Strong Distance of Complete Bipartite Graphs

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Abstract

For two vertices $u, v$ in a strong digraph $D$, the strong distance $sd(u,v)$ between $u$ and $v$ is the minimum size of a strong subdigraph of $D$ containing $u$ and $v$. For a vertex $v$ of $D$, the strong eccentricity $se(v)$ is the maximum strong distance between $v$ and all vertices in $D$. The strong radius $srad(D)$ is the minimum strong eccentricity among all the vertices of $D$, and the strong diameter $sdiam(D)$ is the maximum strong eccentricity among all the vertices of $D$. The lower orientable strong radius $srad(G)$ of a graph $G$ is the minimum strong radius over all strong orientations of $G$. The upper orientable strong radius $SRAD(G)$ of a graph $G$ is the maximum strong radius over all strong orientations of $G$. The lower orientable strong diameter $sdiam(G)$ of a graph $G$ is the minimum strong diameter over all strong orientations of $G$. The upper orientable strong diameter $SDIAM(G)$ of a graph $G$ is the maximum strong diameter over all strong orientations of $G$. This paper provides the lower and upper orientable strong radius and diameter of complete bipartite graphs.

Keywords: Strong distance, complete bipartite graph, strong radius, strong diameter.

1 Introduction

In [2], they defined strong distance $sd(u,v)$ between $u$ and $v$ for two vertices $u, v$ in a strong digraph $D$ as is the minimum size of a strong subdigraph of $D$ containing $u$ and $v$. Follow the definitions in [2], the strong eccentricity $se(v)$ of a vertex $v$ in $D$ is

$$se(v) = \max \{sd(v,x) : x \in V(D)\};$$

and the strong radius $srad(D)$ of $D$ is

$$srad(D) = \min \{se(v) : v \in V(D)\};$$

while the strong diameter $sdiam(D)$ of $D$ is

$$sdiam(D) = \max \{se(v) : v \in V(D)\}.$$  

For a connected graph $G$, we define the lower orientable strong radius $srad(G)$ of $G$ by $srad(G)=\min\{srad(D) : D$ is a strong orientations of $G\}$; while the upper orientable strong radius $SRAD(G)$ of $G$ by $SRAD(G)=\max\{srad(D) : D$ is a strong orientations of $G\}$. We also define the lower orientable strong diameter $sdiam(G)$ of $G$ by $sdiam(G)=\min\{sdiam(D) : D$ is a strong orientations of $G\};$ while the upper orientable strong diameter $SDIAM(G)$ of a graph $G$ by $SDIAM(G)=\min\{sdiam(D) : D$ is a strong orientations of $G\}$.

In [1] they discussed the strong radius and strong diameter of complete graphs, and [2] gave upper bound for the strong diameter of a strong oriented graph. Here we turn our attention to the complete bipartite graphs.

2 Orientable strong radius

The girth $g(G)$ of a graph $G$ with cycles is the length of a smallest cycle in $G$. Lemma 1 is used in the proof of the lower orientable strong radius of the complete bipartite graphs.

Lemma 1 Let $G = (V,E)$ be a connected graph with $n$ vertices and $D$ be an orientation of $G$. Then $srad(D) \geq g(G)$.

Proof: Let $V(G)=\{v_i : 1 \leq i \leq n\}$, and $C = \{v_1, v_2, \ldots , v_i, v_1\}$ be a cycle whose length is $l = g(G)$. Orient the edges of $G$ cyclically so that $(v_j, v_{j+1})$ is an arc in $D$ for $1 \leq j \leq l - 1$ and $(v_1, v_2)$ is an arc in $D$. Then there exists a strong distance $sd(v_r, v_s) = l$ for $1 \leq r, s \leq l$, and $r \neq s$. This implies $srad(D) \geq l = g(G)$. \hfill $\Box$
**Theorem 1** Let $2 \leq m \leq n$. Then $\text{srad}(K_{m,n}) = 4$.

**Proof:** Let $V = \{v_1, v_2, \ldots, v_m\}$, $U = \{u_1, u_2, \ldots, u_n\}$ be two partite sets of $K_{m,n}$. For any strong orientation of $K_{m,n}$, since $g(K_{m,n}) = 4$, by Lemma 1, we have $\text{srad}(G) \geq 4$. Consider an orientation $D_1$ of $K_{m,n}$ such that $(v_i, u_i)$ is an arc in $D_1$ for $1 \leq i \leq m$; $(v_m, u_j)$ is an arc in $D_1$ for $m + 1 \leq j \leq n$, and orient all other edges from $U$ to $V$. Then $\text{srad}(D_1) = 4$, which implies $\text{srad}(K_{m,n}) = 4$. \hfill \Box

Figure 1 shows an orientation $D$ of $K_{3,4}$ such that $\text{srad}(D) = 4$.

Next theorem give the lower bound of the upper orientable strong radius of the complete bipartite graphs.

**Theorem 2** Let $2 \leq m \leq n$. Then

$$\text{SRAD}(K_{m,n}) \geq \begin{cases} m + 1 & \text{if } m \text{ is odd,} \\ m + 2 & \text{if } m \text{ is even.} \end{cases}$$

**Proof:** Let $V = \{v_1, v_2, \ldots, v_m\}$, $U = \{u_1, u_2, \ldots, u_n\}$ be two partite sets of $K_{m,n}$.

Case 1: Let $m = n$. Consider an orientation $D_1$ of $K_{m,m}$ such that $(v_i, u_{i+1})$ is an arc in $D_1$ for $1 \leq i \leq m - 1$, $(v_m, u_1)$ is an arc in $D_1$ for $2 \leq k \leq m$, $l < k$, and orient all other edges from $U$ to $V$. Then $\text{srad}(D_1) = \begin{cases} m + 1 & \text{if } m \text{ is odd,} \\ m + 2 & \text{if } m \text{ is even,} \end{cases}$ which implies $\text{SRAD}(K_{m,n}) \geq \begin{cases} m + 1 & \text{if } m \text{ is odd,} \\ m + 2 & \text{if } m \text{ is even.} \end{cases}$

Examples of the orientation for $K_{5,5}$ and $K_{4,4}$ are shown in Figure 2 and 3.

**3 Orientable strong diameter**

In this section, we provide the upper orientable strong diameter and lower orientable strong diameter for the complete bipartite graphs.

**Lemma 2** For any orientation $D$ of $K_{m,n}$, $\text{sd}(u,v)$ is even for $u,v \in V(D)$.

**Proof:** Let $V = \{v_1, v_2, \ldots, v_m\}$, $U = \{u_1, u_2, \ldots, u_n\}$ be two partite sets of $K_{m,n}$. Let $r$, $s$ be two vertices in $D$, which are in the same partite set. Let $P_1$ be a shortest directed path from $r$ to $s$, $P_2$ be a shortest directed path from $s$ to $r$. We know that both $|P_1|$ and $|P_2|$ are even.

Case 1: $P_1$ and $P_2$ have even or no common edge. Then $\text{sd}(r,s) = |P_1| + |P_2|$ which is even.

Case 2: $P_1$ and $P_2$ have odd common edge. Without loss of generality, assume that there is only one common edge (say $xy$).

![Figure 1: An orientation $D$ of $K_{3,4}$ such that $\text{srad}(D) = 4$.](image1.png)

![Figure 3: An orientation to reach lower orientable strong radius of and $K_{4,4}$.](image3.png)
Figure 2: An orientation to reach lower orientable strong radius of $K_{5,5}$.

Figure 4: An orientation to reach lower orientable strong radius of $K_{5,6}$.

Figure 5: An orientation to reach lower orientable strong radius of $K_{4,5}$.
Subcase 1: $x$ is in the same partite set of vertices $r$ and $s$. Let $(a_1, x)$ be an arc on $P_2$ in $D$, $(b_1, x)$ be an arc on $P_1$ on $D$. Since $(s, b_1)$ must be an arc in $D$, we have a new path $P_2''$, which implies that $P_1$ and $P_2''$ have two common edges. Then $sd(r, s) = |P_1| + |P_2''|$, which is even.

Subcase 2: $x$ is in the other partite set of $r$ and $s$. Let $(y, a_2)$ be an arc on $P_2$, $(y, b_2)$ be an arc on $P_1$. Since $(b_2, r)$ must be an arc in $D$, we have a new path $P_2'$, which implies that $P_1$ and $P_2'$ have two common edges. Then $sd(r, s) = |P_1| + |P_2'|$, which is even.

By Case 1 and 2, we know that $sd(u, v)$ is even for all $u, v \in V(D)$. 

\[ \text{Proposition 1 (From [2]) For every strong orientable graph } D, \]
\[ \text{srad}(D) \leq \text{sdiam}(D) \leq 2\text{srad}(D). \]

\[ \text{Theorem 3 Let } 2 \leq m \leq n. \text{ Then} \]
\[ \text{sdiam}(K_{m,n}) = \begin{cases} 4 & \text{if } m = n \\ 6 & \text{if } m \neq n. \end{cases} \]

\[ \text{Proof:} \text{ Let } V = \{v_1, v_2, \ldots, v_m\}, U = \{u_1, u_2, \ldots, u_n\} \text{ be two partite sets of } K_{m,n}. \]

For $m = n$, since $g(K_{m,n}) = 4$, by Lemma 1, $\text{srad}(D) \geq 4$ for any strong orientation $D$ of $K_{m,n}$. By Proposition 1, such $\text{sdiam}(D) \geq \text{srad}(D)$, hence $\text{sdiam}(D) \geq 4$. Consider an orientation $D_1$ such that $(u_i, v_i)$ is an arc in $D_1$ for $1 \leq i \leq m$, and all other edges are oriented all other edges from $V$ to $U$. Then $\text{sdiam}(D_1) = 4$ which implies that $\text{sdiam}(K_{m,n}) = 4$.

For $m \neq n$, suppose there exists an orientation $D_2$ such that $\text{sdiam}(K_{m,n}) = 4$. For any two vertices $u_i, u_q$ in $U$, $1 \leq p, q \leq n$, there exist $v_r, v_s \in V$, such that $(u_p, v_r), (v_r, u_q), (u_q, v_s), (v_s, u_p)$ are arcs in $D_2$. This implies that it takes at least $n$ vertices in $V$, which is a contradiction. So $\text{sdiam}(K_{m,n}) > 4$, by Lemma 2, we know that $\text{sdiam}(K_{m,n}) \geq 6$. Consider an orientation $D_3$ such that $(u_i, v_i)$ is an arc in $D_3$ for $1 \leq i \leq m$, $(u_j, v_m)$ is an arc in $D_3$ for $m + 1 \leq j \leq n$, and oriente all other edges from $V$ to $U$. Then $\text{sdiam}(D_3) = 6$, which implies that $\text{sdiam}(K_{m,n}) = 6$. 

\[ \text{Theorem 4 Let } 2 \leq m \leq n. \text{ Then} \]
\[ \text{SDIAM}(K_{m,n}) = \begin{cases} 2m & \text{if } m = n, \\ 2m + 2 & \text{if } m \neq n. \end{cases} \]

Figure 6 and 7 show an orientation to reach lower orientable strong diameter of $K_{4,4}$ and $K_{4,5}$.

\[ \text{Figure 6: An orientation to reach lower orientable strong diameter of } K_{4,4}. \]

\[ \text{Proof:} \text{ Let } V = \{v_1, v_2, \ldots, v_m\}, U = \{u_1, u_2, \ldots, u_n\} \text{ be two partite sets of } K_{m,n}. \]

For $m = n$, consider an orientation $D_1$ of $K_{m,n}$. Let $(u_m, v_1)$ be an arc in $D_1$, and let $P_1 : v_1, u_1, v_2, u_2, \ldots, v_m, u_m$ be a directed path whose length is $2m - 1$. Then $\text{sd}(v_1, u_m) = 2m$. Since every vertex of $K_{m,n}$ has lying on directed path $P_1$, we know that $P_1$ is a longest possible directed path over all orientations of $K_{m,n}$. This implies $\text{sdiam}(D) \leq 2m$ for any orientation $D$ of $K_{m,n}$. Consider an orientation $D_2$ such that $(v_i, u_i+1)$ is an arc in $D_2$ for $1 \leq i \leq m - 1$, $(v_k, u_i)$ is an arc in $D_2$ for $2 \leq k \leq m, l < k$, and all other edges are assign directed from $U$ to $V$. Then $\text{sdiam}(D_2) = 2m$ which implies that $\text{SDIAM}(K_{m,n}) = 2m$. An orientation to reach upper orientable strong diameter of $K_{5,5}$ and $K_{4,4}$ are shown in Figure 2 and Figure 3.

For $m \neq n$, consider an orientation $D_3$ such that $(v_i, u_{i+1})$ is an arc in $D_3$ for $1 \leq i \leq m, (v_m, u_j)$ is an arc in $D_3$ for $m + 2 \leq j \leq n$, $(v_k, u_i)$ is an arc in $D_3$ for $2 \leq k \leq m, l < k$, and oriente all other edges from $U$ to $V$. Then $\text{sdiam}(D_3) = 2m + 2$. Assume that $\text{SDIAM}(K_{m,n}) = 2m + 2$. For any orientation $D$ of $K_{m,n}$, let $r, s$ be two vertices which implies that $\text{sd}(r, s) = \text{sdiam}(D)$.

Case 1: $r, s$ are not in the same partite set. Without loss of generality, let $\text{sd}(v_1, u_m) = \text{sdiam}(D)$. Let $(u_n, v_1)$ be an arc in $D$, and $P_2 : v_1, u_1, v_2, u_2, \ldots, v_m, u_m$ be a directed path whose length is $2m - 1$. Since $P_2$ is the longest possible directed path in $D$, we have $\text{sd}(v_1, u_m) = 2m$, which implies that $\text{sdiam}(D) \leq 2m$.

Case 2: $r, s$ are in the same partite set. Without loss of generality, let $\text{sd}(u_1, u_n) = \text{sdiam}(D)$. Let $p$ be the number of vertices in $V$ on $u_1 - u_n$ directed path, $q$ be the number of vertices in $V$ on $u_n - u_1$ directed path. Then $2(p + q) > 2m + 2$, which
implies that \( p + 1 > m + 1 \). Let \( P_3 \) be a directed path from \( u_1 \) to \( u_n \), \( P_4 \) be a directed path from \( u_n \) to \( u_1 \). Let \( x, y \) are two vertices in \( V(D) \) on \( P_3 \) and \( P_4 \).

Subcase 1: Either \((u_1, x)\) or \((y, u_n)\) is an arc in \( D \), and either \((u_1, y)\) or \((x, u_n)\) is an arc in \( D \). Then there exists one vertex (say \( z \)) in \( V \) on \( P_3 \) such that \((u_n, z)\), \((z, u_1)\) are arcs in \( D \). Since the length of the longest possible directed path from \( u_1 \) to \( u_n \) is \( 2m \), we have \( \text{sd}(u_1, u_n) = 2m + 2 \). This implies that \( \text{sdi}(D) \leq 2m + 2 \).

Subcase 2: Both \((u_1, x)\) and \((y, u_n)\) are arcs in \( D \), or both \((u_1, y)\) and \((x, u_n)\) are arcs in \( D \). If both \((u_1, x)\) and \((y, u_n)\) are arcs in \( D \), let \( P_3 : u_1, x, \ldots, y, u_n \) whose length is \( 2k \). Since \((u_n, x)\), \((y, u_1)\) must be arcs in \( D \), let \( P_4 : u_n, x, \ldots, y, u_1 \) whose length is \( 2l \). Then \( \text{sd}(u_1, u_n) = 2k + 2l - (2l - 2) = 2k + 2 \). Since the length of the longest directed path from \( u_1 \) to \( u_n \) is \( 2m \), \( \text{sd}(u_1, u_n) = 2m + 2 \). This implies that \( \text{sdi}(D) \leq 2m + 2 \). If both \((u_1, y)\) and \((x, u_n)\) are arcs in \( D \), similarly, we have \( \text{sdi}(D) \leq 2m + 2 \). By Case 1 and Case 2, we know that \( \text{Sdiam}(K_{m,n}) = 2m + 2 \). \( \square \)

An orientation to reach upper orientable strong diameter of \( K_{5,6} \) and \( K_{4,8} \) are shown in Figure 4 and Figure 5.

References
