	3
2	6	0	2	0	5
3	3	5	6	6	0
0	4	6	6	2	4
1	4	3	1	6	4

Figure 9: An example of a domino antimagic square of order 6.

5 A domino antimagic square using all dominoes

As shown in the section "Domino Antimagic Rectangles", it is impossible to use all 28 dominoes to form a 7×8 antimagic domino rectangle. However, it is possible to use all 28 dominoes to create a 7×8 rectangle containing a 0-pip end lying horizontally on the right column (see Figure 10) which results in a special antimagic square of order 7 in the remaining columns. Using similar arguments about the row, column, and diagonal sums as in the section "Domino Antimagic Rectangles," we can determine that the only possible values for the least sum k are $k = 16, 17$.

4	3	3	5	1	1	6	0
1	2	1	3	3	6	0	0
1	4	1	5	1	6	1	0
4	4	2	2	6	6	2	0
5	6	2	4	2	3	3	0
3	3	4	6	6	2	4	0
2	5	4	5	5	5	5	0

4	3	3	5	1	1	6	0
1	2	1	3	4	6	0	0
1	4	1	5	1	6	1	0
2	3	2	2	2	5	2	0
5	5	4	4	6	2	3	0
5	6	3	3	4	5	4	0
6	3	6	6	4	2	5	0

Figure 10: A special type of domino antimagic square of order 7 with a column of zeros for $k = 16$ and $k = 17$.

DOI 10.2478/rmm-2024-0010

6 Open Puzzles

6.1 Unsettled possibilities for domino antimagic squares and rectangles

Although we have settled which sizes of domino antimagic squares and rectangles are possible using the standard double-6 set of dominoes, the question remains largely open for arbitrary domino sets. Which sizes and sums of domino antimagic rectangles or squares are possible using dominoes from the set using values up to d?

6.2 Variations on domino antimagic squares of odd order

Using a column of zeros as described in the order 7 case is one way to create a special type of domino antimagic square of odd order. Another possible way to create a domino antimagic square of odd order is to include a central hole that is assumed to have value 0. Figure 11 shows examples of domino antimagic squares of order 3 with a central hole. In particular, are there domino antimagic squares of order 5 and 7 with a central hole?

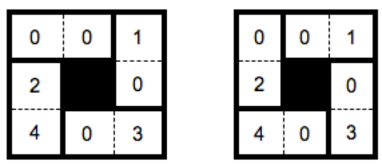

Figure 11: Two different domino antimagic squares of order 3 with a central hole with the same underlying entries.

A related question is that of "hole rearrangeability," by which we mean that a domino antimagic square of odd order with a central hole can be rearranged so that the hole is elsewhere and yet the sums of the new rows, columns, and main diagonals still form a set of distinct, consecutive integers. Are there general conditions which imply hole rearrangeability or prevent it?

6.3 Placing all dominoes within larger square arrays

The domino antimagic square of order 7 with a column of zeros is the only special type of domino antimagic square discussed in this paper which allows all 28 dominoes to be used. By allowing some entries in a square array to be empty and count as 0 (just like a central hole space), all 28 dominoes could be placed into larger square arrays and perhaps have antimagic sum properties. Consider an 8×8 array in which the four corners and four central entries are designated as empty, as shown in Figure 12. Is it possible to place the 28 dominoes in this array such that the sums formed by the rows, columns, and main diagonals are distinct and consecutive integers? For what larger squares (of order up to 12) and empty entry designations (preferably symmetric) can the dominoes be placed in this way?

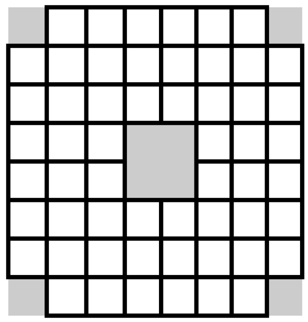

Figure 12: An 8×8 array with designated empty entries to accommodate all 28 dominoes.

References

[Fir11] Firstov, V.E. (2011). Combinatorial theory of domino magic squares. *Chebyshevskii Sb.* 12 (2): 135–150.

[Gar69] Gardner, Martin. (1969). Mathematical Games. *Scientific American.* 221(6): 122-127.

[Lin62] Lindon, J. A. (1962). Anti-Magic Squares. *Recr. Math. Mag.*, 7: 16–19.

[SpG08] Springfield, M., Goddard, W. (2008). The Existence of Domino Magic Squares and Rectangles. *Bull. Inst. Combin. Appli.* 52:101–108.

[Tri69] Trigg, C.W. (1969). The Sums of Third-Order Antimagic Squares. *J. of Rec. Math.* 2:250–254.

[ZaS06] Zulauf, A., Stokes, B. (2006). Magic Domino Squares 1-6, *New Zealand Mathematics Magazine* 43 (2): 54–59.

DOI 10.2478/rmm-2024-0010