The approximability of the minimum border problem

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Abstract

Motivated to approximating the maximum balance connected partition problem, we investigate the minimum border problem. Given a biconnected graph $G = (V, E)$ with nonnegative vertex weight w and two vertices $y_1, y_2 \in V$, the minimum border problem looks for the minimum weight vertex subset $T \supset \{y_1, y_2\}$ such that both the subgraphs induced by T and $V - T$ are connected. In this paper, we show the approximability of the problem: for any $\varepsilon > 0$, the problem cannot be approximated with ratio $|V|^{1-\varepsilon}$ unless NP=P.

1 Introduction

There are many applications that can be modeled as a problem of breaking a connected graph into some "balanced" connected subgraphs. Let $G = (V, E, w)$ be a simple undirected graph, in which w is a nonnegative vertex weight. For $U \subset V$, by $G[U]$ we denote the subgraph induced by U. The maximum balance connected q-partition (BCP_q) problem looks for a q-partition $(V_1, V_2, \dots V_q)$ as even as possible such that the every induced subgraph $G[V_i]$, $1 \le i \le q$, is connected. When $q = 2$, the balance of a bipartition (V_1, V_2) is defined by $\min\{w(V_1), w(V_2)\}$ and we want to maximize the balance, in which $w(V_1) = \sum_{v \in V_1} w(v)$ is the total weight of V_1.

In [1], Chlebíková shows that BCP_2 is NP-hard in the strong sense and cannot be approximated with an absolute error guarantee of $|V|^{1-\varepsilon}$ for any $\varepsilon > 0$ unless NP=P. A $\frac{4}{3}$-approximation algorithm is also given in the paper and it is currently best result of the problem. For $q = 3$ and $q = 4$ there is a 2-approximation proposed by Chataigner et al. [2]. More references and results about the problem can also be found in [2].

For BCP_2, if the input graph, either weighted or unweighted, has articulation vertices, the problem can be reduced to the weighted version on the blocks of the graph. Therefore the most important thing is to solve the weighted problem on biconnected graphs. Let $G = (V, E)$ be a connected graph and (V_1, V_2) a connected partition of G. A vertex u is *movable* if it can be moved to the other part with keeping the result also a connected partition. Chlebíková's 4/3-approximation algorithm on a 2-connected graph is as follows. It starts with choosing the maximum weight vertex into V_1 and leaving all other vertices to V_2. It then repeatedly moves the movable vertex of minimum weight in V_2 if the balance is increased by the moving, and it stops otherwise. The worst case for Chlebíková's algorithm is as follows. There are three heavy vertices and each has weight $w(V)/4$. The remaining vertices have weight $w(V)/4$ in total. One of the subset of the optimal solution contains two of the heavy vertices and some very small weight vertices. The algorithm may get stuck at V_2 contains two heavy vertices and a path between them of total weight $w(V)/8$. In this case, the only two movable vertices in V_2 are the two heavy ones and the result is a bipartition of balance $(3/8)w(V)$, and the error ratio is roughly 4/3.

If we have a good method finding a small weight vertex set which connects two given vertices and keeps the remaining connected, we may improve the approximation ratio of BCP_2. By this motivation, we study the following minimum border problem.

> **Problem**: Minimum Border (MinBorder) Problem on 2-connected graphs
> **Instance**: A 2-connected graph $G = (V, E)$, vertex weight $w : V \to Z^{+0}$, two vertices $y_1, y_2 \in V$ and $(y_1, y_2) \notin E$.
> **Goal**: Find a connected partition (V_1, V_2) of G such that $\{y_1, y_2\} \subset V_2$ and $w(V_2)$ is minimized. We call V_2 a border if (V_1, V_2) is feasible.

More precisely, if there are 1, 2, or 4 vertices of weight larger than $w(V)/5$, we can easily

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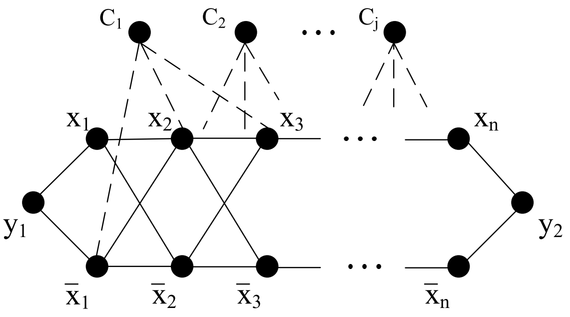

Figure 1: The graph $G = (V, E)$ transformed from 3-SAT problem.

get a 5/4-approximation of the BCP_2 problem by Chlebíková's algorithm. But it can only ensure a ratio of 4/3 when there are 3 such vertices. Good results of the minimum border problem may be helpful.

In this paper, we show the problem is NP-hard and a bad news of its approximability: for any $\varepsilon > 0$, the problem cannot be approximated with ratio $|V|^{1-\varepsilon}$ unless NP=P.

2 NP-hardness and approximability

We show the NP-hardness by transforming from the 3-SAT problem. Let C_j, $1 \le j \le m$ be the clauses of the 3-SAT problem and x_i, $1 \le i \le n$, the variables. We construct a 2-connected graph $G = (V, E, w)$ as follows. Let $V = \{y_1, y_2\} \cup \{x_i, \bar{x}_i | 1 \le i \le n\} \cup \{C_j | 1 \le j \le m\}$. Let $E = E_1 \cup E_2 \cup E_3$, in which $E_1 = \{(y_1, x_1), (y_1, \bar{x_1}, (y_2, x_n), (y_2, \bar{x_n}\}$, $E_2 = \{(x_i, x_{i+1}), (x_i, \bar{x}_{i+1}), (\bar{x}_i, x_{i+1}), (\bar{x}_i, \bar{x}_{i+1}) | \forall 1 \le i < n\}$, and $E_3 = \{(x_i, C_j) | \forall x_i \in C_j\} \cup \{(\bar{x}_i, C_j) | \forall x_i \in C_j\}$. The vertex weights are assigned by $w(x_i) = w(\bar{x}_i) = 1$ and $w(C_j) = n - 1$ for any $1 \le i \le n$ and any $1 \le j \le m$. Let $X = \{x_i, \bar{x}_i | 1 \le i \le n\}$. Figure 1 illustrates the transformation.

Claim 1: If there is a truth assignment satisfying all the clauses, there is a border of weight n.

Proof: For $1 \le i \le n$, let $s_i = x_i$ if x_i is assigned False, and $s_i = \bar{x}_i$ if x_i is assigned True. Let $V^* = \{s_i | 1 \le i \le n\} \cup \{y_1, y_2\}$. The weight of S is n. We now check both S and $V - S$ induce connected subgraphs. Clearly S is a path between y_1 and y_2, and therefore $G[S]$ is connected. We can see that $X - S$ is a path. Since the truth assign satisfies all clauses, each C_j is connected to at least one vertex in $X - S$. Consequently $G[V - S]$ is also connected. □

Claim 2: If there is a border of weight n, there is a truth assignment satisfying all the clauses.

Proof: Suppose S is a border and $w(S) = n$. Since any border contains at least n vertices besides $\{y_1, y_2\}$, S must consists of n vertices in X,i.e., x_i and $\bar{x}_i$ are definitely not in S simultaneously for any $1 \le i \le n$ and there is no any C_j in S. For each $1 \le i \le n$, we assign x_i false if $x_i \in S$ and assign true otherwise. Since S is feasible, each C_j is connected to some x_i or $\bar{x}_i$ which is true by the assignment. Consequentially, there is a truth assignment satisfying all the clauses. □

Theorem 3: The minimum border problem is NP-hard.

Proof: Clearly the transformation can be done in polynomial time. By Claims 1 and 2, if there is a polynomial time algorithm finding the optimal solution of the minimum border problem, the 3-SAT problem can also be solved in polynomial time. Since 3-SAT is NP-complete [3], the minimum border problem is NP-hard. □

Corollary 4: The minimum border problem is NP-hard in the strong sense.

Proof: In the proof of Theorem 3, the constructed graph is weighted. But only C_j has weight larger than one, and all vertices in X have unit weight. We transform the graph G into an unweighted graph G' as follows. For any $v \in X$ and C_j, if there is an edge (v, C_j), we subdivide the edge by inserting $n/2$ internal vertices of degree two. It can be verified that Claims 1 and 2 still hold for G'. □

We then show the inapproximability.

Theorem 5: Unless NP=P, the Minimum Border problem cannot be approximated with ratio $|V|^{1-\epsilon}$ in polynomial time for any $\epsilon > 0$.

Proof: In the proof of Theorem 3, we can see that if there is a border contains no any C_j, there is a true assignment satisfying all C_j. Similar to the proof of Corollary 4, we transform the graph G into an unweighted graph G'. But this time we replace each edge (C_j, v) with a path consisting of $n^k/2$ vertices, in which $k > 1$ is a constant. Then we can claim that all C_j are satisfiable if and only if there is a border of weight less than $n^k + 2$, which implies the border contains no any C_j. That is, if there is a polynomial time approximation algorithm with ratio n^{k-1}, the 3-SAT is polynomial time solvable. Unless NP=P, the minimum border problem cannot be approximated with ratio better than n^{k-1}. Since $|V(G')| = O(mn^k) \leq cn^{k+3}$ for some constant c, $n^{k-1} \geq |V(G')|^{\frac{k-1}{k+3}}$. For any constant $\varepsilon > 0$, let $k = 4/\varepsilon - 3$. The bound of possible approximation ratio of any polynomial time algorithm is $|V(G')|^{1-\epsilon}$.

□

3 Concluding remarks

In this paper, we show the in-approximability of the minimum border problem. But the bad news doesn't imply there is no better approximation algorithm for the balance connected partition problem. How to improve the ratio of BCP_2 is still interesting.

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