Balanced bipancyclicity of hypercubes *

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Abstract

Let G be a bipartite graph. For any two vertices u and v in G, a cycle C is called a *balanced cycle* between u and v if $d_C(u,v) = max\{d_C(x,y) \mid x$ and u are in the same partite set, and y and v are in the same partite set $\}$. A bipartite graph G is *bipancyclic* [3] if it contains a cycle of every even length from 4 to $|V(G)|$ inclusive. A bipartite graph G is *balanced bipancyclic* if for each pair of vertices $u,v \in V(G)$, it contains a balanced cycle of every even length of k satisfying $max\{2d_G(u,v),4\} \leq k \leq |V(G)|$ between u and v. In this paper, we show that Q_n is balanced bipancyclic.
Keywords: hypercube, interconnection networks, edge-bipancyclic, balanced bipancyclic.

1 Introduction

An interconnection network topology is usually represented by a graph where vertices represent processors and edges represent links between processors. There are various kinds of graphs applied to design interconnection networks. Our fundamental graph terminologies refer to [1]. A graph $G = (V,E)$ is bipartite if the node set $V(G) = B \cup W$ is the union of two disjoints node sets B and W (also called the *partite sets*), such that every edge joins B and W. Two vertices, u and v, have the same color if and only if u and v are in the same partite set. We also use $G = (B \cup W, E)$ to denote a bipartite graph. Two vertices a and b are *adjacent* if $(a,b) \in E$. A path is a sequence of adjacent vertices, written as $\langle v_0, P[v_0,v_m], v_m\rangle = \langle v_0, v_1, v_2, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except possibly $v_0 = v_m$. The path $\langle v_0, P[v_0,v_m], v_m\rangle$ could be simply replaced with $P[u,v]$ and P. The two vertices v_0 and v_m are called the *end-vertices* of $P[v_0,v_m]$. The *length* of a path P denoted by $l(P)$ is the number of edges in P. Two paths are vertex-disjoint (also called disjoint) if and only if they do not have any vertices in common. Two edges (u,v) and (w,z) are disjoint if $u \notin \{w,z\}$ and $v \notin \{w,z\}$. Let u and v be two vertices of G. The *distance* between u and v denoted by $d_G(u,v)$ is the length of a shortest path of G joining u and v.

A *cycle* C is a special path with at least three vertices such that the first vertex is the same as the last one. A cycle C is called k-cycle if $l(C) = k$. A path (respectively, cycle) which traverses each vertex of G exactly once is a *hamiltonian path* (respectively, *hamiltonian cycle*). To route a packet from u to v in a k-cycle, one may first breaks the packet into two smaller pieces. Then, route the two pieces along two internal vertex-disjoint paths to the two intermediate vertices v_1, v_2. In the second phase, symmetrically, the two pieces are routed from the intermediate vertices v_1, v_2 to their common destination v. The packet is combined in v until all

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pieces of this packet arrived. Therefore, this kind of transmission delay between u and v in a cycle is determined by the longest path between u and v in this cycle. It is of interest to find a cycle passing through u and v such that lengths of two disjoint paths between u and v in this cycle are as equal as possible.

Definition 1 *Let G be a graph. For any two vertices $u, v \in V(G)$, a cycle C is called a balanced cycle between u and v if $d_C(u,v) = max\{d_C(x,y) \mid x, y \in V(C)\}$.*

Consequently, if C is a balanced k-cycle between u and v, $d_C(u,v) = \lfloor \frac{k}{2} \rfloor$. In a bipartite graph, there are only even cycles and vertex set is divided into two partite sets. Hence we modify definition 1 for bipartite graphs.

Definition 2 *Let $G = (B \cup W, E)$ be a bipartite graph. For any two vertices u and v in G, a cycle C is called a balanced cycle between u and v if $d_C(u,v) = max\{d_C(x,y) \mid x$ and u are in the same partite set, and y and v are in the same partite set. $\}$.*

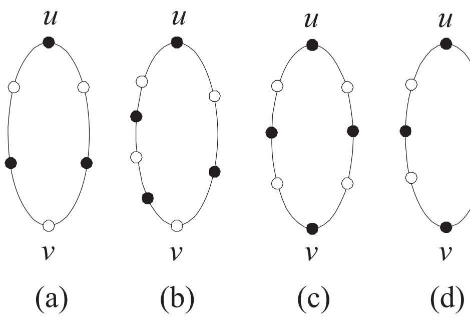

Figure 1: (a) A balanced 6-cycle between u and v that are in different partite sets. (b) A balanced 8-cycle between u and v that are in different partite sets. (c) A balanced 8-cycle between u and v that are in the same partite set. (d) A balanced 6-cycle between u and v that are in the same partite set.

A bipartite graph is *vertex-bipancyclic* [3] if every vertex lies on a cycle of every even length from 4 to $|V(G)|$ inclusive. Similarly, a bipartite graph is *edge-bipancyclic* if every edge lies on a cycle of every even length from 4 to $|V(G)|$ inclusive. Obviously, every edge-bipancyclic graph is vertex-bipancyclic. A bipartite graph G is *balanced bipancyclic* if for each pair of vertices $u, v \in V(G)$, it contains a balanced cycle of every even length of k satisfying $max\{2d_G(u,v), 4\} \leq k \leq |V(G)|$ between u and v.

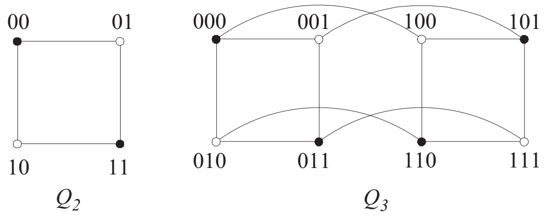

Figure 2: Q_n for $n = 2, 3$.

Let $u = u_{n-1}u_{n-2}\ldots u_1u_0$ be an n-bit binary strings. The *Hamming weight* of u, denoted by $w(u)$, is the number of u_i such that $u_i = 1$. Let $u = u_{n-1}u_{n-2}\ldots u_1u_0$ and $v = v_{n-1}v_{n-2}\ldots v_1v_0$ be two distinct n-bit binary strings. The *Hamming distance* $h(u,v)$ between two vertices u and v is the number of different bits in the corresponding strings of both vertices. The *n-dimensional hypercube*, denoted by Q_n, consists of all n-bit binary strings as its vertices and two vertices u and v are adjacent if and only if $h(u,v) = 1$. Thus, Q_n is a bipartite graph with bipartition $\{u \mid w(u)$ is odd$\}$ and $\{u \mid w(u)$ is even$\}$. Figure 2 shows Q_n for $n = 2$ and $n = 3$. It is observed that $h(u,v)$ is odd if and only if u and v are in different partite sets. For $0 \leq k < n$, we use u^k to denote the binary string $v_{n-1}v_{n-2}\ldots v_1v_0$ such that $v_k = 1 - u_k$ and $u_i = v_i$ if $i \neq k$. An edge (u,v) in $E(Q_n)$ is of *dimension* i if $u = v^i$. It is known that $d_{Q_n}(u,v) = h(u,v)$. The following lemmas are useful in our later proofs.

Lemma 1 *[2] Let u and v be two arbitrary distinct vertices with the same partite set in Q_n for $n \geq 2$. Then, for any vertex w such that $h(w,u)$ is odd, there exists a path joining u and v passing all vertices of Q_n except w.*

Lemma 2 *[2] Let u and v be two arbitrary distinct vertices in Q_n and $h(u,v) = d$, where $n \geq 2$. There are paths formed by $\langle u, P[u,v], v\rangle$ in the Q_n with lengths d,$d+2$,$d+4,\ldots,c$, where $c = 2^n - 1$ if d is odd, and $c = 2^n - 2$ if d is even.*

2 Q_n is balanced bipancyclic

In this section, for any two vertices u and v in Q_n, we will discuss the cycle passing u and v with

some special properties. Let $h(u,v) = d$. We have that if u and v are in the same partite set then d is even, otherwise d is odd. The balanced $2l$-cycle C with $l \geq 2$ between u and v must satisfy one of the following conditions:
(1) d is odd, l is odd, and $d_C(u,v) = l$.
(2) d is odd, l is even, and $d_C(u,v) = l-1$.
(3) d is even, l is even, and $d_C(u,v) = l$.
(4) d is even, l is odd, and $d_C(u,v) = l-1$.

Q_n is balanced bipancyclic if for each pair of vertices $u, v \in V(Q_n)$, it contains a balanced cycle of every even length of $2l$ satisfying $max\{h(u,v), 2\} \leq l \leq 2^{n-1}$ between u and v. The following lemma is useful in the proof of Theorem 1.

Lemma 3 *For $n \geq 2$, let (u,v) and (w,z) be two disjoint edges in Q_n. Then, Q_n can be partitioned into two $(n-1)$-cubes such that one contains (u,v) and the other contains (w,z).*

Proof. The lemma is true when $n = 2$. Let (u,v) and (w,z) be two disjoint edges, and $v = u^i$ and $z = w^k$. Hence u, v, w, and z are four distinct vertices. Without loss of generality, we may assume that $u = 00\ldots0$ and $w = w_{n-1}w_{n-2}\ldots w_{k+1}0w_{k-1}\ldots w_0$. Since $n \geq 3$ as well as u, v, and w are distinct, there exists $j \neq i$ and $j \neq k$ such that $w_j = 1$. One may partition Q_n along dimension j into two $(n-1)$-cubes, Q^0_{n-1} and Q^1_{n-1}, such that Q^0_{n-1} contains u and v as well as Q^1_{n-1} contains w and z. □

Lemma 4 *For any two disjoint edges (u,v) and (w,z) in Q_n with $n \geq 2$, there exist two disjoint paths $P_1[u,v]$ and $P_2[w,z]$, in Q_n where $l(P_1) = 1$, 3, 5 , 7, ..., $2^{n-1}-1$ and $P_2 = 1$, 3, 5 , 7, ..., $2^{n-1}-1$.*

Proof. Let (u,v) and (w,z) be two disjoint edges in Q_n. By Lemma 3, Q_n can be partitioned along dimension j into two $(n-1)$-cubes, Q^0_{n-1} and Q^1_{n-1}, such that Q^0_{n-1} contains (u,v) and Q^1_{n-1} contains (w,z) for some $0 \leq j \leq n-1$. By Lemma 2, there exist paths joining u and v (respectively, w and z) of lengths 1, 3, ..., $2^{n-1}-1$ in Q^0_{n-1} (respectively, Q^1_{n-1}). □

Theorem 1 *Q_n is balanced bipancyclic if $n \geq 2$.*

Proof. Let $u = u_{n-1}u_{n-2}\ldots u_1u_0$ and $v = v_{n-1}v_{n-2}\ldots v_1v_0$ be any two distinct vertices of Q_n and $h(u,v) = d$. To prove the theorem, we will find every balanced $2l$-cycle between u and v where $max\{d, 2\} \leq l \leq 2^{n-1}$. The proof is divided into two parts: $d = 1$ and $d \geq 2$.

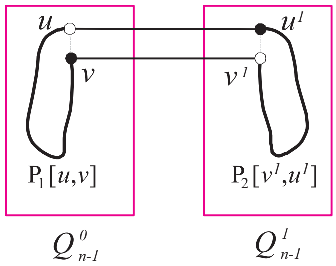

Figure 3: (a) Let $l(P_1) = l(P_2) = k$. Then, a balanced $(2k+2)$-cycle between u and v is constructed, where $k = 1, 3, 5, \ldots, 2^{n-1}-1$. (b) Let $l(P_1) = k+2$ and $l(P_2) = k$. Then, a balanced $(2k+4)$-cycle between u and v is constructed, where $k = 1, 3, 5, \ldots, 2^{n-1}-3$.

Case 1: $d = 1$, i.e. u and v are adjacent. (See Figure 3.)

Without loss of generality, we may assume that (u,v) is an edge of dimension 0. We may partition Q_n along dimension 1 into two $(n-1)$-subcubes such that Q^0_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_1 = 0\}$ and Q^1_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_1 = 1\}$. Therefore, u and v are in the same subcube Q^0_{n-1} or Q^1_{n-1}. Without loss of generality, we suppose that u and v are in Q^0_{n-1}.

Let (u,u^1) and (v,v^1) be two edges of dimension 1. Hence $h(u^1,v^1) = 1$ and $u^1, v^1 \in V(Q^1_{n-1})$. Applying Lemma 2, there are paths formed by $\langle u, P_1[u,v], v\rangle$ in the Q^0_{n-1} with length $k_1 = 1$, 3, 5, 7, ..., $2^{n-1}-1$ and there are paths formed by $\langle v^1, P_2[v^1,u^1], u^1\rangle$ in the Q^1_{n-1} whose lengths are $k_2 = 1, 3, 5, 7, \ldots, 2^{n-1}-1$. We can construct a cycle as $C = \langle u, P_1[u,v], v, v^1, P_2[v^1,u^1], u^1, u\rangle$ of length $l(C) = k_1 + k_2 + 2$ where $k_1 = l(P_1)$ and $k_2 = l(P_2)$. Obviously, the cycle C passes through u and v.

(a). balanced $(2k+2)$-cycle between u and v where $k = 1, 3, 5, \ldots, 2^{n-1}-1$. Let $k_1 = k$ and $k_2 = k$. Then, $l(C) = 2k+2$ where $k = 1, 3, 5, \ldots, 2^{n-1}-1$. Hence $d_C(u,v) = k = \frac{l(C)}{2} - 1$. Since d is odd, $\frac{l(C)}{2}$ is even, and $d_C(u,v) = \frac{l(C)}{2} - 1$, the cycle C is balanced $(2k+2)$-cycle between u and v where $k = 1, 3, 5, \ldots, 2^{n-1}-1$.

(b). balanced $(2k+4)$-cycle between u and v where $k = 1, 3, 5, \ldots, 2^{n-1}-3$. Let $k_1 = k+2$ and $k_2 = k$. Then, $l(C) = 2k+4$ where $k = 1, 3, 5, \ldots, 2^{n-1}-3$. Hence $d_C(u,v) = k+2 = \frac{l(C)}{2}$.

Since d is odd, $\frac{l(C)}{2}$ is odd, and $d_C(u,v) = \frac{l(C)}{2}$, the cycle C is balanced $(2k+4)$-cycle between u and v where $k = 1, 3, 5, \ldots, 2^{n-1} - 3$.

Case 2: $d \geq 2$, i.e. u and v are not adjacent.

We prove this case by induction on n. Obviously, the proof of case 2 holds for $n = 2$. Assume that the proof of case 2 is true for every integer $2 \leq m < n$. Let $u = u_{n-1}u_{n-2}\ldots u_1u_0$ and $v = v_{n-1}v_{n-2}\ldots v_1v_0$ be any two distinct vertices of Q_n and $h(u,v) = d$. Partitioning Q_n along dimension 0, Q_n can be divided into two $(n-1)$-subcubes where Q^0_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_0 = 0\}$ and Q^1_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_0 = 1\}$.

Subcase 2-1: $u, v \in Q^0_{n-1}$ or $u, v \in Q^1_{n-1}$. (See Figure 4 and Figure 5.)

Without loss of generality, we may assume that $u, v \in Q^0_{n-1}$. For the basis of this proof, we consider Q_3. It is clear that Q_3 is balanced bipancyclic (See Figure 4 for an illustration).

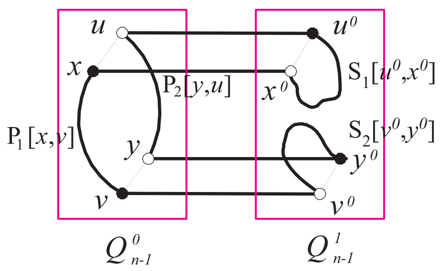

Figure 5: Let $l(S_1) = l(S_2) = k$ where $k = 1, 3, 5, \ldots, 2^{n-1}$. Then, a balanced $(m + 2k + 2)$-cycle between u and v is constructed, where $\langle u, x, P_1[x,v], v, y, P_2[y,u], u\rangle$ is balanced m-cycle between u and v of Q^0_{n-1} where $m \geq 6$.

Suppose that $n \geq 4$. By induction hypothesis, Q^0_{n-1} is balanced bipancyclic. Every balanced $2l$-cycle between u and v in Q_n can be found in Q^0_{n-1} where $d \leq l \leq 2^{n-2}$. Let C be a balanced m-cycle with $m \geq 6$ between u and v in Q^0_{n-1}. Hence we rewrite the cycle C as $\langle u,\ x,\ P_1[x,v],\ v,\ y,\ P_2[y,u],\ u\rangle$. Let (u,u^0), (x,x^0), (v,v^0), and (y,y^0) be four edges of dimension 0. It is observed that u^0, x^0, v^0, and y^0 are four distinct vertices in Q^1_{n-1}, and that (u^0,x^0) and (v^0,y^0) are two disjoint edges in Q^1_{n-1}. Applying Lemma 4, there exist two disjoint paths $S_1[u^0,x^0]$ and $S_2[v^0,y^0]$ in Q^1_{n-1} such that $l(S_1) = l(S_2) = k$ where $k = 1, 3, 5, 7, \ldots, 2^{n-2} - 1$. Therefore, we may construct a cycle $C' = \langle u,\ u^0,\ S_1[u^0,x^0],\ x^0,\ x,\ P_1[x,v],\ v,\ v^0,\ S_2[v^0,y^0],\ y^0,\ y,\ P_2[y,u],\ u\rangle$ passing through u and v. Hence $l(C') = m + 2k + 2$.

Subcase 2-1-1: balanced $(2^{n-1}+2k)$-cycle between u and v where $k = 1, 3, 5, \ldots, 2^{n-2} - 1$. Let $m = 2^{n-1} - 2$. Therefore, $l(C') = 2^{n-1} + 2k$.

(a). Suppose that d is odd. Since C is a balanced $(2^{n-1} - 2)$-cycle between u and v, and $\frac{l(C)}{2} = 2^{n-2} - 1$ is odd, $d_C(u,v) = 2^{n-2} - 1$. It is clearly that $d_{C'}(u,v) = d_C(u,v) + k + 1 = 2^{n-2} + k$ and $\frac{l(C')}{2} = 2^{n-2} + k$. Since d is odd, $\frac{l(C')}{2}$ is odd, and $d_{C'}(u,v) = 2^{n-2} + k = \frac{l(C')}{2}$, the cycle C' is balanced $(2^{n-1} + 2k)$-cycle between u and v in Q_n where $k = 1, 3, 5, \ldots, 2^{n-2} - 1$.

(b). Suppose that d is even. Since C is a balanced $(2^{n-1}-2)$-cycle between u and v, and $\frac{l(C)}{2} = 2^{n-2} - 1$ is odd, $d_C(u,v) = 2^{n-2} - 2$. It is clearly that $d_{C'}(u,v) = d_C(u,v) + k + 1 = 2^{n-2} + k - 1$ and $\frac{l(C')}{2} = 2^{n-2} + k$. Since d is even, $\frac{l(C')}{2}$ is odd, and $d_{C'}(u,v) = 2^{n-2} + k - 1 = \frac{l(C')}{2} - 1$, the cycle C' is balanced $(2^{n-1} + 2k)$-cycle between u and v in Q_n where $k = 1, 3, 5, \ldots, 2^{n-2} - 1$.

Subcase 2-1-2: balanced $(2^{n-1}+2k+2)$-cycle between u and v where $k = 1, 3, 5, \ldots, 2^{n-2} - 1$. Let $m = 2^{n-1}$. Therefore, $l(C') = 2^{n-1} + 2k + 2$.

(a). Suppose that d is odd. Since C is a balanced 2^{n-1}-cycle between u and v, and $\frac{l(C)}{2} = 2^{n-2}$ is even, $d_C(u,v) = 2^{n-2} - 1$. It is clearly that $d_{C'}(u,v) = d_C(u,v) + k + 1 = 2^{n-2} + k$ and $\frac{l(C')}{2} = 2^{n-2} + k + 1$. Since d is odd, $\frac{l(C')}{2}$ is even, and $d_{C'}(u,v) = 2^{n-2} + k = \frac{l(C')}{2} - 1$, the cycle C' is balanced $(2^{n-1} + 2k + 2)$-cycle between u and v in Q_n where $k = 1, 3, 5, \ldots, 2^{n-2} - 1$.

(b). Suppose that d is even. Since C is a balanced 2^{n-1}-cycle between u and v, and $\frac{l(C)}{2} = 2^{n-2}$ is even, $d_C(u,v) = 2^{n-2}$. It is clearly that $d_{C'}(u,v) = d_C(u,v) + k + 1 = 2^{n-2} + k + 1$ and $\frac{l(C')}{2} = 2^{n-2} + k + 1$. Since d is even, $\frac{l(C')}{2}$ is even, and $d_{C'}(u,v) = 2^{n-2} + k + 1 = \frac{l(C')}{2}$, the cycle C' is balanced $(2^{n-1} + 2k + 2)$-cycle between u and v in Q_n where $k = 1, 3, 5, \ldots, 2^{n-2} - 1$.

Subcase 2-2: $u \in Q^0_{n-1}$ and $v \in Q^1_{n-1}$ (or $v \in Q^0_{n-1}$ and $u \in Q^1_{n-1}$).

Without loss of generality, we may assume that $u \in Q^0_{n-1}$ and $v \in Q^1_{n-1}$. Let (u,u^0) and (v,v^0) be two edges of dimension 0. Hence $u^0 \in V(Q^1_{n-1})$ and $v^0 \in V(Q^0_{n-1})$, and $h(u,v^0) = h(v,u^0) = d - 1$.

Subcase 2-2-1: d is even, i.e. u and v are in the same partite set. (See Figure 6.) Hence u^0 and v are in different partite sets. Simi-

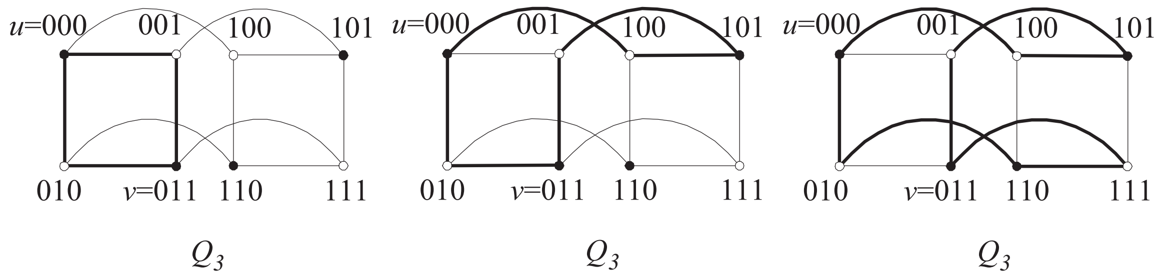

Figure 4: Three balanced cycles between $u = 000$ and $v = 011$ in Q_3.

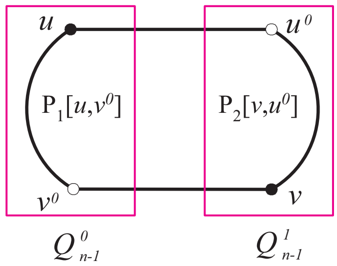

Figure 6: $h(u,v) = d$ is even. (a) Let $l(P_1) = l(P_2) = k$. Then, a balanced $(2k+2)$-cycle between u and v is constructed, where $k = d-1, d+1, d+3, \ldots, 2^{n-1}-1$. (b) Let $l(P_1) = k+2$ and $l(P_2) = k$. Then, a balanced $(2k+4)$-cycle between u and v is constructed, where $k = d-1, d+1, d+3, \ldots, 2^{n-1}-3$.

larly, v^0 and u are in different partite sets. By Lemma 2, there exists a path $P_1[u, v^0]$ (respectively, $P_2[v, u^0]$) connecting u and v^0 (respectively, v and u^0) where $l(P_1) = d-1, d+1, d+3, \ldots, 2^{n-1}-1$ (respectively, $l(P_2) = d-1, d+1, d+3, \ldots, 2^{n-1}-1$). The cycle C can be constructed as $\langle u, P_1[u, v^0], v^0, v, P_2[v, u^0], u^0, u\rangle$. Therefore, the cycle C passing through u and v, and $l(C) = k_1 + k_2 + 2$ where $k_1 = l(P_1)$ and $k_2 = l(P_2)$.

(a). balanced $(2k+2)$-cycle between u and v where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-1$. Let $k_1 = k$ and $k_2 = k$ where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-1$. Therefore, $l(C) = 2k+2$. One can observe that $\frac{l(C)}{2} = k+1$ and $d_C(u,v) = k+1$. Since d is even, $\frac{l(C)}{2}$ is even, and $d_C(u,v) = \frac{l(C)}{2}$, the cycle C is balanced $(2k+2)$-cycle between u and v where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-1$.

(b). balanced $(2k+4)$-cycle between u and v where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-3$. Let $k_1 = k+2$ and $k_2 = k$ where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-3$. Therefore, $l(C) = 2k+4$. One can observe that $\frac{l(C)}{2} = k+2$ and $d_C(u,v) = k+1$. Since d is even, $\frac{l(C)}{2}$ is odd, and $d_C(u,v) = \frac{l(C)}{2}-1$, the cycle C is balanced $(2k+4)$-cycle between u and v where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-3$.

Subcase 2-2-2: d is odd, i.e. u and v are in different partite sets. (See Figure 7.) Hence u^0 and v are in the same partite set. Similarly, v^0 and u are in the same partite set. By Lemma 2, there exists a paths $P_1[u, v^0]$ (respectively, $P_2[v, u^0]$) connecting u and v^0 (respectively, v and u^0) where $l(P_1) = d-1, d+1, d+3, 2^{n-1}-2$ (respectively, $l(P_2) = d-1, d+1, d+3, \ldots, 2^{n-1}-2$). The cycle C can be constructed as $\langle u, P_1[u, v^0], v^0, v, P_2[v, u^0], u^0, u\rangle$. Therefore, the cycle C passing through u and v, and $l(C) = k_1 + k_2 + 2$ where $k_1 = l(P_1)$ and $k_2 = l(P_2)$.

(a). balanced $(2k+2)$-cycle between u and v where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-2$. Let $k_1 = k$ and $k_2 = k$ where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-2$. Therefore, $l(C) = 2k+2$. One can observe that $\frac{l(C)}{2} = k+1$ and $d_C(u,v) = k+1$. Since d is odd, $\frac{l(C)}{2}$ is odd, and $d_C(u,v) = \frac{l(C)}{2}$, the cycle C is balanced $(2k+2)$-cycle between u and v where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-2$.

(b). balanced $(2k+4)$-cycle between u and v where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-4$. Let $k_1 = k+2$ and $k_2 = k$ where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-4$. Therefore, $l(C) = 2k+4$. One can observe that $\frac{l(C)}{2} = k+2$ and $d_C(u,v) = k+1$. Since d is odd, $\frac{l(C)}{2}$ is even, and $d_C(u,v) = \frac{l(C)}{2}-1$, the cycle C is balanced $(2k+4)$-cycle between u and v where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-4$.

(c). balanced 2^n-cycle between u and v. Let

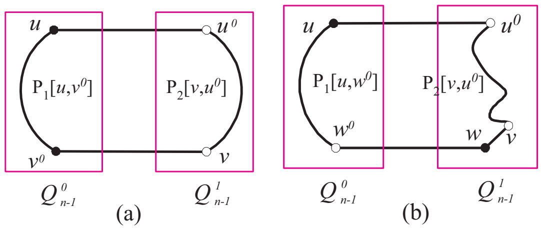

Figure 7: $h(u, v) = d$ is odd. (a.1) Let $l(P_1) = l(P_2) = k$. Then, a balanced $(2k + 2)$-cycle between u and v is constructed, where $k = d-1, d+1, d+3, \ldots, 2^{n-1} - 2$. (a.2) Let $l(P_1) = k+2$ and $l(P_2) = k$. Then, a balanced $(2k + 4)$-cycle between u and v is constructed, where $k = d-1, d+1, d+3, \ldots, 2^{n-1} - 4$. (b) A balanced hamiltoian cycle between u and v where $l(P_1) = 2^{n-1} - 1$ and $l(P_2) = 2^{n-1} - 2$.

$w \in V(Q^1_{n-1})$ and $h(w, v) = 1$. It is observed that $h(w, u^0)$ is odd. By Lemma 1, there exists a path $P[v, u^0]$ of length $2^{n-1} - 2$ joining v and u^0 passing all vertices of Q^1_{n-1} except w. Let (w, w^0) be an edge of dimension 0. Hence w^0 is in Q^0_{n-1}, and w^0 and u are in different partite sets. By Lemma 2, there exists a hamiltonian path $P_1[u, w^0]$ joining u and w^0 in Q^0_{n-1}. Therefore, longest cycle C between u and v in Q_n can be constructed as $\langle u, P_1[u, w^0], w^0, w, v, P_2[v, u^0], u^0, u\rangle$. Therefore, the cycle C passing through u and v, $l(C) = 2^{n-1} - 1 + 1 + 1 + 2^{n-1} - 2 + 1 = 2^n$, and $d_C(u, v) = 2^{n-1} - 1 = \frac{l(C)}{2} - 1$. Since d is odd, $\frac{l(C)}{2}$ is even, and $d_C(u, v) = \frac{l(C)}{2} - 1$, the cycle C is balanced cycle between u and v. The theorem is proved. □

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