Colouring games on outerplanar graphs and trees

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Abstract

Suppose f is a graph function which assigns to each graph H a positive integer $f(H) \leq |V(H)|$. An f-colouring of G is a mapping $c : V(G) \to N$ such that every subgraph H of G receives at least $f(H)$ colours, i.e., $|c(H)| \geq f(H)$. The f-chromatic number, $\chi(f, G)$, is the minimum number of colours used in an f-colouring of G. The parameter $\chi(f, G)$ is introduced by Nesetril and Ossona de Mendez and they proved that if $f(H) \leq \min\{p, td(H)\}$, where p is a constant and $td(H)$ is the tree-depth of H, then for any proper minor closed class $\mathcal{K}$ of graphs, $\chi(f, G)$ is bounded by a constant for all $G \in \mathcal{K}$. In this paper, we study the game version of f-colouring of graphs. Suppose G is a graph and X is a set of colours. Two players, Alice and Bob, take turns colour the vertices of G with colours from the set X. A partial colouring of G is legal (with respect to graph function f) if for any subgraph H of G, the sum of the number of colours used in H and the number of uncoloured vertices of H is at least $f(H)$. Both Alice and Bob must colour legally (i.e., the partial colouring produced needs to be legal). The game ends if either all the vertices are coloured or there are uncoloured vertices but there is no legal colour for any of the uncoloured vertices. In the former case, Alice wins the game. In the latter case, Bob wins the game. The f-game chromatic number of G, $\chi_g(f, G)$, is the least number of colours that the colour set X needs to contain so that Alice has a winning strategy. In this paper, we prove that if $f(K_2) = 2$, $f(C_n) = 3$ for any $n \geq 3$ and $f(H) = 1$ otherwise, then for any outerplanar graph G, $\chi_g(f, G) \leq 7$. If $i \geq 6$ and ϕ_i is the graph function with $\phi_i(K_2) = 2$, $\phi_i(P_i) = 3$ and $\phi_i(H) = 1$ otherwise, then for any tree T, $\chi_g(\phi_i, T) \leq 10$. On the other hand, if $i \leq 5$, then for any integer k, there is a tree T such that $\chi_g(\phi_i, T) \geq k$.

1 Introduction

Many variations of the chromatic number of graphs have been studied extensively in the literature. As an unification of many variants of chromatic number, Nešetřil and Ossona de Mendez in [18] introduced the following generalization of chromatic number of graphs. Suppose f is a graph function, which assigns to each graph H a positive integer $f(H) \leq |V(H)|$. An f-colouring of a graph G is a mapping c which assigns to each vertex of G a colour so that any subgraph H of G receives at least $f(H)$ colours. The f-chromatic number, $\chi(f, G)$, is the least number of colours used in an f-colouring of G. For example, if $f_1(C_n) = 2$ for any n, and $f_1(H) = 1$ otherwise, then $\chi(f_1, G)$ is the *point-arboricity* of G, i.e., the smallest size of vertex partition whose parts induces forests. If $f_2(K_2) = 2$ and $f_2(H) = 1$ otherwise, then $\chi(f_2, G)$ is the same as $\chi(G)$. If $f_3(K_2) = 2$, $f_3(C_n) = 3$ for any $n \geq 3$, and $f_3(H) = 1$ otherwise, then $\chi(f_3, G)$ is the *acyclic chromatic number* of G, i.e., the minimum number of colours needed to colour the vertices so that each colour class is an independent set, and the union of any two colour classes induces a forest. If $f_4(K_2) = 2, f_4(P_4) = 3$ (where P_4 is the path on 4 vertices) and $f_4(H) = 1$ otherwise, then $\chi(f_4, G)$ is the *star-chromatic number* of G, i.e., the minimum number of colours needed to colour the ver-

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tices so that each colour class is an independent set, and the union of any two colour classes induces a star forest. If $f_5(K_{1,d+1}) = 2$ and $f_5(H) = 1$ otherwise, then $\chi(f_5, G)$ is the *d-relaxed chromatic number* of G, i.e., the minimum number of colours needed to colour the vertices of G so that each colour class induces a subgraph of maximum degree at most d. A class $\mathcal{K}$ of graphs is called a proper minor closed class of graphs, if $G \in \mathcal{K}$ implies that $G' \in \mathcal{K}$ for any minor G' of G, and $\mathcal{K}$ does not contain all finite graphs. Many other variants of chromatic number of graphs are known to be bounded by a constant on any proper minor closed class of graphs. Nešetřil and Ossona de Mendez studied the problem that for which graph function f, the parameter $\chi(f, g)$ is bounded by a constant on any proper minor closed class of graphs. They proved that $\chi(f, G)$ is bounded by a constant on any proper minor closed class of graphs if and only if there is a constant p such that $f(H) \leq \min\{p, td(H)\}$, where $td(H)$ is the tree-depth of H defined as follows: Suppose T is a rooted tree. The *height* of T is the number of vertices in a longest path from the root to a leaf. The *closure* of T is the graph Q on $V(T)$ in which $x\ sim_Q y$ if x is an ancestor of y or y is an ancestor of x. The *tree-depth* of a connected graph G is the smallest height of a rooted tree T such that G is a subgraph of the closure of T. If G is disconnected, then the tree-depth of G is the maximum of the tree-depth of its components.

In this paper, we are interested in the game version of f-colourings. Suppose G is a graph, f is a graph function and X is a set of colours. The *f-colouring game* on G with colour set X is the following two-person game: Two players, Alice and Bob, take turns colour the vertices of G, with Alice takes the first turn. Suppose some vertices of G are coloured. A colour $\alpha \in X$ is legal for an uncoloured vertex u if by assigning colour α to u, the resulting partial colouring has the following property: For any subgraph H of G, the sum of the number of colours assigned to vertices of H and the number of uncoloured vertices of H is at least $f(H)$. On each turn, a player chooses one uncoloured vertex u of G and one colour α from X which is legal for u, and assign colour α to u. The game ends if either all the vertices are coloured or there is no legal colour for any of the remaining uncoloured vertices. If all the vertices are coloured, then Alice wins the game. Otherwise Bob is the winner. So Alice and Bob have opposite goals. Alice wants to produce an f-colouring of G, and Bob tries to prevent this from happening. The *f-game chromatic number* of G, $\chi_g(f, G)$, is the minimum number of colours in the colour set X such that Alice has a winning strategy in the colouring game described above. Observe that if $|X| = |V(G)|$, then Alice always wins. So the parameter $\chi_g(f, G)$ is well-defined.

In case f is the graph function defined as $f(K_2) = 2$ and $f(H) = 1$ otherwise, then $\chi_g(f, G)$ is just the game chromatic number of G, and is denoted by $\chi_g(G)$. About twenty-five years ago, Steven J. Brams invented the coloring game for plane maps, and asked what is the minimum number of colours needed so that Alice always has a winning strategy when the game is played on a plane map. Brams' question is equivalent to ask what is the maximum game chromatic number of planar graphs. Brams' game was published by Martin Gardner in his column "Mathematical Games" in *Scientific American* in 1981 [7]. It remained unnoticed by the graph-theoretic community until ten years later, when it was reinvented by Hans L. Bodlaender [1] in a wider context of general graphs. He considered the game in which Alice and Bob color the vertices of a graph G and introduced the game chromatic number $\chi_g(G)$ of a graph G. Since then the problem has been analyzed in serious combinatorial journals. The benchmark problem in this area is the maximum game chromatic number of planar graphs, which is studied in a sequence of papers [13, 3, 22, 10, 21]. The presently best known upper bound for the game chromatic number of planar graphs is 17 [21]. The game chromatic number of some other classes of graphs have also been studied in the literature, including forests [6, 13], outerplanar graphs [8], partial k-trees [23, 15], etc.

If f is the graph function defined as $f(K_{1,d+1}) = 2$ and $f(H) = 1$ otherwise, then $\chi_g(f, G)$ is called the *d-relaxed game chromatic number* of G, and is denoted by $\chi_g^{(d)}(G)$. The d-relaxed game chromatic number of graphs was introduced in [2], and has been studied in [5, 4, 9, 19, 20]. It is known that if G is a forest, then for $d = 0, 1, 2$, $\chi_g^{(d)}(G) \leq 4 - d$. If G is an outerplanar graph, then for $d = 0, 1, 2, 3, 4$, $\chi_g^{(d)}(G) \leq 7 - d$ and if $d \geq 6$, then $\chi_g^{(d)}(G) \leq 2$. If G is a planar graph, then for $d \geq 93$, $\chi_g^{(d)}(G) \leq 6$, and for $d \geq 132$, $\chi_g^{(d)}(G) \leq 3$. If G is a partial k-tree and $d \geq 4k - 1$, then $\chi_g^{(d)}(G) \leq k + 1$.

There are some other variations of game chromatic number have been studied in the literature. These include game chromatic number of oriented graphs [17, 14], coloring game in which a move can colour more than one vertices [11, 12, 16], and game colouring number (which we shall define in Section 2).

Suppose $\mathcal{K}$ is a class of graphs and f is a graph function. A natural question is whether the f-game chromatic number $\chi_g(f, G)$ is bounded by a constant for all $G \in \mathcal{K}$. If $\chi_g(f, G)$ is bounded from above for all $G \in \mathcal{K}$, then we would like to find the smallest upper bound. In this paper, we consider some special graph functions, and consider the case that $\mathcal{K}$ is either the class of outerplanar graphs or the class of forests.

First we consider the case that $f(K_2) = 2$ and $f(C_n) = 3$ for all $n \geq 3$, and $f(H) = 1$ for all other H. In other words, Alice's goal is to produce an acyclic colouring of G. For this graph function f, we call $\chi_g(f, G)$ the *acyclic game chromatic number* of G and denote it by $\chi_g^a(G)$. We observe that $\chi_g^a(G)$ is not bounded for series-parallel graphs, however, $\chi_g^a(G) \leq 7$ for any outer planar graph G. Then we consider graph functions ϕ_i $(i \geq 3)$ defined $\phi_i(K_2) = 2$, $\phi_i(P_i) = 3$ and $\phi_i(H) = 1$ for other graphs H. The question we are interested is whether $\chi_g(\phi_i, T)$ is bounded by a constant all trees T. We shall prove that for $i \geq 6$, $\chi_g(\phi_i, T) \leq 10$ for any tree T, and for $i \leq 5$, $\chi_g(\phi_i, T)$ is not bounded by a constant on trees.

2 Acyclic game chromatic number of outerplanar graphs

This section studies the acyclic game chromatic number of graphs. First we observe that $\chi_g^a(G)$ is unbounded for series-parallel graphs.

Example 1 *For any integer n, there is a series-parallel graph G with $\chi_g^a(G) \geq n$.*

Proof. Let G be the graph with vertex set $\{a, b, c, x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n\}$ and edge set $\{x_ia, x_ib, y_ib, y_ic : i = 1, 2, \cdots, n\}$. We shall prove that with n colours, Bob has a winning strategy. In Bob's first two moves, he makes either a, b or b, c be coloured by the same colour. This is certainly possible, no matter what vertices are coloured by Alice. Now with n colours, Bob wins the game, because if all vertices are coloured, at least two of the x_i's are coloured by the same colour, and at least two of the y_i's are coloured by the same colour. In any case, there is a 2-coloured cycle, and hence is not an f-colouring of G. ∎

Now we shall prove that $\chi_g^a(G)$ is bounded for outer planar graphs. First we prove an easy lemma.

Lemma 1 *Suppose G is an outerplanar graph, C is a cycle of G and uxv are three consecutive vertices of C. Let P_{uv} be the shortest path of $G - x$ connecting u and v. Then all the vertices of P_{uv} are contained in C.*

Proof. Assume $w \in P_{uv}$ is not contained in C. Let z, z' be the two vertices of P_{uv} on the two sides of w in P_{uv} that are closest to w and lies on C. Then the segment of C connecting z, z' contains at least one vertex, say w', because P_{uv} is a shortest path. Now we can contract edge of the union $C \cup P_{uv}$ so that z, z' become adjacent to x, w, w'. So $K_{2,3}$ is a minor of G, in contrary to the assumption that G is outerplanar. ∎

The game colouring number of a graph defined through the following two person game: Alice and Bob alternately marks vertices of G, with Alice takes the first turn. Each move marks one unmarked vertex. The game ends when all vertices are marked. When the game ends, let $m : V(G) \to N$ be defined as $m(v) = k$ if v is marked at the kth move (counting both Alice's move and Bob's move). Let $s(v) = |\{u : u \sim v, m(u) < m(v)\}|$ be the number of neighbours of v that are marked before v. The *score* of the game is $\max\{s(v) : v \in V(G)\}$. The *game colouring number* of a graph G, denoted by $\mathrm{col}_g(G)$, is the least integer k such that Alice has a strategy in playing the marking game so that the score is at most $k - 1$. We shall need the following result proved in [8]:

Theorem 1 *If G is an outerplanar graph, then $\mathrm{col}_g(G) \leq 7$.*

Now we are ready to prove the main result of this section.

Theorem 2 *If G is an outerplanar graph, then $\chi_g^a(G) \leq 7$.*

Proof. Assume G is an outerplanar graph. By Theorem 1, Alice has a strategy for choosing the vertices to be coloured in her moves so that each uncoloured vertex has at most 6 coloured neighbours. Alice uses this strategy to choose the vertex to be coloured. When a vertex to be coloured has been chosen, Alice uses any legal colour to colour that vertex. It remains to show that any uncoloured vertex x has a legal colour.

We assume that G is embedded in the plane so that all the vertices lie on the facial cycle of the infinite face. Assume x is an uncoloured vertex. Let $u_1, u_2, \cdots, u_s$ be the coloured neighbours of x, ordered according to the embedding of G, in the anti-clockwise direction. By the previous paragraph, $s \le 6$.

We choose a set S of colours as follows: First of all, S contains all the colours used on $u_1, u_2, \cdots, u_s$. For $i = 1, 2, \cdots, s-1$, we do the following: If u_i and u_{i+1} are coloured the same colour, and the shortest path P_i in $G-x$ connecting u_i and u_{i+1} exist and are all coloured, then choose one colour c_i used on vertices of P_i that is distinct from the colour of u_i, add colour c_i to S.

By the construction of S, we know that S contains at most $s \le 6$ colours. As there are 7 colours, so there is a colour $c \notin S$. We claim that c is a legal colour for x. First of all, c is distinct from all the colours of the coloured neighbours of x. So by colouring x with colour c, there is no monochromatic edge. Assume there is a 2-coloured cycle C. Let u_i, u_j be the two neighbours of x in C, with $i < j$. As $u_1, u_2, \cdots, u_s$ are ordered according to the outerplanar embedding of G, we conclude that $C-x$ contains all the vertices $u_i, u_{i+1}, \cdots, u_j$, and all the vertices on the paths $P_i, P_{i+1}, \cdots, P_{j-1}$. But c is distinct from two colours used on P_i. Hence C cannot be a 2-coloured cycle. ∎

3 colouring trees

In this section, let ϕ_i be the graph function defined as $\phi_i(K_2) = 2$ and $\phi_i(P_i) = 3$ and $\phi_i(H) = 1$ for all other graph H. Here P_i is a path with i vertices.

Theorem 3 *If T is a tree and $i \ge 6$, then $\chi_g(\phi_i, T) \le 10$.*

In the following, $T = (V, E)$ is a tree, and X is a colour set with $|X| = 10$. We shall only prove Theorem 3 for the case that $i = 6$. The case that $i > 6$ can be proved in the same way.

Choose a vertex u of T as a root, and consider T as a rooted tree. Then each vertex v of T other than u has a unique father, which we denote by $f(v)$. For convenience, we let $f(u) = u$. For a vertex v of T, let $S(v)$ be the set of sons of v, and let $S^2(v) = \cup_{w \in S(v)} S(w)$. Let $N^*(v) = \{f(v), f(f(v))\} \cup S(v) \cup S^2(v)$.

Suppose the tree T is partially coloured. We denote by C the set of coloured vertices and denote by U the set of uncoloured vertices. For a coloured vertex $v \in C$, we denote the colour of v by $c(v)$. A colour α is *legal* for an uncoloured vertex v if

- α is not used by any coloured neighbour of v.
- There is no path P of on 6 vertices containing v such that all vertices of P other than v are coloured by α and β for some colour β.

A colour α is called a *permissible colour* for an uncoloured vertex v if

- α is a legal colour for v.
- α is not used by any vertex in $N^*(v)$.

Alice will colour a vertex with a permissible colour only. However, Bob can colour a vertex with any legal colour.

Assume x is a vertex of T (either coloured or uncoloured) and $f(x) \in C$. Let $P = p_0 \cdots p_r$ be the longest 2-coloured path such that $p_0 = x$ and $p_{j+1} = f(p_j)$ for all $0 \le j \le r-1$. We call the path P the *bi-coloured path* above x, call the vertex x_p the *bi-father* of x and denote by $f_b(x)$, and call the colour $c(p_{r-1})$ of p_{r-1} the *safe colour* of x. Note that the bi-coloured path of x and the safe colour of x changes during the play of the game. When we use this terms, we are referring to the particular moment of the game of our consideration. Also note that if $f(f(x))$ is uncoloured, then P contains two vertices only, and if moreover, x is uncoloured then the safe colour of x does not exists and hence is not defined.

During the play of the game, Alice will keep record of a set A of *active vertices*. When an vertex v is put into A, we say that v is *activated*. Once a vertex is activated, it will remain active forever. Initially Alice activates and colours u and colours u.

Suppose Bob has just coloured a vertex x with colour α. Alice's response is divided into two stages.

Initial Stage

Alice first activates vertex x if it is not active yet, i.e., let $A := A \cup \{x\}$.

- If $f(x) \notin A$, then go to the Recursive Stage.
- If $f(x) \in A \cap U$, then colour $f(x)$ with a permissible colour.
- Otherwise,
  1. If $f(f_b(x)) \notin C$, colour $f(f_b(x))$ with a permissible colour.
  2. If $f(f_b(x)) \in C$ and $f_b(x)$ has an uncoloured son v such that the safe colour of x is permissible to v, then colour v with the the safe colour of x.

3. Otherwise, let v be an uncoloured vertex all whose ancestors are coloured, and colour v with a permissible colour.

Recursive Stage

Assume we arrive at a vertex v. First, activate v.

- If $f(v)$ is not active, then set $v = f(x)$ and repeat the Recursive Stage.
- If $f(v) \in A \cap U$, then colour $f(v)$ with a permissible colour.
- If $f(v) \in C$ and $f(f(v)) \in U$, then colour $f(f(v))$ with a permissible colour.
- Otherwise,
  1. If $f(f_b(v)) \notin C$, colour $f(f_b(v))$ with a permissible colour.
  2. If $f(f_b(v)) \in C$, then colour v with a permissible colour.

This completes the description of Alice's strategy. Now we show that this is a winning strategy for Alice. For this purpose, it suffices to show that at any moment, any uncoloured vertex has a permissible colour.

Lemma 2 *Suppose Alice has finished a move and z is an uncoloured vertex. Then z has at most 1 active sons.*

Proof. When the first son of z is activated, z is activated. If z has two active sons, then when the second son of z is activated, Alice should have coloured z. ∎

Lemma 3 *Suppose Alice has finished a move, z is an uncoloured vertex and y is a son of z. Then y has at most 2 active sons.*

Proof. When the second son of y is activated, y is coloured. If y has three active sons, then when the third son of y is activated, Alice should have coloured z. ∎

Lemma 4 *Assume Alice has just finished a move. If z, $f(z)$ and $f(f_b(z))$ are not all coloured, then z has at most 3 active sons.*

Proof. Assume to the contrary that z has 4 active sons. When the third son is activated, Alice colours $f(z)$. When the fourth son is activated, Alice should have coloured $f(f_b(z))$. ∎

Lemma 5 *At any moment of the game, any uncoloured vertex x has a permissible colour.*

Proof. Assume at a certain moment of the game, x is an uncoloured vertex. Let $z = f(x)$. We denote by $N^2(x)$ the set of vertices that have distance at most 2 from x, i.e., $N^2(x) = \{v, d_T(x,v) \leq 2\}$. Thus

$$N^2(x) = \{z, f(z)\} \cup S(z) \cup S(x) \cup S^2(x) = N^*(x) \cup S(z).$$

First we consider the case that at least one of $z, f(z), f(f_b(z))$ is not coloured. We assume that Alice has just finished a move. By Lemma 4, z has at most 3 coloured sons, because every coloured vertex is active. By Lemma 2, x has at most one coloured son. By Lemma 3, $S^2(x)$ contains at most 2 coloured vertices. This implies that $N^2(x)$ contains at most 8 coloured vertices. If Bob has just finished a move, then the number of coloured vertices in $N^2(x)$ increases by at most 1. Thus in any case (i.e., either Alice has just finished a move or Bob has just finished a move) there is a colour β not used by any vertices in $N^2(x)$. It is obvious β is a permissible colour for x.

Next we assume that all the vertices $z, f(z)$ and $f(f_b(z))$ are coloured. Let $Y \subseteq S(z)$ be the set of sons of z that were activated before the last vertex of $z, f(z), f(f_b(z))$ was coloured. It follows from Lemma 4 that $|Y| \leq 4$ (note that Lemma 4 assumed that it is after Alice has just finished a move. In Bob's next move, he can colored a descendent of z, so Alice may activate one more son of z before colouring $f(f_b(z))$).

Assume that Alice has just finished a move. If $x \notin Y$, then $S(x) \sup S^2(x)$ contains no coloured vertices, because when a descendent of x is coloured, Alice should have coloured x by following the strategy. In this case $Y \cup N^*(x)$ contains at most 6 coloured vertices. If $x \in Y$, then by Lemma 3, $S(x) \cup S^2(x)$ contains at most 3 coloured vertices and hence $Y \cup N^*(x)$ contains at most 8 coloured vertices. In Bob's next move, the number of coloured vertices in $Y \cup N^*(x)$ increases by at most 1. So in any case (i.e., either $x \in Y$ or $x \notin Y$, and either Alice has just finished a move or Bob has just finished a move), $Y \cap N^*(x)$ contains at most 9 coloured vertices. Let β be a colour not used by any vertex in $Y \cap N^*(x)$. We shall prove that β is a legal colour for x, which implies that β is a permissible colour for x. Assume to the contrary that β is not a legal colour for x. Then there is a 2-coloured path $P = (w_1, w_2, w_3, w_4, w_5)$ such that $x \sim w_1$, and w_2, w_4 are coloured by colour β. Since $w_2 \notin N^*(x)$, we conclude that $w_2 \in S(z)$, and $w_1 = z$. Let α be the colour of $z = w_1, w_3, w_5$.

Observe that z has no other sons coloured with β, for otherwise before we colour x, the partial coloring is already illegal.

We divide the discussion into a few cases.

Case 1 w_2 is coloured by Bob. Since $w_2 \notin Y$, when any descendent of w_2 is activated, Alice will colour w_2, provided that it is not coloured. Therefore w_2 has no active sons when Bob colour w_2. After Bob colours w_2, the bi-path above w_2 ends at z, and β is the safe colour of w_2. By Alice's strategy, she colours a son of z with colour β (observe that at that time, β is certainly a legal colour for x, because w_2 has no coloured sons). This is in contrary to our conclusion that z has no other sons coloured with colour β.

Case 2 w_2 is coloured by Alice. By the same argument as in Case 1, w_2 has at most 1 active son when Alice colours w_2. Note that w_2 is coloured before w_4, for otherwise β is not a permissible colour for w_2. Moreover, w_4 is coloured by Bob, because β is not a permissible colour for w_4. Since z is coloured before w_3, α is not a permissible colour for w_3. Thus w_3 is also coloured by Bob.

Assume w_3 is coloured after w_2. Then w_3 is coloured before w_4, for otherwise after Bob colours w_4, Alice would have coloured w_3. At the time w_3 is coloured, the bi-path above w_3 ends at z, and the safe colour of w_3 is β. By Alice's strategy, she should have coloured a son of z with colour β, because at that moment, w_4 is not coloured yet, and β must be a legal colour for x and hence a permissible colour for x. (Note that if β is not a legal colour for x at that moment, then it must be the case that w_3 has a son $w'_4 \neq w_4$ coloured with colour β and w'_4 has a son w'_5 coloured with colour α. Then before we colour x, the partial colouring is already illegal, because $(w'_5, w'_4, w_3, w_4, w_5)$ is a 2-coloured P_5.)

Assume w_3 is coloured before w_2. Then w_3 has no active sons before w_2 is coloured. In particular, w_5 and w_4 are activated after w_2 is coloured. This implies that w_5 is coloured by Bob, because α is not a permissible colour for w_5. At the time w_4 is coloured, w_4 has no active sons, for otherwise w_4 would have been coloured by Alice. In particular, w_5 is coloured after w_4. At the moment Bob colours w_4, the bi-path above w_4 ends at z, and the safe colour of w_4 is β. By Alice's strategy, she should have coloured a son of z other than w_2 by colour β, because at that moment, w_5 is not coloured yet, and β is a legal colour for x. This is again in contrary to our conclusion that z has no other son coloured with colour β. This completes the proof of Lemma 5, as well as the proof of Theorem 3 for the case $i = 6$. ∎

If F is a forest, it is easy to see that the argument presented in this section also apply. Thus we have the following result.

Theorem 4 *If F is a forest and $i \geq 6$, then $\chi_g(\phi_i, F) \leq 10$.*

4 Some open questions

In this section, we first prove that for any integer n, for any $3 \leq i \leq 5$, there is a tree T with $\chi_g(\phi_i, T) \geq n$. We shall only prove this for the case that $i = 5$. The cases $i = 3, 4$ can be proved in the same way.

Theorem 5 *For any positive integer n, there is a tree T_n for which $\chi_g(\phi_5, T_n) \geq n$.*

Proof. Let T_n be the rooted tree with root vertex u which has $2n+1$ sons, $u_1, \cdots, u_{2n+1}$, and each of u_i has one son v_i for $i = 1, \cdots, 2n+1$. We will show that if Alice and Bob play the ϕ_5-coloring game on T_n with $n-1$ colours, then Bob has a winning strategy.

Suppose Alice colours u with colour α in her first move. Then Bob chooses an uncoloured vertex v_i such that u_i is uncoloured and colours v_i with the colour α. Then at least $n+1$ vertices of $\{v_i\}$ are coloured with α.

Suppose Alice colours a vertex $y \neq u$ with the colour β in her first move. Then Bob colours u with the colour $\alpha \neq \beta$ in his first move. Then Bob chooses an uncoloured vertex v_i such that u_i is uncoloured and colours v_i with the colour α. Thus at least n vertices of $\{v_i\}$ are coloured with α. Let $A = \{i : c(v_i) = \alpha\}$. In both cases, $|A| \geq n$. If all the vertices of T_n are coloured, then since there are only $n-1$ colours, there are two vertices u_i, u_j coloured the same colour and $v_i, v_j \in A$. Then (v_i, u_i, u, u_j, v_j) is a 2-coloured path. ∎

In this paper, we have only studied some very special graph functions f, and the classes of graphs are also very restricted: outerplanar graphs or forests. Many fundamental questions remain open. We call a graph function f a *game bounded graph function* if each proper minor closed class $\mathcal{K}$ of graphs, there is a constant C such that $\chi_g(f, G) \leq C$ for any $G \in \mathcal{K}$.

Question 1 *Which graph functions are game bounded?*

We know that for any proper minor closed class $\mathcal{K}$ of graphs, the acyclic chromatic number $\chi_a(G)$ is bounded by a constant for all $G \in \mathcal{K}$. As $\chi_g(G) \leq \chi_a(G)(\chi_a(G)+1)$ [3], this implies that $\chi_g(G)$ is bounded by a constant for all $G \in \mathcal{K}$. In other words, if $f(K_2) = 2$ and $f(H) = 1$ otherwise, then f is a game bounded graph function. It is also easy to show that for any $d \geq 0$, if $f(K_{1,d+1}) = 2$ and $f(H) = 1$ otherwise, then f is a game bounded graph function. If $f(C_n) = 2$ for any $n \geq 3$ and $f(H) = 1$ otherwise, then f is a game bounded graph function.

Let ϕ_i be the graph function defined as in Section 3.

Question 2 *Does there exist an integer i and a constant C such that for any outerplanar graph G, $\chi_g(\phi_i, G) \leq C$? Does there exist an integer i and a constant C such that such that for any planar graph G, $\chi_g(\phi_i, G) \leq C$? Does there exist an integer i such that ϕ_i is a game bounded graph function?*

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