On the Panconnected Property of Hierarchical Crossed Cube *

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Abstract

The Hierarchical Crossed Cube is a new Hierarchical interconnetion network, proposed by [Hong-Chun Hsu, Chang-Hsiung Tsai, Cheng-Hsien Tsai, Guei-Jhuang Wu, Shin-Hao Liu, and Pao-Lien Lai, Hierarchical Crossed Cube: A New Interconnection Topology, Proceedings of the 25th Workshop on Combinatorial Mathematics and Computation Theory pp. 70-79, (2008)]. For $n \leq 2$, the $HCC(k,n)$ is bipartite and has no cycle of odd length. Hence, we investigate the panconnected property of $HCC(k,n)$ for $n \geq 3$ in this paper. For any two adjacent vertices connected by an external edge, there exists a path of length l, with $l = 1, 7$ and $9 \leq l \leq 2^{k+2n} - 1$, joining them. Let $d_{HCC(k,n)}(u,v)$ denote the distance between u and v in $HCC(k,n)$. For any two vertices that the shortest path joining them consists of external edges, there exists a path of length l with $l = d_{HCC(k,n)}(u,v), d_{HCC(k,n)}(u,v) + 2$ and $d_{HCC(k,n)}(u,v) + 4 \leq l \leq 2^{k+2n} - 1$ joining them. For any distinct two vertices u, v except the above mentioned conditions, there exists a path of length l with $l = d_{HCC(k,n)}(u,v)$ and $d_{HCC(k,n)}(u,v) + 2 \leq l \leq 2^{k+2n} - 1$, joining them.

1 Introduction

Recently, Hierarchical interconnection networks have obtained lots of attention. There are many research about the Hierarchical structure [6, 10, 12, 14, 17]. A new interconnection network is proposed and called Hierarchical Crossed Cube (HCC) [8]. The HCC has many advantages such that lower degree, smaller diameter and maximum fault tolerance in same number of vertices. The interconnection network can be expressed as a graph. For the graph definition and notation we follow [2, 7]. The vertices on the graph represent processors and edges represent links between processors. Let $G = (V, E)$ be an undirected graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set.

For convenience, $\bar{b}$ represnets the complement string of binary string b. The terms binary string and binary number are used interchangeably in the following discussions. A path in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. We use (u,v)-path or $\langle v_0, P, v_n\rangle$ or $P(v_0, v_n)$ to denote path $P = \langle v_0, v_1 \ldots v_n\rangle$. Let $len(P)$ denote the length of path P that is the number of edges in P. A cycle is a path with at least three vertices such that the first vertex is the same as the last one. The distance between u and v, denoted by $d_G(u,v)$, is the length of shortest path of G joining u and v.

A graph G is said to be *hamiltonian connected* if there exists a hamiltonian path between any two vertices of G [13]. A graph G is *panconnected* if there exists a path of length l joining any two different vertices u and v with $d_G(u,v) \leq l \leq |V(G)| - 1$ [1]. The panconnectivity is an important property to determine if the topology of a network is suitable for some applications where mapping paths of any length into the topology of network is required. The pan-

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connectivity of interconnection networks have attracted much research interest in the recent years. A large amount of related work appeared in the literature [4, 5, 11, 15, 16]. For $n \leq 2$, the $HCC(k,n)$ is bipartite and has no cycle of odd length. Hence, we investigate the panconnected property of $HCC(k,n)$ for $n \geq 3$ in this paper. For any two adjacent vertices connected by an external edge, there exists a path of length l with $l = 1, 7$ and $9 \leq l \leq 2^{k+2n} - 1$ joining them. For any two vertices that the shortest path joining them consists of external edges, there exists a path of length l with $l = d_{HCC(k,n)}(u,v), d_{HCC(k,n)}(u,v) + 2$ and $d_{HCC(k,n)}(u,v) + 4 \leq l \leq 2^{k+2n} - 1$ joining them. For any distinct two vertices u, v except the above mentioned conditions, there exists a path of length l with $l = d_{HCC(k,n)}(u,v)$ and $d_{HCC(k,n)}(u,v) + 2 \leq l \leq 2^{k+2n} - 1$ joining them.

The rest of this paper is organized as follows. In Section 2, we first introduce the necessary preliminaries of Hierarchical Crossed Cube. Then the panconnected properties of Hierarchical Crossed Cube are discussed in Section 3. Finally, some conclusions are given in Section 4.

2 Preliminary

We need to define Crossed Cube before introducing Hierarchical Crossed Cubes. To defind Crossed Cube [3], a notion called "pair related" relation is listed as follows.

Definition 1 *Let $R = \{(00,00),(10,10),(01,11),(11,01)\}$. Two digit binary strings $u = u_1u_0$ and $v = v_1v_0$ are pair related, denoted as $u \sim v$, if and only if $(u,v) \in R$.*

The following is the recursive definition of the n-dimensional Crossed Cube CQ_n.

Definition 2 *The Crossed Cube CQ_1 is a complete graph with two vertices labelled by 0 and 1, respectively. For $n \geq 2$, a n-dimensional Crossed Cube CQ_n consists of two $(n-1)$-dimensional sub-Crossed Cubes, CQ^0_{n-1} and CQ^1_{n-1}, and a perfect matching between the vertices of CQ^0_{n-1} and CQ^1_{n-1} according to the following rule:*

Let $V(CQ^0_{n-1}) = \{0u_{n-2}u_{n-3}\ldots u_0 : u_i \in (0,1)\}$ and $V(CQ^1_{n-1}) = \{1v_{n-2}v_{n-3}\ldots v_0 : v_i \in (0,1)\}$. The vertex $u = 0u_{n-2}u_{n-3}\ldots u_0 \in V(CQ^0_{n-1})$ and the vertex $v = 1v_{n-2}v_{n-3}\ldots v_0 \in V(CQ^1_{n-1})$ are adjacent in CQ_n if and only if

1. *$u_{n-2} = v_{n-2}$ if n is even, and*
2. *$(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R$, for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$.*

For convenience, let $v = v_{n-1}v_{n-2}v_{n-3}\ldots v_0$ be the i-dimensional neighbor, $0 \leq i \leq n-1$, of $u = u_{n-1}u_{n-2}u_{n-3}\ldots u_0$ if the following conditions are satisfied: (1) $u_j = v_j$ for $i+1 \leq j \leq n-1$, (2) $u_i \neq v_i$, and (3) $(u_{2j+1}u_{2j}, v_{2j+1}, v_{2j}) \in R$ and $u_{i-1} = v_{i-1}$ if i is odd, for $0 \leq j < \lfloor \frac{i-1}{2} \rfloor$. Moreover, let u^i, $0 \leq i \leq n-1$, denote the i-dimensional neighbor of vertex u in CQ_n. A graph G is i-fault Hamiltonian connected if $G - F$ is also Hamiltonian connected, for any faulty set F where $F \subset V(G) \cup E(G)$ and $|F| \leq i$. Some well-known properties of CQ_n are necessary in the following discussion and listed as follows.

Lemma 1 *The diameter of CQ_n is $\lceil \frac{n+1}{2} \rceil$.*

Lemma 2 *[4, 5, 15] For any two vertices u and v with distance $d_{CQ_n}(u,v)$ in CQ_n with $n \geq 2$, CQ_n contains an (u,v)-path of every length l from $d_{CQ_n}(u,v)$ to $2^n - 1$ except for $d_{CQ_n}(u,v) + 1$.*

Lemma 3 *[9] CQ_n is $(n-3)$-fault Hamiltonian connected for $n \geq 3$.*

Shown as Fig. 1.b, CQ_3 is symmetric. Hence, it is not difficult to verify the two useful properties in Lemmas 4, 5 and it may be possible to omit the detail proofs.

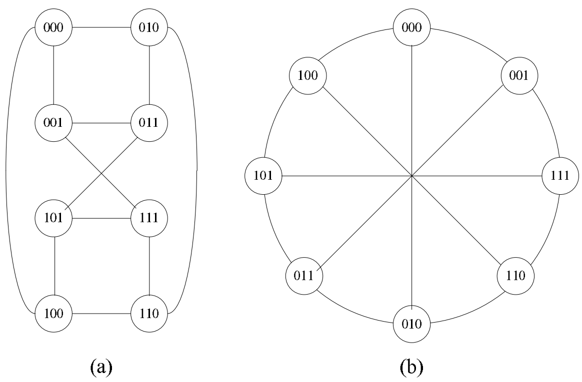

Figure 1: Illustrations of CQ_3.

Lemma 4 *Let u and v be two distinct vertices of CQ_3. CQ_3 contains an (u,v)-path of every length l from 3 to 7.*

Lemma 5 *Let u, s, t be three successive vertices of CQ_3 with $d_{CQ_3}(u,s) = 1$, $d_{CQ_3}(s,t) = 1$ and $d_{CQ_3}(u,t) = 2$. There exists a path of length 6 joining s and t in $CQ_3 - u$.*

We now introduce the Hierarchical Crossed Cubes for the following discussion. Let k, n be two positive integers and $HCC(k,n)$ denote Hierarchical Crossed Cubes. $HCC(k,n)$ is a graph $G=(V,E)$ on 2^{k+2n} vertices, where $V=\{v|v=v_{k+2n-1}v_{k+2n-2}\ldots v_1v_0|v_i\in(0,1), 0\le i\le k+2n-1\}$ and $E=E_{int}\cup E_{ext}$. The label of a vertex u is divided into three parts, say $u=u_Xu_Yu_Z$, where $u_X=u_{k+2n-1}u_{k+2n-2}\ldots u_{2n}$, $u_Y=u_{2n-1}u_{2n-2}\ldots u_n$, and $u_Z=u_{n-1}u_{n-2}\ldots u_0$. The set of edges E is the union of two sets E_{int} and E_{ext}, which are the sets of internal and external edges, respectively, as the following equations.

1. $E_{int}=\{(u,v)|u_X=v_X, u_Y=v_Y$, and $(u_Z,v_Z)\in E(CQ_n)\}$ and
2. $E_{ext}=\{(u,v)|(u_X,v_X)\in E(Q_k), u_Y=v_Z$, and $u_Z=v_Y\}$.

The structure of $HCC(k,n)$ consists of three levels of hierarchy. At the highest level of hierarchy, we have 2^k classes that connected in a Hypercube fashion, denoted as $S_{HCC(k,n)}(x)$, and each class consists 2^n cubes constitute the second level hierarchy, denoted as $CQ_n^{(x,y)}$. In final level, each cube is formed as a CQ_n of 2^n vertices. Figure 2 shows the structure of $HCC(2,3)$. Moreover, $HCC(k,n)$ is constructed with two $HCC(k-1,n)$'s, called $HCC(k-1,n)^0$ and $HCC(k-1,n)^1$, respectively, where the label of vertex is starting with 0 and 1, respectively. For example, $u=0u_{k+2n-2}u_{k+2n-3}\ldots u_1u_0\in V(HCC(k-1,n)^0)$ and $v=1v_{k+2n-2}v_{k+2n-3}\ldots v_1v_0\in V(HCC(k-1,n)^1)$.

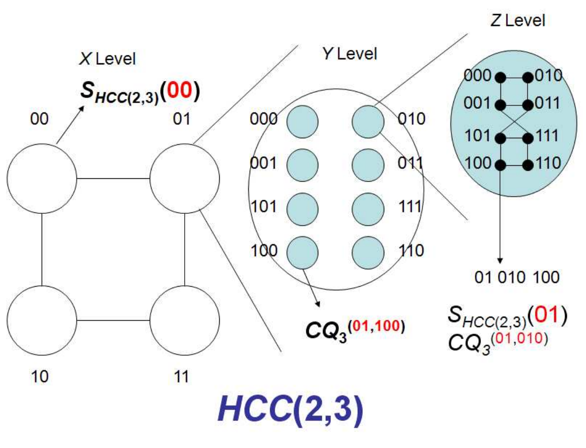

Figure 2: Structure of **HCC(2,3)**.

Let $u=u_Xu_Yu_Z$ and $v=v_Xv_Yv_Z$ be two distinct vertices of $HCC(k,n)$. Let $d^E(u,v)$ and $d^I(u,v)$ denote the number of external edge(s) and internal edge(s), respectively, in a shortest path joining u,v in $HCC(k,n)$. By Algorithms in [8], we have the distance and diameter results for $HCC(k,n)$ as Lemmas 6 and 7.

Lemma 6 *Let k and n be two positive integers. Let $u=u_Xu_Yu_Z$ and $v=v_Xv_Yv_Z$ be two distinct vertices of $HCC(k,n)$. Then $d_{HCC(k,n)}(u,v)=d^E(u,v)+d^I(u,v)$ with*

(1) *$d^E(u,v)=0$ and $d^I(u,v)=d_{CQ_n}(u_Z,v_Z)$ if $u_X=v_X$ and $u_Y=v_Y$; or*

(2) *$d^E(u,v)=2$ and $d^I(u,v)=d_{CQ_n}(u_Y,v_Y)+d_{CQ_n}(u_Z,v_Z)$ if $k=1$, $u_X=v_X$ and $u_Y\neq v_Y$; or*

(3) *$d^E(u,v)=d_{Q_k}(u_X,v_X)$ and $d^I(u,v)=d_{CQ_n}(u_Y,v_Y)+d_{CQ_n}(u_Z,v_Z)$ if $d_{Q_k}(u_X,v_X)$ is even; or*

(4) *$d^E(u,v)=d_{Q_k}(u_X,v_X)$ and $d^I(u,v)=d_{CQ_n}(u_Y,v_Z)+d_{CQ_n}(u_Z,v_Y)$ if $d_{Q_k}(u_X,v_X)$ is odd.*

Lemma 7 *Let k and n be two positive integers. Then the diameter of $HCC(k,n)$ is equal to $2+2\lceil\frac{n+1}{2}\rceil$ as $k=1$ and equal to $k+2\lceil\frac{n+1}{2}\rceil$ for $k\ge 2$.*

For convenience, let $\oplus$ denote the bitwise Exclusive or (XOR) operation on binary numbers. Let $u=u_Xu_Yu_Z$ and $v=v_Xv_Yv_Z$ be two vertices of $HCC(k,n)$. We call v be the I_i-dimensional (respectively, the E_j-dimensional) neighbor of u if $u_X=v_X$, $u_Y=v_Y$, and $v_Z=(u_Z)^i$ (respectively, if $u_X=v_X\oplus 2^j$, $u_Y=v_Z$, and $u_Z=v_Y$). Moreover, let symbols u^{I_i}, $0\le i\le n-1$, and u^{E_j}, $0\le j\le k-1$, denote the I_i-dimensional and the E_j-dimensional neighbors of vertex u, respectively, in $HCC(k,n)$. Let $L(k+n)=\{I_0,I_1,\cdots,I_{n-1},E_0,\cdots,E_{k-1}\}$ denote the set consisting of $k+n$ edges outgoing from a vertex in $HCC(k,n)$. The elements of $L(k+n)$ can be used to represent a path in $HCC(k,n)$ explicitly. For instance, we write $0000000[I_0,I_1,E_0]1011000$ to denote the path $\langle 0000000, 0000001, 0000011, 1011000\rangle$ in $HCC(1,3)$.

For $HCC(k,n)$, we list some useful properties as Observations 1-11 and omit proof here because of space limit. By Algorithms in [8], we also have the shortest path property as Observation

1. Moreover, two results on adjacent vertices of $HCC(k,n)$ are presented as Observations 2 and 3 .

Observation 1 *Let k and n be two positive integers and let $u = u_X u_Y u_Z$ and $v = v_X v_Y v_Z$ be two vertices of $HCC(k,n)$ with $d_{Q_k}(u_X, v_X) \geq 2$. Then there exists such shortest path as $P_s\langle u, P'_s, v^{E_{k-1}}, v\rangle$ that all vertices of P'_s belong to the same subgraph $HCC(k-1,n)^i$, $i = 0,1$, of $HCC(k,n)$ and v belongs to another subgraph $HCC(k-1,n)^{\bar{i}}$ of $HCC(k,n)$.*

Observation 2 *Let k,n be two positive integers and let u,v be two adjacent vertices of $HCC(k,n)$ with $v = u^{I_i}$, $0 \leq i \leq n-1$. There is a (u,v)-path of length 7 as $u[E_j, I_i, E_j, I_i, E_j, I_i, E_j]v$, $0 \leq j \leq k-1$, in $HCC(k,n)$.*

Observation 3 *Let k,n be two positive integers and let u,v be two adjacent vertices of $HCC(k,n)$ with $v = u^{E_j}$, $0 \leq j \leq k-1$. There is a (u,v)-path of length 7 as $u[I_i, E_j, I_i, E_j, I_i, E_j, I_i]v$, $0 \leq i \leq n-1$, in $HCC(k,n)$.*

To simplify the explanation, if no ambiguity arises, consider each subgraph $CQ_n^{(x,y)}$, $(x = x_{k-1}x_{k-2}\ldots x_1x_0 | x_i \in \{0,1\}, 0 \leq i \leq k-1, y = y_{n-1}y_{n-2}\ldots y_1y_0 | y_j \in \{0,1\}, 0 \leq j \leq n-1)$, as a super vertex, and consider $HCC(1,n)$ as a complete bipartite graph $K_{2^n,2^n}$ of super vertex. Clearly, $HCC(k,n)$ is also a bipartite graph of super vertex for $k \geq 2$. Herein, the super vertex of $CQ_n^{(x,y)}$ is also labeled by (x,y) or xy. A cycle (respectively, path) consists of super vertices of $HCC(1,n)$ is called a *super cycle* (respectively, *super path*). Then, we have the following observations on super cycle and super path.

Observation 4 *There exists the super cycle containing even number of super vertices from 4 to 2^{n+1} in $HCC(1,n)$ for $n \geq 1$.*

Observation 5 *Let $k \geq 2$, $n \geq 3$, and let u,v be two adjacent vertices, connected by an internal edge, of $HCC(k,n)$. Then $d_{HCC(k,n)}(u^{E_{k-1}}, v^{E_{k-1}}) = 3$. (See Fig. 3)*

Observation 6 *Let k,n be two positive integers with $k \geq 2$, and let u,v be two distinct vertices of $HCC(k,n)$ with $v = (u^{E_i})^{E_{i'}}$, $0 \leq i \neq i' \leq k-1$. There is a (u,v)-path of length 4 as $u[I_j, E_{i'}, E_i, I_j]v$, $0 \leq j \leq n-1$, in $HCC(k,n)$ (See Figure 4).*

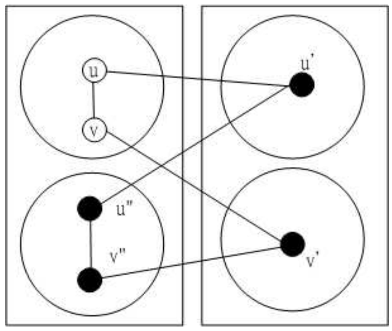

Figure 3: Illustration of Observation 5.

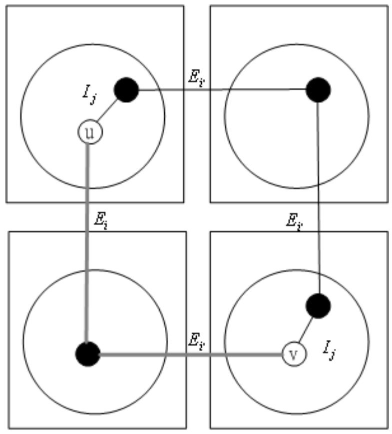

Figure 4: Illustration of Observation 6.

Observation 7 *Let n be one positive integer and let S^0, S^1 be two partite sets of the same number 2^n of super vertices in $HCC(1,n)$. Let s^0, s^1 be two super vertices with $s^0 \in S^0$ and $s^1 \in S^1$. There exists a super path of odd length from 1 to $2^{n+1}-1$ joining s^0 and s^1 in $HCC(1,n)$.*

Observation 8 *Let n be one positive integer and let S^0, S^1 be two partite sets of the same number 2^n of super vertices in $HCC(1,n)$. Let s^0, s^1 be two super vertices with $s^0, s^1 \in S^0$ (or $s^0, s^1 \in S^1$). There exists a super path of even length from 2 to $2^{n+1}-2$ joining s^0 and s^1 in $HCC(1,n)$.*

Let $P = \langle u_0 = u'_0, P^0, u_1, u'_1, P^1, \ldots, u'_m, P^m, u_{m+1}\rangle$ be a (u_0, u_{m+1})-path of $HCC(k,n)$ where path P^i, $0 \leq i \leq m$, contains only internal edge(s) ($u_{i+1} = u'_i$ if $len(P_i) = 0$), and $(u_i, u'_i) \in E_{ext}(HCC(k,n))$ for $1 \leq i \leq m$. P^i is called an *internal subpath* of P. Note that each super path contains no internal edge(s) but external edge(s). Hence, we can get a longer path by augmenting a super path with valid internal subpath(s). By the structure of $HCC(1,n)$ and Observations 7 and 8, we have Observations 9 and 10. Moreover, Observation 11 follows by Lemma 2.

Observation 9 *Let $n \geq 3$ and let S^0, S^1 be two partite sets of the same number 2^n of super vertices in $HCC(1,n)$. Let $s^0 \in S^0, s^1 \in S^1$ and let $u \in CQ_n^{s^0}$ and $v \in CQ_n^{s^1}$ be two distinct vertices of $HCC(1,n)$. Let $P(s^0, s^1)$ be a super path of odd length p, $3 \leq p \leq 2^{n+1}-1$, in $HCC(1,n)$. There exists a (u,v)-path consists of p external edges and at least $p-1$ internal subpath(s) of nonzero length in $HCC(1,n)$.*

Observation 10 *Let $n \geq 3$ and let S^0, S^1 be two partite sets of the same number 2^n of super vertices in $HCC(1,n)$. Let s^0, s^1 be two super vertices with $s^0, s^1 \in S^0$ (or $s^0, s^1 \in S^1$)and let $u \in CQ_n^{s^0}$ and $v \in CQ_n^{s^1}$ be two vertices of $HCC(1,n)$. Let $P(s^0, s^1)$ be a super path of even length p, $2 \leq p \leq 2^{n+1}-2$ in $HCC(1,n)$. There exists such (u,v)-path consists of p external edges and at least $p-2$ internal subpath(s) of nonzero length in $HCC(1,n)$.*

Observation 11 *Let $n \geq 3$ and let S^0, S^1 be two partite sets of the same number 2^n of super vertices in $HCC(1,n)$. Let u, v be two vertices of $HCC(1,n)$ and let $P(u,v)$ be a path of length l_0 and consist of p external edges and $m \geq 1$ internal subpath(s) of nonzero length. Then there exists a (u,v)-path of length l with $l = l_0$ and $l_0 + 2 \leq l \leq p + m(2^n - 1)$ in $HCC(1,n)$.*

3 The Panconnected property of $HCC(k,n)$

In this section, we will first discuss the panconnected property of $HCC(1,n)$ as Theorem 1 and then use induction to demonstrate the panconnected property of $HCC(k,n)$ as Theorem 2. Note that we only investigate the panconnected property of $HCC(k,n)$ for $n \geq 3$ as $HCC(k,n)$ is bipartite and has no cycle of odd length for $n \leq 2$. Lemmas 8-13 provide necessary path properties for Theorem 1.

Lemma 8 *Let $n \geq 3$ and let S^0, S^1 be two partite sets of the same number 2^n of super vertices in $HCC(1,n)$. Let $s^0 \in S^0, s^1 \in S^1$ and let $u \in CQ_n^{s^0}$ and $v \in CQ_n^{s^1}$ be two adjacent vertices of $HCC(1,n)$. There exists the (u,v)-path of length l with $l = 1, 7$ and $9 \leq l \leq 2^{1+2n}-1$ in $HCC(1,n)$.*

Proof: By Oberservation 3, there exists a path $P(u,v) = u[I_0, E_0, I_0, E_0, I_0, E_0, I_0]v$ of length 7 joining u and v in $HCC(1,n)$. Clearly, there are 3 external edges and 4 internal subpaths of nonzero length on $P(u,v)$(See Fig. 5.a). By Observation 11, we can get the longer path of length from 9 to $3 + 4(2^n - 1) = 4(2^n) - 1$ joining u and v. By Observation 9, there exists the $P(u,v)$ consists of $2m-1$ external edges and $2m$ internal paths of nonzero length for $2 \leq m \leq 2^n$ in $HCC(1,n)$(See Fig. 5.b). By Lemmas 1 and 7, $len(P(u,v)) \leq 2m - 1 + 2m(\lceil \frac{n+1}{2} \rceil)$. By Observation 11, we can get the longer path of length from $2m - 1 + 2m(\lceil \frac{n+1}{2} \rceil) + 2$ to $2m2^n - 1$. Note that $2m - 1 + 2m(\lceil \frac{n+1}{2} \rceil) + 2 \leq (2m-2)2^n$ for $n \geq 3$ and $m \geq 2$. The lemma holds. □

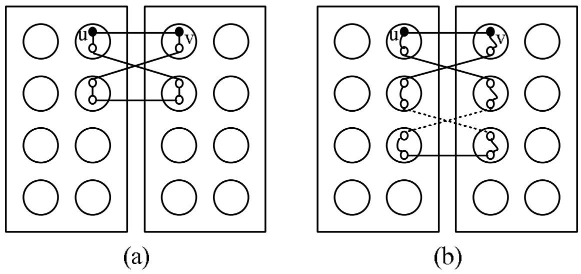

Figure 5: Illustration of Lemma 8.

Lemma 9 *Let $n \geq 3$ and let S^0, S^1 be two partite sets of the same number 2^n of super vertices in $HCC(1,n)$. Let $s^0 \in S^0, s^1 \in S^1$ and let $u \in CQ_n^{s^0}$ and $v \in CQ_n^{s^1}$ be two non-adjacent vertices of $HCC(1,n)$. There exists the (u,v)-path of length l with $l = d_{HCC(1,n)}(u,v)$ and $d_{HCC(1,n)}(u,v) + 2 \leq l \leq 2^{1+2n} - 1$ in $HCC(1,n)$.*

Proof: Clearly, the shortest path joining u and v has at least one external edge and one internal subpath of nonzero length(See Fig. 6.a and 6.b). By Observation 11, we can get the longer path of length from $d_{HCC(k,n)}(u,v) + 2$ to 2^n. Moreover, there exists the path $P(u,v)$ consists of 3 external edges and 2 internal subpaths of nonzero length in $HCC(1,n)$ as Fig. 6.c. Clearly, we can get the longer path of length from $3 + 2(\lceil \frac{n+1}{2} \rceil) + 2$ to $2^{n+1}+1$. Note that $3 + 2(\lceil \frac{n+1}{2} \rceil) + 2 \leq 2^n + 1$. By Observation 9, we can get the (u,v)-path, denoted by P, consists of $2m-1$ external edges and $2m$ internal subpaths of nonzero length for $2 \leq m \leq 2^n$ in $HCC(1,n)$(See Fig. 6.d). By Lemmas 1, 7 and Observation 11, we can get the longer path of length from $2m - 1 + 2m(\lceil \frac{n+1}{2} \rceil) + 2$ to $2m2^n - 1$.

Note that $2m-1+2m(\lceil\frac{n+1}{2}\rceil)+2 \leq (2m-2)2^n$ for $n \geq 3$ and $m \geq 2$. The lemma holds. □

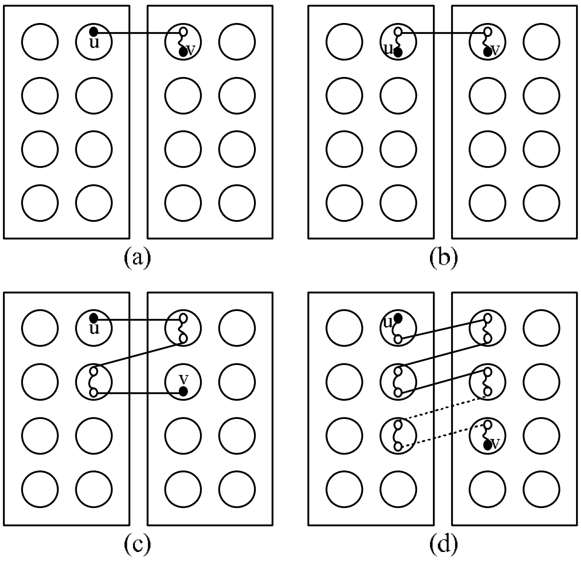

Figure 6: Illustration of Lemma 9

By these similar proof techniques in Lemmas 8 and 9 with Lemmas 1, 2, 4 and Oberservations 2, 4, 8, 10, and 11, we can prove Lemmas 10 and 11.

Lemma 10 *Let $n \geq 3$ and let S^0, S^1 be two partite sets of the same number 2^n of super vertices in $HCC(1,n)$. Let $x \in S^0$ (or $x \in S^1$) be a super vertex and let $u, v \in CQ_n^x$ be two distinct vertices of $HCC(1,n)$. There exists the (u,v)-path of length l with $l = d_{HCC(1,n)}(u,v)$ and $d_{HCC(1,n)}(u,v)+2 \leq l \leq 2^{1+2n}-1$ in $HCC(1,n)$.*

Lemma 11 *Let $n \geq 3$ and let S^0, S^1 be two partite sets of the same number 2^n of super vertices in $HCC(1,n)$. Let $s^0, s^1 \in S^0$ (or $s^0, s^1 \in S^1$) be two distinct super vertices and let $u \in CQ_n^{s^0}$ and $v \in CQ_n^{s^1}$ be two vertices of $HCC(1,n)$. There exists the (u,v)-path of length l with $l = d_{HCC(1,n)}(u,v)$ and $d_{HCC(1,n)}(u,v)+2 \leq l \leq (2^{1+n}-1)2^n-1$ in $HCC(1,n)$.*

Let s^0, s^1 be two distinct super vertices in the same partite set of $HCC(1,n)$. For $u \in CQ_n^{s^0}$ and $v \in CQ_n^{s^1}$, we construct the other necessary (u,v)-paths in Lemma 12 and Lemma 13 for $HCC(1,3)$ and $HCC(1,n)$, $n \geq 4$, respectively.

Lemma 12 *Let $u \in CQ_3^{x,y}$ and $v \in CQ_3^{x,y'}$ be two vertices of $HCC(1,3)$ with $y \neq y'$. There exists the (u,v)-path of length $54 \leq l \leq 127$ in $HCC(1,3)$.*

Proof: Let $s, t \in CQ_3^{x,y}$ be two adjacent vertices with $d_{CQ_3}(u_Z, s_Z) = 1$, $d_{CQ_3}(s_Z, t_Z) = 1$, $d_{CQ_3}(u_Z, t_Z) = 2$, and $v_Z \neq t_Z$. By Lemma 5, there exists a (s,t)-path of length 6 in $CQ_3^{x,y} - u$. Let $P(xy, xy')$ be a path consists of 16 external edges and 16 internal subpaths of nonzero length as Fig. 7. By Lemmas 1, 2 and Oberservation 11, we can construct the (u,v)-path of length $16+6+15(\lceil\frac{3+1}{2}\rceil)+2 \leq l \leq 127$ in $HCC(1,3)$. Note that $16+6+15(\lceil\frac{3+1}{2}+2\rceil) = 54$. This lemma holds. □

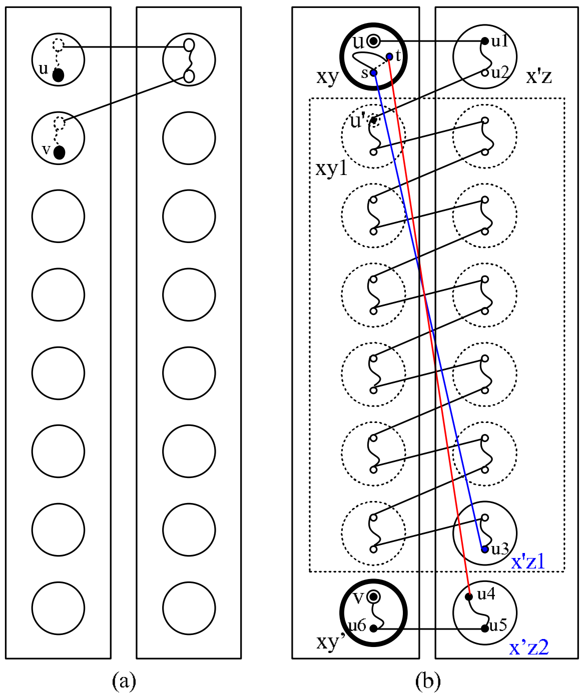

Figure 7: Illustration of Lemma 12.

Lemma 13 *Let $n \geq 4$ and let S^0, S^1 be two partite sets of the same number 2^n of super vertices in $HCC(1,n)$. Let $s^0, s^1 \in S^0$ (or $s^0, s^1 \in S^1$) be two distinct super vertices and let $u \in CQ_n^{s^0}$ and $v \in CQ_n^{s^1}$ be two vertices of $HCC(1,n)$. There exists the (u,v)-path of length l with $l = d_{HCC(1,n)}(u,v)$ and $d_{HCC(1,n)}(u,v)+2 \leq l \leq 2^{1+2n}-1$ in $HCC(1,n)$.*

Proof: Let $s, t \in CQ_n^{s^0}$ be two distinct vertices with $u \neq s, t$ and $v_Z \neq t_Z$. By Lemma 3, there exists a (u,v)-path of length 2^n-2 in $CQ_n^{s^0} - u$. By the similar proof techniques in Lemmas 8, 9, and 12, we can construct the (u,v)-path of length l with $l = d_{HCC(1,n)}(u,v)$ and $d_{HCC(1,n)}(u,v)+2 \leq$

$l \leq 2^{1+2n} - 1$ in $HCC(1,n)$ and this lemma holds. $\square$

We now list the panconnected results for $HCC(1,3)$ as Lemmas 14 and 15.

Lemma 14 *Let u, v be two vertices of $HCC(1,3)$ with $v \neq u^{E_0}$. There exists the (u,v)-path of length l with $l = d_{HCC(1,3)}(u,v)$ and $d_{HCC(1,3)}(u,v) + 2 \leq l \leq 2^7 - 1$ joining them in $HCC(1,3)$.*

Lemma 15 *Let u, v be two adjacent vertices of $HCC(1,3)$ with $v = u^{E_0}$. There exists the (u,v)-path of length l with $l = 1, 7$ and $9 \leq l \leq 2^7 - 1$ in $HCC(1,3)$.*

By Lemmas 8-15, the panconnected result for $HCC(1,n)$ is presented as Theorem 1.

Theorem 1 *Let $n \geq 3$, and let u and v be two vertices of $HCC(1,n)$. $HCC(1,n)$ contains the (u,v)-path of length l with*

(1) $l = 1, 7$ *and* $9 \leq l \leq 2^{1+2n} - 1$ *if* $v = u^{E_0}$; *or*

(2) $l = d_{HCC(1,n)}(u,v)$ *and* $d_{HCC(1,n)}(u,v) + 2 \leq l \leq 2^{1+2n} - 1$ *if* $v \neq u^{E_0}$.

We then consider the necessary paths in $HCC(k,n)$ for $k \geq 2$ as Lemmas 16 and 17. By the symmetry of Q_k, any two distinct vertices $u = u_X u_Y u_Z$ and $v = v_X v_Y v_Z$ of $HCC(k,n)$ with $v_X \neq \overline{u_X}$, there exists such automorphism σ that $\sigma(u)$ and $\sigma(v)$ belong to the same subgraph $HCC(k-1,n)^i$, $i = 0, 1$, of $\sigma(HCC(k,n))$. Hence, we only discuss the two conditions: (1) $v_X \neq \overline{u_X}$ and (2) $v_X = \overline{u_X}$ for (u,v)-paths in $HCC(k,n)$. Lemma 16 is an important induction property for condition (1). Lemma 17 presents the necessary result for condition (2).

Lemma 16 *Let $k \geq 2$ and $n \geq 3$ and let u, v be two vertices belong to the same subgraph $HCC(k-1,n)^i$, $i = 0, 1$, of $HCC(k,n)$. There exists the (u,v)-path of length l with $2^{k-1+2n} \leq l \leq 2^{k+2n} - 1$ in $HCC(k,n)$ if there is the (u,v)-path of length l_0 with $max(9, d_{HCC(k-1,n)^i}(u,v) + 4) \leq l_0 \leq 2^{k-1+2n} - 1$ in $HCC(k-1,n)^i$.*

Proof: Without loss of generality, let $i = 0$ and let $P^0(u,v)$ be the path of length l_0 with $max(9, d_{HCC(k-1,n)^0}(u,v) + 4) \leq l_0 \leq 2^{k-1+2n} - 1$ in $HCC(k-1,n)^0$. Let (s,t) be an external edge on $P^0(u,v)$ and $P^0(u,v) = \langle u, P^{00}, s, t, P^{01}, v \rangle$. Clearly, $(s^{E_{k-1}}, t^{E_{k-1}})$ is an external edge in $HCC(k-1,n)^1$. By the induction hypothesis, there is the path $P^1(s^{E_{k-1}}, t^{E_{k-1}})$ of length l_1 with $9 \leq l_1 \leq 2^{k-1+2n} - 1$ in $HCC(k-1,n)^1$. Let $P(u,v) = \langle u, P^{00}, s, s^{E_{k-1}}, P^1(s^{E_{k-1}}, t^{E_{k-1}}), t^{E_{k-1}}, t, P^{01}, v \rangle$. Note that $max(9, d_{HCC(k-1,n)^0}(u,v) + 4) + 8 \leq len(P(u,v)) \leq 2^{k+2n} - 1$. By Lemmas 1 and 7, $max(9, d_{HCC(k-1,n)^0}(u,v) + 4) + 8 \leq max(9, 2(k-1) + 2\lceil \frac{n+1}{2} \rceil + 4) + 8 \leq 2^{k-1+2n}$ for $k \geq 2$ and $n \geq 3$. Clearly, $P(u,v)$ has length l with $2^{k-1+2n} \leq l \leq 2^{k+2n} - 1$ in $HCC(k,n)$ for $k \geq 2$. $\square$

Lemma 17 *Let $k \geq 2$ and $n \geq 3$ and let u, v be two vertices of $HCC(k,n)$ with $v_X = \overline{u_X}$. For $d_{HCC(k,n)}(u,v) = d^E(u,v) \geq 2$, there exists a path of length l with $l = d_{HCC(k,n)}(u,v), d_{HCC(k,n)}(u,v) + 2$ and $d_{HCC(k,n)}(u,v) + 4 \leq l \leq 2^{k+2n} - 1$ joining them in $HCC(k,n)$. For $d_{HCC(k,n)}(u,v) > d^E(u,v)$, there exists a path of length l with $l = d_{HCC(k,n)}(u,v)$ and $d_{HCC(k,n)}(u,v) + 2 \leq l \leq 2^{k+2n} - 1$ joining them in $HCC(k,n)$.*

Proof: Without loss of generality, let $u \in HCC(k-1,n)^0$ and $v \in HCC(k-1,n)^1$. We prove this lemma by induction on k. By Theorem 1, this lemma is true for $k = 1$. Assume that $HCC(k-1,n)$ is true. We now prove that this lemma still holds in $HCC(k,n)$. Then, consider $P(u,v)$ with the following two cases: (1) $d_{HCC(k,n)}(u,v) = d^E(u,v) = 2$, and (2) $d_{HCC(k,n)}(u,v) = d^E(u,v) > 2$ or $d_{HCC(k,n)}(u,v) > d^E(u,v)$.

Case 1: $d_{HCC(k,n)}(u,v) = d^E(u,v) = 2$.

By Oberservation 6, there exists a path $P_4(u,v) = u[I_0, E_{i'}, E_i, I_0]v = \langle u, u', t, v', v \rangle$ of length 4 joining u and v. Clearly, $len(P_4(u,v)) = d_{HCC(k,n)}(u,v)) + 2$ and there are 2 external edges and 2 internal edges on $P_4(u,v)$. Let $P^0(u,u')$ be a path of length l_0 with $d_{HCC(k-1,n)}(u,u') + 2 = 3 \leq l_0 \leq 2^{k-1+2n} - 1$. Let $P^1(v,t)$ be a path of length l_1 with $d_{HCC(k-1,n)}(v,t) + 2 = 4 \leq l_1 \leq 2^{k-1+2n} - 1$. Let $P(u,v) = \langle u, P^0(u,u'), u', v, P^1(v,t), t \rangle$ be a path of length $l = l_0 + l_1 + 1$. Then $d_{HCC(k-1,n)^0}(u,u') + 2 + d_{HCC(k-1,n)^1}(v,t) + 2 + 1 \leq l \leq 2^{k+2n} - 1$. Note that $d_{HCC(k-1,n)}(u,u') + 2 + d_{HCC(k-1,n)}(v,t) + 2 + 1 = d_{HCC(k,n)}(u,v) + 4$. Thus, there exists a path of length l with $l = d_{HCC(k,n)}(u,v), d_{HCC(k,n)}(u,v) + 2$ and $d_{HCC(k,n)}(u,v) + 4 \leq l \leq 2^{k+2n} - 1$ joining them in $HCC(k,n)$.

Case 2: $d_{HCC(k,n)}(u,v) = d^E(u,v) > 2$ or $d_{HCC(k,n)}(u,v) > d^E(u,v)$.

By Observation 1, we can find a shortest path as $P_s(u,v) = \langle u, \ldots, v'', v', v\rangle$ (see Figure 8). By the induction hypothesis, there exists a path $P^0(u,v')$ of length l_0 with $d_{HCC(k-1,n)^0}(u,v') + c \le l_0 \le 2^{k-1+2n}-1$, $c = 2,4$, in $HCC(k-1,n)^0$. Note that $d_{HCC(k-1,n)^0}(u,v') = d_{HCC(k,n)}(u,v) - 1$. Then we can obtain a path $P(u,v) = \langle u, P^0(u,v'), v', v\rangle$ of length $d_{HCC(k,n)}(u,v) + c \le len(P(u,v)) \le 2^{k-1+2n}$.

We know that $d_{HCC(k-1,n)^0}(u,v'') = d_{HCC(k,n)}(u,v) - 2$. Let $P^0(u,v'')$ be a path of length l_0 with $d_{HCC(k-1,n)^0}(u,v'') + c \le l_0 \le 2^{k-1+2n} - 1$, $c = 2,4$, in $HCC(k-1,n)^0$ and let $P^1(v,t)$ be a path of length l_1 between v,t in $HCC(k-1,n)^1$. By Observations 4 and Observation 5, we know that $v = t^{E_i}$, $0 \le i \le k-2$, or $d_{HCC(k-1,n)^1}(v,t) = 3 > d^E(v,t) = 1$. This implies that $l_1 = 1,7$, or $9 \le l_1 \le 2^{k-1+2n}$ if $v = t^{E_i}$ and implies that $l_1 = 3$ or $5 \le l_1 \le 2^{k+2n-1}$ if $d_{HCC(k-1,n)^1}(v,t) = 3$. Let $P(u,v) = \langle u, P^0(u,v''), v'', t, P^1(t,v), v\rangle$ have length $l = l_0 + l_1 + 1$. Then $d_{HCC(k-1,n)^0}(u,v'') + c + 1 + 5 \le l \le 2^{2n+k} - 1$. Note that $(d_{HCC(k,n)}(u,v) - 2) + 10 \le 2^{k-1+2n} + 1$ for $k \ge 2$ and $n \ge 3$. This completes the proof. □

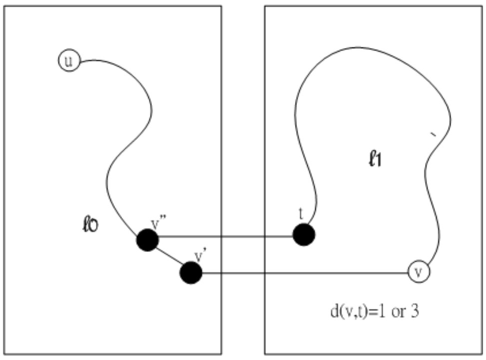

Figure 8: Illustration of **Lemma 17 Case 2**.

Finally, the main result, Theorem 2, is immediate from Theorem 1 and Lemmas 16 and 17.

Theorem 2 *Let $k \ge 1$ and $n \ge 3$ and let u and v be two vertices of $HCC(k,n)$. $HCC(k,n)$ contains the (u,v)-path of length l with*

(1) $l = 1,7$ *and* $9 \le l \le 2^{k+2n} - 1$ *if* $v = u^{E_i}$, $0 \le i \le k-1$; *or*

(2) $l = d_{HCC(k,n)}(u,v), d_{HCC(k,n)}(u,v) + 2$ *and* $d_{HCC(k,n)}(u,v) + 4 \le l \le 2^{k+2n} - 1$ *if* $d_{HCC(k,n)}(u,v) = d^E(u,v) \ge 2$; *or*

(3) $l = d_{HCC(k,n)}(u,v)$ *and* $d_{HCC(k,n)}(u,v) + 2 \le l \le 2^{k+2n} - 1$, *otherwise.*

4 Conclusion

The $HCC(k,n)$(Hierarchical Crossed Cube) is a new interconnection network. $HCC(k,n)$ has many interesting properties such as low degree, logarithmic diameter, simple routing algorithm, and maximum connectivity. In this paper, we show the panconnected property of $HCC(k,n)$ for $n \ge 3$. For any two adjacent vertices connected by an external edge, there exists a path of length l with $l = 1,7$ and $9 \le l \le 2^{k+2n} - 1$ joining them. For any two vertices that the shortest path joining them consists of external edges, there exists a path of length l with $l = d_{HCC(k,n)}(u,v), d_{HCC(k,n)}(u,v) + 2$ and $d_{HCC(k,n)}(u,v) + 4 \le l \le 2^{k+2n} - 1$ joining them. For any distinct two vertices u,v except the above mentioned conditions, there exists a path of length l with $l = d_{HCC(k,n)}(u,v)$ and $d_{HCC(k,n)}(u,v) + 2 \le l \le 2^{k+2n} - 1$ joining them.

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