The degree and extremal number of edges in hamiltonian connected graphs

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Abstract

Assume that $n$ and $\delta$ are positive integers with $3 \leq \delta < n$. Let $hc(n, \delta)$ be the minimum number of edges required to guarantee an $n$-vertex graph $G$ with $\delta(G) \geq \delta$ to be hamiltonian connected. Any $n$-vertex graph $G$ with $\delta(G) \geq \delta$ is hamiltonian connected if $|E(G)| \geq hc(n, \delta)$. We prove that $hc(n, \delta) = C(n - \delta + 1, 2) + \delta^2 - \delta + 1$ if $\delta \leq \lfloor \frac{n+\delta^2(n \mod 2)}{\delta} \rfloor + 1$, $hc(n, \delta) = C(n - \lceil \frac{n}{\delta} \rceil + 1, 2) + \lfloor \frac{n}{\delta} \rfloor^2 - \lfloor \frac{n}{\delta} \rfloor + 1$ if $\lfloor \frac{n+\delta^2(n \mod 2)}{\delta} \rfloor + 1 < \delta \leq \lceil \frac{n}{\delta} \rceil$, and $hc(n, \delta) = \lceil \frac{n}{\delta} \rceil$ if $\delta > \lceil \frac{n}{2} \rceil$.

1 Introduction

For the graph definitions and notations, we follow [1]. Let $G = (V, E)$ be a graph if $V$ is a finite set and $E$ is a subset of $\{(u, v) \mid (u, v) \in V\}$. We say that $V$ is the vertex set and $E$ is the edge set. Two vertices $u$ and $v$ are adjacent if $(u, v) \in E$. The complete graph $K_n$ is the graph with $n$ vertices such that any two distinct vertices are adjacent. The degree of a vertex $u$ in $G$, denoted by $\deg_G(u)$, is the number of vertices adjacent to $u$. We use $\delta(G)$ to denote $\min\{\deg_G(u) \mid u \in V(G)\}$. A path of length $m - 1$, $(v_0, v_1, \ldots, v_{m-1})$, is an ordered list of distinct vertices such that $v_i$ and $v_{i+1}$ are adjacent for $0 \leq i \leq m - 2$. A cycle is a path with at least three vertices such that the first vertex is the same as the last one. A hamiltonian cycle of $G$ is a cycle that traverses every vertex of $G$ exactly once. A graph is hamiltonian if it has a hamiltonian cycle. A hamiltonian path is a path of length $|V(G)| - 1$. A graph $G$ is hamiltonian connected if there exists a hamiltonian path between any two distinct vertices of $G$. It is easy to see that a hamiltonian connected graph with at least three vertices is hamiltonian.

It is proved by Moon [9] that the degree of any vertex in a hamiltonian connected graph with at least four vertices is at least 3, so it is natural to consider the $n$-vertex graph $G$ with $n \geq 4$ and $\delta(G) \geq 3$. Assume that $n$ and $\delta$ are positive integers with $3 \leq \delta < n$. Let $hc(n, \delta)$ be the minimum number of edges required to guarantee an $n$-vertex graph with $\delta(G) \geq \delta$ to be hamiltonian connected. Any $n$-vertex graph $G$ with $\delta(G) \geq \delta$ is hamiltonian connected if $|E(G)| \geq hc(n, \delta)$. We will prove the following main theorem.

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Theorem 1 Assume that \( n \) and \( \delta \) are positive integers with \( 3 \leq \delta < n \). Then \( \text{hc}(n, \delta) = C(n - \delta + 1, 2) + 2^{\frac{\delta}{2}} - 1 \) if \( \delta \leq \left\lfloor \frac{n+3}{2} \right\rfloor \), \( \text{hc}(n, \delta) = C(n - \left\lfloor \frac{n}{2} \right\rfloor + 1, 2) + 2^{\frac{\delta}{2}} - \left\lfloor \frac{n}{2} \right\rfloor + 1 \) if \( \left\lfloor \frac{n+3}{2} \right\rfloor + 1 < \delta \leq \left\lfloor \frac{n}{2} \right\rfloor \), and \( \text{hc}(n, \delta) = \left\lceil \frac{n - 3}{2} \right\rceil \) if \( \delta > \left\lfloor \frac{n}{2} \right\rfloor \).

We defer the proof of Theorem 1. In Section 2, we present the mathematical background. Finally, we give the proof of Theorem 1 in Section 3.

2 Preliminary

The following theorem is proved by Ore [10].

Theorem 2 [10] Let \( G \) be an \( n \)-vertex graph with \( \delta(G) > \left\lfloor \frac{n}{2} \right\rfloor \). Then \( G \) is hamiltonian connected.

The following theorem is given by Lick [8].

Theorem 3 [8] Let \( G \) be an \( n \)-vertex graph. Assume that the degree \( d_i \) of \( G \) satisfy \( d_1 \leq d_2 \leq \ldots \leq d_n \). If \( d_{j-1} \leq j \leq n/2 \Rightarrow d_{n-j} \geq n-j+1 \), then \( G \) is hamiltonian connected.

To our knowledge, no one has ever discussed the sharpness of the above theorem. In the following, we give a logically equivalent theorem.

Theorem 4 Let \( G \) be an \( n \)-vertex graph. Assume that the degree \( d_i \) of \( G \) satisfy \( d_1 \leq d_2 \leq \ldots \leq d_n \). If \( G \) is non-hamiltonian connected, then there exist at least one integer \( 2 \leq m \leq n/2 \) such that \( d_{m-1} \leq m \leq n/2 \) and \( d_{n-m} \leq n-m \).

To discuss the sharpness of Theorem 4, we introduce the following family of graphs. Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs. The union of \( G_1 \) and \( G_2 \), written \( G_1 \cup G_2 \), has edge set \( E_1 \cup E_2 \) and vertex set \( V_1 \cup V_2 \) with \( V_1 \cap V_2 = \emptyset \). The join of \( G_1 \) and \( G_2 \), written \( G_1 \vee G_2 \), obtained from \( G_1 + G_2 \) by joining each vertex of \( G_1 \) to each vertex of \( G_2 \).

The degree sequence of an \( n \)-vertex graph is the list of vertices degree, in nondecreasing order, as \( d_1 \leq d_2 \leq \ldots \leq d_n \). For \( 2 \leq m \leq n/2 \), let \( H_{m,n} \) denote the graph \( (K_{m-1} + K_{n-2m+1}) \cup K_m \). The graphs \( H_{3,11} \) and \( H_{4,12} \) are shown in Figure 1. Obviously, the degree sequence of \( H_{m,n} \) is \( \left\{ \left( m, m, \ldots, m \right), n-m, n-m, \ldots, n-m, \right. \left. n-1, n-1, \ldots, n-1 \right\} \).

A sequence of real numbers \( (p_1, p_2, \ldots, p_n) \) is said to be majorised by another sequence \( (q_1, q_2, \ldots, q_n) \) if \( p_i \leq q_i \) for \( 1 \leq i \leq n \). A graph \( G \) is degree-majorised by a graph \( H \) if \( |V(G)| = |V(H)| \) and the nondecreasing degree sequence of \( G \) is majorised by that of \( H \). For instance, the 5-cycle is degree majorised by the complete bipartite graph \( K_{2,3} \) because \( (2,2,2,2,2) \) is majorised by \( (2,2,2,3,3) \).

Lemma 1 Let \( G = (V, E) \) be a graph, \( X \) be a subset of \( V \), and \( u, v \) be any two distinct vertices in \( X \). Suppose that there exists a hamiltonian path between \( u \) and \( v \). Then there are at most \( |X| - 1 \) connected components of \( G - X \).

Let \( S \) be the subset of \( V(H_{m,n}) \) corresponding to the vertex of \( K_m \). Since \( 2 \leq m \leq n/2 \), \( |S| \geq 2 \). Let \( u \) and \( v \) be any two distinct vertices in \( S \). Obviously, there are \( m \) connected components of \( H_{m,n} - S \). By Lemma 1, \( H_{m,n} \) does not have a hamiltonian path between \( u \) and \( v \). Thus, \( H_{m,n} \) is not hamiltonian connected. In other words, the result in Theorem 4 is sharp.

So we have the following corollary.

Corollary 1 The graph \( H_{m,n} \) is not hamiltonian connected where \( n \) and \( m \) are integers with \( 2 \leq m \leq n/2 \).

Thus, the following theorem is equivalent to Theorem 4.

Theorem 5 If \( G \) is an \( n \)-vertex non-hamiltonian connected graph, then \( G \) is degree-majorised by some \( H_{m,n} \) with \( 2 \leq m \leq n/2 \).

Corollary 2 Let \( n \geq 6 \). Assume that \( G \) is an \( n \)-vertex non-hamiltonian connected graph. Then
δ(G) ≤ ⌊ n 2 ⌋ and |E(G)| ≤ max{ |E(ℋδ(G),n)|, |E(ℋ( ⌊ n 2 ⌋ +1),n)| }.

Proof. Let G be any n-vertex non-hamiltonian connected graph. With Theorem 2, δ(G) ≤ ⌊ n 2 ⌋. By Theorem 5, G is degree-majorised by some ℋm,n. Since δ(Hm,n) = m, δ(G) ≤ m ≤ ⌊ n 2 ⌋. Therefore |E(G)| ≤ max{|E(Hm,n)|, |δ(G)| ≤ m ≤ ⌊ n 2 ⌋}. Since |E(Hm,n)| = 1 2 (m(m−1)+(n−2m+1)(n−m)+m(n−1)) is a quadratics function with respect to m and the maximum value of it occurs at the boundary m = δ(G) or m = ⌊ n 2 ⌋, |E(G)| ≤ max{ |E(ℋδ(G),n)|, |E(ℋ( ⌊ n 2 ⌋ +1),n)| }.

By Corollary 2, we have the following corollary.

Corollary 3 Let G be an n-vertex graph with n ≥ 6. If |E(G)| ≥ max{ |E(ℋδ(G),n)|, |E(ℋ( ⌊ n 2 ⌋ +1),n)| } + 1, then G is hamiltonian connected.

Lemma 2 Let n and k be integers with n ≥ 6 and 3 ≤ k ≤ ⌈ n 2 ⌉. Then |E(ℋk,n)| ≥ |E(ℋ( ⌈ n 2 ⌉ +1),n)| if and only if 3 ≤ k ≤ ⌊ (n+3)(n mod 2) + 1 2 ⌋ or k = ⌈ n 2 ⌉.

Proof. We first prove the case that n is even. We claim that |E(ℋk,n)| ≥ |E(ℋ( ⌈ n 2 ⌉ +1),n)| if and only if 3 ≤ k ≤ ⌈ n 2 ⌉ + 1 or k = ⌈ n 2 ⌉. Suppose that |E(ℋk,n)| < |E(ℋ( ⌈ n 2 ⌉ +1),n)|. Then |E(ℋk,n)| = 1 4 (k(k−1)+(n−2k+1)(n−k)+k(n−1)) < |E(ℋ( ⌈ n 2 ⌉ +1),n)| = 1 4 (( ⌈ n 2 ⌉ − 1)( ⌈ n 2 ⌉ )+( ⌈ n 2 ⌉ )(n−1)+1). This implies 3k 2 − (2n+3)k+3 ⌈ n 2 ⌉ n 2 − n 2 < 0, which means (k− ⌊ n 2 ⌋ )(3k− ⌊ n 2 ⌋ −3) < 0. Thus |E(ℋk,n)| < |E(ℋ( ⌈ n 2 ⌉ +1),n)| if and only if ⌈ n 2 ⌉ +1 < k < ⌊ n 2 ⌋. Note that n and k are integers with n is even, n ≥ 6, and 3 ≤ k ≤ ⌈ n 2 ⌉. Therefore, |E(ℋk,n)| ≥ |E(ℋ( ⌈ n 2 ⌉ +1),n)| if and only if 3 ≤ k ≤ ⌊ n 2 ⌋ + 1 or k = ⌈ n 2 ⌉.

For odd integer n, using the same method, we can prove that |E(ℋk,n)| < |E(ℋ( ⌈ n 2 ⌉ −1),n)| if and only if n+1 2 +1 < k < n−1 2 . Given that n ≥ 7, and 3 ≤ k ≤ n−1 2 , then |E(ℋk,n)| ≥ |E(ℋ( ⌈ n 2 ⌉ −1),n)| if and only if 3 ≤ k ≤ n+1 2 + 1 or k = n−1 2 . Therefore, the result follows.

3 Proof of Theorem 1

Now, we give the proof of Theorem 1.

By brute force, we can check that hc(4,3) = 6, hc(5,3) = 8, and hc(5,4) = 10. Therefore, the theorem holds for n = 4, 5.

Then, we consider that 3 ≤ δ ≤ ⌊ n 2 ⌋ and n ≥ 6.

Suppose that 3 ≤ δ ≤ ⌊ (n+3)(n mod 2) + 1 2 ⌋ or δ = ⌈ n 2 ⌉. By Lemma 2, |E(ℋδ,n)| ≥ |E(ℋ( ⌊ n 2 ⌋ +1),n)|. Let G be any n-vertex graph with δ(G) ≥ δ and |E(G)| ≥ |E(ℋδ,n)| + 1. By Corollary 3, G is hamiltonian connected. By the definition of ℋm,n, we know that |E(ℋδ,n)| + 1 = C(n−δ+1,2)+δ 2 −δ+1. Therefore, hc(n,δ) ≤ C(n−δ+1,2)+δ 2 −δ+1. By Corollary 1, ℋδ,n is not hamiltonian connected. Thus, hc(n,δ) > |E(ℋδ,n)| = C(n−δ+1,2)+δ 2 −δ+1. Hence, hc(n,δ) = C(n−δ+1,2)+δ 2 −δ+1.

Finally, we consider the case that δ > ⌊ n 2 ⌋ and n ≥ 6. Let G be any graph with δ(G) ≥ δ > ⌊ n 2 ⌋. By Theorem 2, G is hamiltonian connected. Obviously, |E(G)| ≥ ⌊ n 2 ⌋. Thus, hc(n,δ) = ⌊ n 2 ⌋.

The proof is complete.

References


