Fault-free vertex-pancyclicity in faulty augmented cubes

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Abstract

The n-dimensional augmented cube, denoted as AQn, a variation of the hypercube, possesses some properties superior to those of the hypercube. In this paper, we show that every vertex in AQn lies on a fault-free cycle of every length from 3 to 2n, even if there are up to n − 1 link faults. We also show that our result is optimal.

1 Introduction

A graph G is a triple consisting of a vertex set V(G), an edge set E(G), and a relation that associates with each edge two vertices called its endpoints [22]. We usually use a graph to represent the topology of an interconnection network (network for short). The hypercube is one of the most versatile and efficient interconnection network (network for short). The hypercube is ideally suited to both special-purpose and general-purpose tasks, and can computation. The hypercube is ideally suited to both special-purpose and general-purpose tasks, and can efficiently simulate many other same sized networks [15]. We usually use Qn to denote an n-dimensional hypercube. Many variants of the hypercube have been proposed. The augmented cube, recently proposed by Choudum and Sunitha [5], is one of such variations. An n-dimensional augmented cube AQn can be formed as an extension of Qn by adding some links. For any positive integer n, AQn is a vertex transitive, (2n − 1)-regular, and (2n − 1)-connected graph with 2n vertices. AQn retains all favorable properties of Qn since Qn ⊂ AQn. Moreover, AQn possesses some embedding properties that Qn does not. Previous works relating to the augmented hypercube can be found in [2], [5], [11], [12], [13], [16], [17], [20], [21].

Linear arrays and rings, two of the most fundamental networks for parallel and distributed computation, are suitable for developing simple algorithms with low communication costs. Many efficient algorithms designed based on linear arrays and rings for solving a variety of algebraic problems and graph problems can be found in [15]. The pancyclicity of a network represents its power of embedding rings of all possible lengths. A graph G is called pancyclic whenever G contains a cycle of each length l for 3 ≤ l ≤ |V(G)|. The arrangement graph [7], the hypercomplete network [6], the WK-recursive network [8], the alternating group graph [14], and the hyper-de Bruijn networks [9] are all pancyclic. A graph G is vertex-pancyclic (respectively, edge-pancyclic) if every vertex (respectively, edge) lies on a cycle of every length from 3 to |V(G)|. It is clear that if a graph G is edge-pancyclic, then it is vertex-pancyclic. The recursive circulant graphs with some condition [1], the alternating group graph [3], and the augmented cube [17] are edge-pancyclic.

Since faults may occur to networks, the fault tolerance of networks is an important issue in designing network topologies. Let Fe ⊂ E(G) (respectively, Fe ⊂ V(G)) denote the faulty edges (respectively, the faulty vertices) in a graph G and let F = Fc ∪ Fe. Suppose that G − F is P, where P is pancyclic, vertex-pancyclic, or edge-pancyclic. Then, we call G |F| fault-tolerant P. In addition, G is |F|-edge fault-tolerant P (respectively, |F|-vertex fault-tolerant P) if F = Fe (respectively, if F = Fc). Note that if G is |F|-fault-tolerant P, then G is |F|-edge fault-tolerant P and |F|-vertex fault-tolerant P. Previously, the pancyclicity on various faulty networks was studied in [4], [10], [17], [18], [19], [20]. In [20], AQn has been shown to be (2n − 3) fault-tolerant pancyclic, where n ≥ 4. Up to now, there is no research to concern fault-tolerant vertex-pancyclicity or fault-tolerant edge-pancyclicity. In this paper, we show that AQn − Fe is vertex-pancyclic if |Fe| ≤ n − 1, where n ≥ 2. That is, we show that AQn is (n − 1)-edge fault-tolerant vertex-pancyclic, where n ≥ 2. In addition, we also show that our result is optimal.

The rest of this paper is organized as follows. In Section 2, the structure of the augmented hypercube is elaborated, and some definitions, notations, and properties used throughout this paper are introduced. In Section 3, we show that AQn is (n − 1)-edge fault-tolerant vertex-pancyclic. In Section 4, this paper concludes with some remarks.

2 Preliminaries

Let G be a graph and let u, v ∈ V(G). The degree of vertex v in G, written as degG(v), is the number of edges incident to v in G. In addition, δ(G) = min{degG(v)} v ∈ V(G). A path P[x0, x] = {x0, x1, ..., x} is a sequence of nodes such that two consecutive nodes are adjacent. t is the distance between nodes x0 and x; if P[x0, x] is a shortest path in G. We use dG(x0, x) to denote the distance between x0 and x; in G, and use (u, v) to denote an edge whose endpoints are u and v.
and \( v \). Moreover, a path \( \langle x_0, x_1, \ldots, x_d \rangle \) may contain other subpaths, denoted as \( \langle x_0, x_1, \ldots, x_d, P(x_i, x_j), x_{j+1}, \ldots, x_d \rangle \), where \( P(x_i, x_j) = \langle x_i, x_{i+1}, \ldots, x_j \rangle \). A cycle is a path with \( x_0 = x_d \) and \( t \geq 3 \). A cycle (respectively, path) in \( G \) is called a Hamiltonian cycle (respectively, Hamiltonian path) if it contains every vertex of \( G \) exactly once.

An \( n \)-dimensional hypercube (\( n \)-cube for short) \( Q_n \) is an undirected graph with \( 2^n \) nodes each labeled with a distinct binary string of length \( n \). Nodes \( b = b_1b_2\ldots b_n \) and \( b' = b_1'\ldots b_n' \) are connected by an edge if and only if \( b_i \neq b_i' \) for some \( i \leq n \) and \( b_i \) represents the one complement of \( b_i' \).

An \( n \)-dimensional augmented hypercube \( AQ_n \) is \( Q_n \) augmented by adding more links among its nodes (thus, \( V(AQ_n) = V(Q_n) \)). For a node \( b = b_1b_2\ldots b_n \), it has \( n-1 \) more links to connect to nodes \( b_i' = b_1b_2\ldots b_{i-1}b_{i+1}\ldots b_n \) and \( b_i = b_1b_2\ldots b_i\ldots b_n \) in addition to its original \( n \) links, for all \( i \in \{1, 2, \ldots, n-1\} \). \( AQ_n \) has \( 2^n \) (\( n-1 \)) more links than \( Q_n \). Note that \( AQ_1 \) is isomorphic to \( Q_1 \). Let \( AQ_n^{+1} \) (respectively, \( AQ_n^{-1} \)) be the subgraph of \( AQ_n \) induced by \( \{0b_1b_2\ldots b_n\} \) \( b_i = 0 \) or 1 for \( 2 \leq i \leq n \) (respectively, \( \{b_1b_2\ldots b_n: b_i = 0 \) or 1 for \( 2 \leq i \leq n \)) and \( b' = b_1\ldots b_{i-1}b_{i+1}\ldots b_n \) in addition to its original \( n \) links, for all \( i \in \{1, 2, \ldots, n-1\} \). \( AQ_n \) has \( 2^n-1 \) (\( n-1 \)) more links than \( Q_n \). It is easy to see that \( AQ_n^{+1} \) (respectively, \( AQ_n^{-1} \)) is isomorphic to \( AQ_{n-1} \). In addition, \( AQ_n \) can be recursively constructed by adding \( 2^n \) edges between \( AQ_n^{+1} \) and \( AQ_n^{-1} \). A vertex \( b = 0b_1b_2\ldots b_n \) \( b_i = 0 \) for \( 2 \leq i \leq n \) is joined to two vertices in \( AQ_n^{+1} \), which are \( b^+ = b^+_1b_2\ldots b_n \) and \( b^- = b^-_1b_2\ldots b_n \). It is easy to see that an edge joins nodes \( b^+ \) and \( b^- \). The structures of \( AQ_n, AQ_n^{+1}, \) and \( AQ_n^{-1} \) are shown in Figure 1. It is known that \( AQ_n \) is a vertex-transitive and \( (2n-1) \)-regular graph [1]. The following three lemmas are needed to derive our main result.

**Lemma 3.** Let \( (x, y) \in E(AQ_n^{+1}) \). Then \( (x^+, y^+) \) and \( (x^-, y^-) \) are also edges in \( AQ_n^{-1} \).

**Proof.** It is trivial if \( y = x^\prime \), where \( i \in \{2, 3, \ldots, n\} \). Now, consider that \( y = x_i^\prime \) for some \( i \in \{2, 3, \ldots, n-1\} \). Suppose that \( x = 0_{x_1x_2\ldots x_n} \). Then, \( y = x_{x_1x_2\ldots x_n} \), \( x^\prime = 1_{x_1x_2\ldots x_n} \), \( y^\prime = 1_{x_1x_2\ldots x_n} \), \( x^\prime = 1_{x_1x_2\ldots x_n} \), and \( y^\prime = (x^\prime)^\prime \). The result follows. \( \square \)

### 3 Fault-free vertex-pancyclicity

In this section, we show that \( AQ_n \) is \((n-1)\)-edge fault-tolerant vertex-pancyclic. We format the theorem as follows.

**Theorem 1.** Let \( F \subseteq E(AQ_n) \) denote the faulty edge set of \( AQ_n \), where \( n \geq 2 \). \( AQ_n - F \) is vertex-pancyclic if \( |F| \leq n-1 \).

**Proof.** We proceed by induction on \( n \). It is easy to see that the theorem holds for \( AQ_2 \). As per our induction hypothesis, assume that the result holds for \( AQ_{n-1} \) for some \( n \geq 3 \). Consider that \( AQ_n \) and \( F \subseteq E(AQ_n) \), where \( n \geq 3 \) and \( |F| \leq n-1 \). For simplicity, we may assume \( |F| = n-1 \). In addition, since \( AQ_n \) is vertex-symmetric, we only need to show that \( z = 0^n \) (\( n \) consecutive 0’s) lies on a cycle of length \( l \) in \( AQ_n - F \), where \( 3 \leq l \leq 2^n \). Let \( F_0 = F \cap E(AQ_n^{+1}) \), \( F_1 = F \cap E(AQ_n^{-1}) \), and \( F_2 = F \cap \{(b, b') | b \in V(AQ_n^{+1}) \cup \{(b, b') | b \in V(AQ_n^{-1})}) \). The following cases are considered:

**Case 1:** \(|F_0| = n-1 \). Thus, \(|F_1| = |F_2| = 0 \). Two cases are further considered:

**Case 1.1:** \( 3 \leq l \leq 2^{n-1} + 1 \). Remember that an edge joins \( z^\prime = (0^{n-1}) \) and \( z^\prime = (1^n) \); that is, \( d_{AQ_n^{+1}}(z^\prime, z^\prime') = 1 \). By Lemma 1, there exists a path \( P(z^\prime, z^\prime') \) of length \( l \) in \( AQ_n^{+1} \), where \( 1 \leq l \leq 2^{n-1} - 1 \). Therefore, the desired
cycle of length $l = l_1 + 2 \in \{3, 4, \ldots, 2^{n+1} \}$ can be constructed by $(x, z', P[z', z'], z', z)$ (see Figure 2(a)).

**Case 1.2:** Let $(x', y') \in E_0$ such that $(x', y') \notin E(C_0)$ or $(x', y') \notin C_0$ or $(x', y') \notin C_0$. If $(x', y') \notin E(C_0)$, then randomly select an edge $(x, y)$ from $E(C_0)$. If $(x', y') \in E(C_0)$, then let $(x, y) = (x', y')$. Next, let $P[x, y] = C_0 - \{x, y\}$. Clearly, $z \in P[x, y]$ is of length $2^{n+1} - 1$, and $d_{n_{2^{n-1}}}(y^x, x^y) = 1$. By Lemma 1, there exists a path $P[y^x, x^y]$ of length $l_1$ in $AQ_{n-1}$, $1 \leq l_1 \leq 2^{n+1} - 1$. Therefore, the desired cycle of length $l = l_1 + 2 \in \{2^{n+1} + 2, 2^{n+1} + 3, \ldots, 2^n \}$ can be constructed by $(x, P[x, y], y, y^x, P[y^x, x^y], x^y, x)$ (see Figure 2(b)).

**Case 2:** If $|F_0| < n - 2$. Two cases are further considered:

**Case 2.1:** $3 \leq l \leq 2^{n+1}$. By the induction hypothesis, there exists a cycle $C$ of the required length $l \in \{3, 4, \ldots, 2^{n+1} \}$ in $AQ_{n-1} - F_0$ such that $z \in E(C)$. Clearly, $C$ is the desired cycle.

**Case 2.2:** Let $(x', y') \in F_0$ such that $2^{n+1} + 2 \leq l \leq 2^{n+1} + 2$. By the induction hypothesis, there exists a cycle $C_0$ in $AQ_{n-2} - F_0$ such that $z \in C$. Clearly, $C$ is the desired cycle.

Since $|E(C)| \geq 2^{n-1} - 1$, we have at least $2^{n-1} - 1$ choices. If such an edge does not exist, then $|F| > (2^{n-1} - 1) > n - 1$, which is a contradiction.
Case 2.3: $2^n + 1 \leq l \leq 2^n$. First, consider that $n = 3$. We have $l \in \{7, 8\}$, $|F| = 2$, and $|F_0| \leq 1$. By Lemma 2 (since $|F| = 2$), there exists a cycle $C$ of length 8 in $AQ_2 - F$; obviously, $C$ contains $z$ and $C$ is the desired cycle of length 8. If $|F_0| \leq 1$, then by Lemma 2 (since $|F_0| \leq 1$) there exists a cycle $C_0$ of length 4 in $AQ_2$. Let $(x, y) \in E(C_0)$ such that $(x', x), (y', y) \notin F$, or $(x, x'), (y, y') \notin F$. Then, let $x' \in \{x', x\}$ and $y' \in \{y', y\}$ such that $(x, x'), (y, y') \notin F$. In addition, we can construct a path $P[y', x']$ of length 2 in $AQ_2$. Since $|F_0| = 0$, by Lemma 2, there exists a Hamiltonian path $P[y', x']$ in $AQ_2 - F_0$. The desired cycle of length 7 can be constructed by $(x, P[x, y], y, y', P[y', x'], x, x')$. If $|F_0| = 2$ (thus $|F_0| = |F_0| = 0$), then let $P[x, y]$ be a path of length 2 in $AQ_2 - F_1$. Since $|F_0| = 0$, by Lemma 2, there exists a Hamiltonian path $P[y', x']$ in $AQ_2 - F_0$. The desired cycle of length 7 can be constructed by $(x, P[x, y], y, y', P[y', x'], x, x')$.

Now consider that $n \geq 4$. Since $|F_0| \leq |F| \leq n - 1 = 2(n - 1) - 3$, by Lemma 2, there exists a Hamiltonian cycle $C_1$ in $AQ_2 - F_1$. Let $(x, y)$ be the subpath of $C_1$ with length $l_1$ such that $(x, x'), (y, y') \notin F$, where $2 \leq l_1 \leq 2^n - 1$. Since $|F_0| \leq n - 2 \leq 2(n - 1) - 4$, by Lemma 2, there exists a Hamiltonian path $P[y', x']$ in $AQ_2 - F_0$ (certainly, $z \in V(P[y', x'])$). The desired cycle of length $l = 2^n + 1 - 1 + 2 + l_1 \in \{2^n + 1, 2^n + 2, ..., 2^n + 2^m\}$ can be constructed by $(x, P[x, y], y, y', P[y', x'], x, x')$ (see Figure 2(d)).

The result is optimal with respect to the number of edge faults tolerated since there are distributions of $n$ edge faults over $AQ_n$ such that some vertex can not lie on a fault-free cycle of length three in the faulty $AQ_n$. Consider that a vertex $u = 0^n$ ($n$ consecutive 0’s). Suppose that $F = \{u, u_i\}$, where $i \in \{1, 2, ..., n - 1\}$ and $i \neq j$, $u_i$ cannot lie on a fault-free cycle of length 3.

4 Discussion and conclusion

Since $|E(C_0)| = 4$, we have four choices. If such an edge does not exist, then $|F_1| \geq 4$, which is a contradiction.

It is easy to verify that we can construct a path of length 2 between two arbitrary vertices in $AQ_2 - F$, where $F \subset E(AQ_2)$ and $|F| \leq 1$.

It is easy to verify that we can find a path of length 2 in $AQ_2 - F$, where $F \subset E(AQ_2)$ and $|F| = 2$.

Since $|E(C_0)| = 2^n - 1$, we have at least $2^{n-1}/2 (=2^n - 2)$ choices. If such a path does not exist, then $|F| \geq 2^{n-2} > n - 1$ when $n \geq 4$, which is a contradiction.

Linear arrays and rings, two of the most fundamental networks for parallel and distributed computation, are suitable for developing simple algorithms with low communication costs. The pancyclicity of a network represents its power of embedding rings of all possible lengths. In this paper, using inductive proofs, we showed that $AQ_n$ is $(n - 1)$-edge fault-tolerant vertex-pancyclic, where $n \geq 2$. That is, every vertex of an $AQ_n$ with at most $n - 1$ faulty edges lies on a fault-free cycle of every length from 3 to $2^n$. In addition, we also showed that our result is optimal.

$AQ_n$ is (2$n - 3$)-fault-tolerant pancyclic (thus, (2$n - 3$)-edge fault-tolerant pancyclic), where $n \geq 4$ [20]. We have shown that $AQ_n$ is $(n - 1)$-edge fault-tolerant vertex-pancyclic but not $n$-edge fault-tolerant vertex-pancyclic. Therefore, $AQ_n$ is not $n$ fault-tolerant vertex-pancyclic and not $n$-edge fault-tolerant edge-pancyclic. A topic for further research is to explore the vertex-pancyclicity and/or edge-pancyclicity of augmented cubes in the presence of hybrid faults.

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References


