Minimum Feedback Arc Sets in Rotator Graphs

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Abstract

A feedback vertex set/arc set (abbreviated to FVS/FAS) is a vertex/arc subset of a graph whose removal induces the remaining graph acyclic. A minimum FVS/FAS is an FVS/FAS which contains the smallest number of vertices/arscs. Hsu and Lin [8] first proposed an algorithm which can find a minimum FVS in rotator graphs. In this paper, we present two efficient algorithms both of which construct FAS’s for a rotator graph with n! nodes in $O(nm!)$ time and also prove that the FAS’s are minimum. In other words, these algorithms derive the optimal results with linear time complexity in terms of the number of arcs in the rotator graph.

Keywords: Feedback arc set; Rotator graph; Generator; Directed cycle

1 Introduction

A feedback vertex set/arc set (abbreviated to FVS/FAS) is a vertex/arc subset of a graph whose removal induces the remaining graph acyclic. The FVS/FAS researches can be applied to deadlock prevention in operating systems [13] and scheduling [4]. A minimum FVS/FAS is an FVS/FAS which contains the smallest number of vertices/arcscs. The both problems of finding a minimum FVS for general graphs and FAS for general directed graphs are NP-hard problems [6]. Therefore, many researches about FVS have been proposed for special graphs such as mesh and butterfly [1,11], hypercube [5], star graph [15], rotator graph [8], shuffle-based interconnection networks [9], and so on. FVS can be studied for both undirected and directed graphs, but only few FAS researches have been proposed for directed graphs such as reducible flow graphs[12], cyclically reducible flow graphs[14], tournaments[2] and bipartite tournaments[7].

Rotator graph, a directed graph and a family of Cayley graph, is first proposed in 1992 [3]. In recent years, many related papers have been published because of the rich topological properties of the rotator graph, such as symmetric structure, recursive construction, low diameter, unique shortest path routing, and so on. In 2006, Hsu and Lin [8] first proposed an algorithm for finding a minimum FVS in rotator graphs. Subsequently, Kuo and Hsu [10] proposed an algorithm for finding a minimum FVS in rotator graphs with linear time complexity in terms of the number of nodes in the rotator graph. In this paper, we present two efficient algorithms which find minimum FAS’s in rotator graphs also with linear time complexity in terms of the number of arcs in the rotator graph.

2 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ and $s_1s_2s_3\ldots s_n$ be a permutation, where $s_i \in N$ for $1 \leq i \leq n$. A rotator graph of scale $n$, denoted by $n$-rotator graph, has $n!$ nodes labeled with a unique permutation $s_1s_2s_3\ldots s_n$. Let $v = s_1s_2s_3\ldots s_n$ be a node in an $n$-rotator graph and $s_i$ be the $i^{th}$ symbol of $v$. Generator $g_i$ inserts the first symbol of a node to the $i^{th}$ position to form an adjacent node of $v$. Node $v * g_i$ denotes the adjacent node of applying $g_i$ to node $v$. Node $v$ links to adjacent nodes $v * g_i$ by the outgoing edges represented as generators $g_i$ for $2 \leq i \leq n$. A 3-rotator graph can be represented by either Fig. 1 or Fig. 2 where edge $g_2$ is represented by directed and undirected edges, respectively. Note that node and vertex are identical in this paper, so are arc and edge.

Definition 1. $(v, g_i)$ denotes the outgoing arc $g_i$ of node $v$. 

**Definition 2.** Node \( v \neq \delta \) denotes the resultant of applying \( \delta \) to node \( v \), where \( \delta \) can be a single generator or a sequence of generators.

**Definition 3.** Let \( g_i \) and \( g_j \) be two generators. If \( i > j \), we call that \( g_i \) is a higher dimension generator than \( g_j \) and denoted by \( g_i \succ g_j \).

**Definition 4.** Let \( V_\delta \) of a rotator graph be the set of all nodes whose \( \delta \)-symbol is \( j \).

For instance, \( V_{2,1} = \{ 1234, 2143, 3124, 3412, 4123, 4312 \} \) for a 4-rotator graph.

**Definition 5.** A directed cycle \( C \) in a rotator graph can be represented as \( \pi(v, R) \), where \( v \) is a node of \( C \) and \( R \) is the generator sequence of \( C \) starting from \( v \). Two directed cycles \( C_1 \) and \( C_2 \) are said to be equivalent, denoted by \( C_1 = C_2 \), if \( C_1 \) and \( C_2 \) contain the same nodes and the same arcs.

For instance, \( \pi(1234, 4g_3g_4g_2g_3g_4g_3g_4) \) is a directed cycle of \( 1234 \rightarrow 2341 \rightarrow 3421 \rightarrow 4213 \rightarrow 2143 \rightarrow 1432 \rightarrow 3124 \rightarrow 2134 \rightarrow 1324 \). Clearly, a directed cycle can be represented in different expressions by starting from different nodes.

**Definition 6.** Let \( \delta_1 \) and \( \delta_2 \) be two sequences of generators. A generator sequence \( R = \delta_1 \oplus \delta_2 \) denotes the concatenation of two sub sequences \( \delta_1 \) and \( \delta_2 \). A generator sequence \( R \oplus \delta_1 = \delta_2 \), denotes removing the leading sub sequence \( \delta_1 \) from \( R \).

The following lemma 1 is cited from [10] for the completeness of the whole proof.

**Lemma 1.** If directed cycle \( C_1 = \pi(v, R) \) where \( R = \delta_1 \oplus \delta_2 \) and directed cycle \( C_2 = \pi(v * \delta_1, \delta_2 \oplus \delta_1) \), then \( C_1 = C_2 \).

**Proof.** \( C_2 = \pi(v * \delta_1, \delta_2 \oplus \delta_1) \)

For example, let \( v = 1234 \) and \( R = (g_2g_4g_3) \oplus (g_3g_4g_2g_3) \). Suppose directed cycle \( C_1 = \pi(v, R) = 1234 \rightarrow 2134 \rightarrow 1342 \rightarrow 3412 \rightarrow 4132 \rightarrow 3124 \rightarrow 1234 \). Directed cycle \( C_2 = \pi(v * (g_2g_4g_3), (g_3g_4g_2g_3)) = \pi(3412, 3g_4g_2g_3g_4g_3g_2g_3) = 3124 \rightarrow 1234 \rightarrow 2134 \rightarrow 1324 \rightarrow 3124 \). Clearly, \( C_1 = C_2 \) because they contain the same nodes and arcs.

**Lemma 2.** Let \( C = \pi(v, R) \) be a directed cycle in an \( n \)-rotator graph, where \( v = s_1s_2s_3...s_n \) and \( g_i \) is the first generator in \( R \). For all symbol \( s_i \) with \( 1 \leq i \leq j \), there exists at least one node in the directed cycle \( C \) whose first symbol is \( s_i \).

**Proof.** The directed cycle \( C \) starts from and ends with the node \( v \), and the next node of \( v \) in \( C \) is \( v * g_j \), which is \( s_jS_{j+1}...S_1s_2s_3...s_n \), so all symbols of \( s_2, s_3, ..., s_j \) must be shifted to the first position in at least one of the nodes in the directed cycle \( C \) such that the symbol \( s_k \) for \( 2 \leq i \leq j \), can be relocated to the \( k \)-index symbol position of \( v \).

For instance, let \( C = \pi(v, R) \), where \( v = 2341 \) and \( R = g_3g_4g_3g_4g_3g_4g_3g_4 \). Since the first generator is \( g_3 \), symbols \( s_1 = 2, s_2 = 3 \) and \( s_3 = 4 \) exist in the first symbol of the nodes in the directed cycle \( C = 2341 \rightarrow 3421 \rightarrow 4213 \rightarrow 2143 \rightarrow 1432 \rightarrow 3124 \rightarrow 2134 \rightarrow 1324 \rightarrow 2341 \).

**Lemma 3.** Let \( C = \pi(v, R) \) be a directed cycle in an \( n \)-rotator graph, where \( v = s_1s_2s_3...s_n \) and \( g_k \) is the highest dimension generator in \( R \). For any node \( v' \) in the directed cycle \( C \), the \( k \)-index symbol of \( v' \in \{ s_1, s_2, ..., s_k \} \) for \( 1 \leq i \leq k \).
Proof. Case \( k = n \). Obviously, all symbols of \( \nu' \in \{ \nu_1, \nu_2, \ldots, \nu_n \} \), and it is true. Case \( k < n \). Because \( g_k \) is the highest dimension generator in \( R \) \((v' \neq g_2) \) denotes that generator \( g_k \) inserts the first symbol of node \( v' \) into the \( k \)th position, the \( k \)th symbol of all nodes in the directed cycle \( C \) is \( s_j \) for \((k + 1) \leq j \leq n \). In other words, the \( k \)th symbol of all nodes in the directed cycle \( C \in \{ s_1, s_2, \ldots, s_k \} \) for \( 1 \leq i \leq k \).

Lemma 4. Let \( C = \pi(v, R) \) be a directed cycle in an \( n \)-rotator graph, where \( v = s_1s_2s_3 \ldots, s_n \) and \( g_k \) is the highest dimension generator in \( R \). For all symbol \( s_i \) with \( 1 \leq i \leq k \), there exists at least one node in the directed cycle \( C \) whose first symbol is \( s_i \). Proof. By Lemma 1, there exists a directed cycle \( C_1 = \pi(v_1, R_1) = C \), where \( v_1 = s'_1s'_2s'_3 \ldots, s'_n \), and the first generator of \( R_1 \) is the highest dimension \( g_k \). By Lemma 2, for all symbol \( s'_i \), with \( 1 \leq i \leq k \), there exists at least one node in directed cycle \( C_1 \) whose first symbol is \( s'_i \). By Lemma 3, all symbol \( s'_i \in \{ s_1, s_2, \ldots, s_k \} \) for \( 1 \leq i \leq k \), so for all symbol \( s_i \) with \( 1 \leq i \leq k \), there exists at least one node in directed cycle \( C_1 \) whose first symbol is \( s_i \).

For example, let \( C = \pi(v, R) \) be a directed cycle in an \( n \)-rotator graph, where \( v = 23415 \) and \( R = g_3 \). Since the highest generator is \( g_4 \), symbols \( s_2 = 2, s_3 = 3, s_4 = 4, \) and \( s_5 = 1 \) exist in the first symbol of nodes in the directed cycle \( C = 23415 \rightarrow 34215 \rightarrow 42135 \rightarrow 21435 \rightarrow 14325 \rightarrow 43125 \rightarrow 31245 \rightarrow 12345 \rightarrow 23415 \).

Definition 7. Let \( C = \pi(v, R) \) be a directed cycle in an \( n \)-rotator graph and \( g_k \) be the highest dimension generator in \( R \). A node \( \nu' = s_1s_2s_3 \ldots, s_n \) in the directed cycle \( C \) is a Cut-node in the directed cycle \( C \) if \( s_1 = \min \{ s_1, s_2, \ldots, s_k \} \).

Lemma 5. Let \( C = \pi(v, R) \) be an arbitrary directed cycle in an \( n \)-rotator graph, where \( v = s_1s_2s_3 \ldots, s_n \) and \( g_k \) is the highest dimension generator in \( R \). There exists at least one Cut-node in the directed cycle \( C \).

Proof. Let \( s_1 = \min \{ s_1, s_2, \ldots, s_k \} \). By Lemma 3 and Lemma 4, we know that there exists at least one node \( \nu' = s'_1s'_2s'_3 \ldots, s'_n \) in the directed cycle \( C \), where \( s'_1 = s_1, \) and all symbol \( s'_j \in \{ s_1, s_2, \ldots, s_k \} \) for \( 1 \leq j \leq k \). Because \( s'_1 = s_1 = \min \{ s_1, s_2, \ldots, s_k \} = \min \{ s'_1, s'_2, \ldots, s'_k \} \), \( \nu' \) is a Cut-node in the directed cycle \( C \).

3 Algorithm

This section proposes two efficient algorithms for finding an FAS in a rotator graph. The first algorithm for efficiently finding an FAS in rotator graphs whose edge \( g_2 \) is represented by a directed edge is shown as follows:

Algorithm FAS-Rotator-I

Let \( FAS_1 \) be an FAS in an \( n \)-rotator graph

For each node \( v = s_1s_2s_3 \ldots, s_n \) in an \( n \)-rotator graph do

\[
\text{flag} = \text{true}
\]

For \( i = 2 \) to \( n \)

If \( s_j < s_i \) and \( \text{flag} \)

Add \( (v, g_j) \) to \( FAS_n \)

Else

\[
\text{flag} = \text{false}
\]

Endif

Enddo

Enddo

Theorem 1. Algorithm FAS-Rotator-I can find an FAS in an \( n \)-rotator graph correctly, where edge \( g_2 \) is represented as a directed edge.

Proof. Let \( C = \pi(v, R) \) be a directed cycle in an \( n \)-rotator graph and \( g_k \) be the highest dimension generator in \( R \). By Lemma 1 and Lemma 5, there exists a directed cycle \( C_1 = \pi(v_1, R_1) = C \), where \( v_1 = s_1s_2s_3 \ldots, s_n \) is a Cut-node in the directed cycle \( C \) and \( s_1 = \min \{ s_1, s_2, \ldots, s_k \} \). Let the first generator of \( R_1 \) be \( g_k \). Because \( g_k \) is the highest dimension generator among \( R_1 \), there only exist \( g_j \) for \( 2 \leq j \leq k \). Let all \((v_1, g_j)\)'s, for \( 2 \leq j \leq k \), be denoted by \( F\)-arcs. Because \( s_1 = \min \{ s_1, s_2, \ldots, s_k \} \), Algorithm FAS-Rotator-I adds the \( F\)-arcs to \( FAS_1 \) and removes the \( F\)-arcs from the \( n \)-rotator graph. Thus, directed cycle \( C_1 \) must be disconnected at the Cut-node \( v_1 \). Since Algorithm FAS-Rotator-I adds all \( F\)-arcs’s to \( FAS_1 \) and removes the \( F\)-arcs from the \( n \)-rotator graph, all directed cycles in the \( n \)-rotator graph become disconnected.

For instance, let directed cycle \( C_1 = \pi(4231, g_3) \) \( g_2, g_3, g_2) = 4231 \rightarrow 2341 \rightarrow 3241 \rightarrow 2431 \)
Theorem 2. The size of the feedback arc set for an $n$-rotator graph generated by Algorithm \textit{FAS-Rotator-I} is $\sum_{i=2}^{n} \frac{n!}{i!}$.

Proof. We will prove this theorem by induction on the size $n$ of the rotator graph.

Case $n=3$. Algorithm \textit{FAS-Rotator-I} generates $\text{FAS}_3 = \{(123, g_2), (123, g_3), (132, g_2), (132, g_3), (231, g_2)\}$. Hence $|\text{FAS}_3| = 3!/2 + 3!/3 = 5$.

Case $n=4$. Algorithm \textit{FAS-Rotator-I} generates $\text{FAS}_4 = \{(1234, g_2), (1234, g_3), (1234, g_4), (1243, g_2), (1243, g_3), (1243, g_4), (1324, g_2), (1324, g_3), (1324, g_4), (1342, g_2), (1342, g_3), (1342, g_4), (2341, g_2), (2341, g_3), (2341, g_4), (2431, g_2), (2431, g_3), (2431, g_4), (3421, g_2), (3421, g_3), (3421, g_4)\}$. Hence $|\text{FAS}_4| = 4!/2 + 4!/3 + 4!/4 = 26$.

Suppose case $n=k$ that $|\text{FAS}_k| = k!/2 + k!/3 + ... + k!/k$ is true.

Now we would like to prove case $n=k+1$ that $|\text{FAS}_{k+1}| = \sum_{i=2}^{k+1} \frac{n!}{i!}$ is true. A $(k+1)$-rotator graph is constructed by $(k+1)$ $k$-rotator graphs, so the total number of $g_i$'s in $\text{FAS}_{k+1}$ for $2 \leq j \leq k$, is $(k+1)(k!/2 + k!/3 + ... + k!/k)$. Let $v = s_1s_2...s_{k+1}$ be a node in a $(k+1)$-rotator graph. Because symbol $v$ is the smallest symbol of $s_1, s_2, ..., s_{k+1}$. Algorithm \textit{FAS-Rotator-I} does not add $(v, g_{k+1})$ to $\text{FAS}_{k+1}$, where $s_j = 1$ for $2 \leq j \leq k$. Consequently, arcs $g_{k+1}$'s in $\text{FAS}_{k+1}$ are $(v, g_{k+1})$'s where $v \in V_{k+1}$, and the total number of arcs $g_{k+1}$'s is $k!$. Therefore, $|\text{FAS}_{k+1}| = (k+1)(k!/2 + k!/3 + ... + k!/k) + 1 = (k+1)!/2 + (k+1)!/3 + ... + (k+1)!/k + 1 = (k+1)!/2 + (k+1)!/3 + ... + (k+1)!/k$, which is true.

Theorem 3. The size of the feedback arc set for a rotator generated by Algorithm \textit{FAS-Rotator-I} is minimum.

Proof. Let $v$ be a node in an $n$-rotator graph and $\delta$ be a generator sequence that is $i$ successive generators $g_i$'s for $2 \leq i \leq n$. Since $v = v * \delta$, $i$ successive arcs $g_i$'s construct a directed cycle, denoted by $G_i$-cycle, for $2 \leq i \leq n$. Any arc $g_i$ in an $n$-rotator graph only belongs to one of the $G_i$-cycle's and $i$ successive arcs $g_i$'s construct a $G_i$-cycle, so there are $n!/i$ $G_i$-cycle's, and each of these $G_i$-cycle's is arc disjoint, for $2 \leq i \leq n$. Then, an $n$-rotator graph contains $n!/2$, $n!/3$, $n!/4$, ..., and $n!/n$ are disjoint directed cycles of length $2, 3, 4, ..., n$, respectively. In order to break these directed cycles, we have to remove at least one arc from each disjoint directed cycle. Thus, we have to remove at least $n!/2 + n!/3 + n!/4 + ... + n!/n = \sum_{i=2}^{n} \frac{n!}{i!}$ arcs. The result of Algorithm \textit{FAS-Rotator-I} is equivalent to this low bound, and therefore the result is minimum.

Subsequently, we discuss how to find the FAS in a rotator graph whose edge $g_2$ is represented by an undirected edge as follows:

Now we consider the case that each edge $g_2$ is treated as an undirected edge in a rotator graph, and thus $g_2$ is not a generator sequence in a directed cycle. Let $\text{FAS}_3$ be a FAS generated by Algorithm \textit{FAS-Rotator-I}. If we remove the $\text{FAS}_3$ except $g_2$'s from the 3-rotator graph as shown in Fig. 2, the directed cycle, $123 \rightarrow 213 \rightarrow 132 \rightarrow 312 \rightarrow 132$, exists in the 3-rotator graph. Nevertheless, we remove the $\text{FAS}_3$ from the 3-rotator graph, and no directed cycle exists in the 3-rotator graph because arcs $(123, g_2)$ and $(132, g_2)$ have been removed. In order to generate a minimum FAS, only one of the arcs $(123, g_2)$ and $(132, g_2)$ has to be removed from the 3-rotator graph, and we choose the $(123, g_2)$.

The modified algorithm for finding FAS in rotator graphs is as follows:

Algorithm \textit{FAS-Rotator-II}

Let $\text{FAS}'_n$ be an FAS in an $n$-rotator graph

For each node $v = s_1s_2s_3...s_n$ in an $n$-rotator graph do:

If $s_1 < s_2 < s_3 ... < s_n$ and flag $= \text{true}$ 

Else 

Enddo

Enddo

Phase 2:

For $s_1 < s_2 < s_3$ Add $(v, g_2)$ to $\text{FAS}'_n$

Enddo

Definition 8. $\text{RG}_n$ denotes an $n$-rotator graph; $\text{FAS}_n$ denotes the FAS generated by Algorithm \textit{FAS-Rotator-I} for the $\text{RG}_n$; $\text{FAS}'_n$ denotes the arc set generated by phase 1 of Algorithm \textit{FAS-Rotator-II} for the $\text{RG}_n$. For $\text{RG}_n$.

Theorem 4. There only exists the type of
directed cycle \( C = \pi(v, R) \) in the \((R_G, -FAS-1', \cdot)\), where \( g_k \) is the highest dimension generator among \( R = g_3 g_2 g_2 g_2 \ldots = 3 \leq a, b \leq k \), and \( v = s_3 s_2 s_1 \ldots s_n \) is a Cut-node in the directed cycle \( C \).

**Proof.** By Theorem 1, the directed cycle \( C \) must be disconnected at the Cut-node \( v \) in the \((R_G = -FAS)\) because all \((v, g)\)'s in the \((R_G = -FAS)\) with \( 2 \leq j \leq k \), are removed. Let \( G_2 \) be an arc set of all arcs \( g_j \)'s in the \( R_G \). Similarly, the directed cycle \( C \) must be disconnected at the Cut-node \( v \) in the \((R_G = -FAS-1', -G_2)\). If we do not remove any arc \( g_2 \) from the \( R_G \), then there must exist an arc \( g_2 \) at the Cut-node \( v \) in the \((R_G = -FAS-1')\), and the directed cycle \( C \) is not disconnected at the Cut-node \( v \) in the \((R_G = -FAS-1')\) by \((v, g_2)\) existing. Let \( C' = \pi(v', R') \) be the directed cycle in the \((R_G = -FAS-1')\), where \( g_k \) be the highest dimension generator in \( R' \). By Lemma 1 and Lemma 5, there exists a directed cycle \( C = \pi(v, R) = C' \). So, only \( \pi(v, g_2, \ldots) \)'s exist in the \((R_G = -FAS-1')\). Since \( v \) is a Cut-node in the directed cycle \( C, s_1 \) is the smallest symbol among \( s_1, s_2, \ldots \) and \( s_k \). Let \( v_1 = v_1 * g_2, \) and then \( s_1 \) is the second symbol of \( v_1 \). Because \( s_1 \) is the smallest symbol among \( s_1, s_2, \ldots \) and \( s_k \), the first symbol of \( v_1 \) is greater than \( s_1 \) and then \( \pi(v_1, g_2, \ldots) \)'s exist in the \((R_G = -FAS-1')\), where \( 3 \leq a \leq k \). Let \( v_2 = v_2 \ast g_2, \) and then \( s_1 \) is the first symbol of \( v_2 \). Because \( s_1 \) is the first symbol of \( v_2 \) and \( s_1 \) is the smallest symbol among \( s_1, s_2, \ldots \) and \( s_k \), \( v_2 \) is also a Cut-node in the directed cycle \( C \), and only \( \pi(v_2, g_2, \ldots) \)'s exist in the \((R_G = -FAS-1')\). Let \( v_3 = v_3 \ast g_2, \) and then \( s_1 \) is the second symbol of \( v_3 \). Because \( s_1 \) is the smallest symbol among \( s_1, s_2, \ldots \) and \( s_k \), \( s_1 \) is the first symbol of \( v_3 \) and then \( \pi(v_3, g_2, \ldots) \)'s exist in the \((R_G = -FAS-1')\), where \( 3 \leq b \leq k \). By Lemma 1, \( \pi(v, g_2 g_2 g_2 g_2 \ldots) = \pi(v, g_2, g_2, \ldots) \) and then \( \pi(v, g_2, g_2, g_2, g_2 \ldots) \)'s exist in the \((R_G = -FAS-1')\), where \( v \) is a Cut-node in the directed cycle \( C \).

**Definition 9.** A node \( v = s_1 s_2 s_3 \ldots s_n \) in an \( n \)-rotator graph is a \( G \)-permutation if \( s_1 < s_2 < s_3 \), where \( n > 2 \).

**Theorem 5.** Let \( C = \pi(v, R) \) be a directed cycle in an \( n \)-rotator graph, where \( g_k \) is the highest dimension generator among \( R = g_2 g_3 g_2 g_2 \ldots = 3 \leq a, b \leq k \), and \( v = s_1 s_2 s_3 \ldots s_n \) is a Cut-node in the directed cycle \( C \). There exist at least one node which is a Cut-node in the directed cycle \( C \) and a \( G \)-permutation.

**Proof.** By Lemma 4, all symbols \( s_1, s_2, \ldots \) and \( s_k \) must be shifted to the first symbol position in at least one of the nodes in the directed cycle \( C \). Let \( v_1 = v_1 \ast g_2 = s_3 s_2 s_1 \ldots s_n, v_2 = v_2 \ast g_2 = s_3 s_2 s_1 \ldots s_n, v_3 = v_3 \ast g_2 = s_3 s_2 s_1 \ldots s_n, \) and \( \ldots \). In other words, all symbols \( s_2, \ldots \) and \( s_k \) must be shifted to the second symbol position in at least one of the nodes in the directed cycle \( C \) and then are shifted to the first symbol position in one of the nodes in the directed cycle \( C \) by generator \( g_2\). In addition, the symbol \( s_1 \) must be shifted to the first position in the node when the symbol \( s_j \) is shifted to the second symbol position in one the nodes in the directed cycle \( C \) for \( 2 \leq j \leq k \). Let \( s_1 \) be the smallest symbol of \( s_2, s_3, \ldots, \) and \( s_k \). Since \( v \) is a Cut-node in the directed cycle \( C, s_1 \) is the smallest symbol among \( s_1, s_2, \ldots \) and \( s_k \). Thus, we know that at least there exists a node \( v' = s_1 s_2, \ldots \) in the directed cycle \( C, s_1 \) is the smallest symbol among \( s_1, s_2, \ldots \) and \( s_k \). Let \( v' = s_1 s_2, \ldots \) be the directed cycle \( C \) and a \( G \)-permutation.

**Theorem 6.** Algorithm \( FAS-Rotator-II \) can find an FAS in an \( n \)-rotator graph correctly, where edge \( g_2 \) is presented as an undirected edge.

**Proof.** By Theorem 4, we know that there only exists the type of directed cycle \( C = \pi(v, R) \) in the \((R_G = -FAS-1')\), where \( g_k \) be the highest dimension generator among \( R, R = g_2 g_3 g_2 \ldots = 3 \leq a, b \leq k \), and \( v \) is a Cut-node in the directed cycle \( C \). Let \( C-G \) be a Cut-node in the directed cycle \( C \) and a \( G \)-permutation. By Theorem 5, we know that there exists at least one Cut-G-node in the directed cycle \( C \). Because there exist \( (v, g_2) \)'s in the Cut-node’s, the directed cycle \( C \) is not disconnected at the Cut-node’s in the \((R_G = -FAS-1')\). Let \( G_2 \)-arcs be an arc set of arcs \( g_2 \)'s of all \( G \)-permutation’s in the \( R_G \). Thus, there exists at least one Cut-G-node whose arc \( g_2 \) is removed from the directed cycle \( C \) in the \((R_G = -FAS-1' - G_2\)-arcs\), and the directed cycle \( C \) must be disconnected at the Cut-G-node in the directed cycle \( C \) in the \((R_G = -FAS-1' - G_2\)-arcs\). Consequently, Algorithm \( FAS-Rotator-II \) adds the \( FAS-1' \) and the \( G_2\)-arcs to \( FAS_\cdot \) and removes the \( FAS_\cdot \) from the \( n \)-rotator graph, and then all directed cycles in the \((R_G = -FAS-1' - G_2\)-arcs\) become disconnected.

For example, \( (24531, 4) \rightarrow 24351 \rightarrow 24531 \rightarrow 24351 \rightarrow 24531 \rightarrow 24351 \rightarrow 24531 \), and all arcs \( g_2 \)'s of nodes 24531 and 24351 are in \( FAS_\cdot \) and removed for \( 2 \leq i \leq 4 \). Thus, the directed cycle in this example is disconnected at these above nodes.

**Theorem 7.** The size of the feedback arc set for an \( n \)-rotator graph generated by Algorithm \( FAS-Rotator-II \) is \((n!6 + \sum_{3 \leq a \leq k} n! / i \cdot i)\), which is minimum.

**Proof.** Let \( v = s_1 s_2 s_3 \ldots s_n \) be a node in an \( n \)-rotator graph. \( |FAS_\cdot| = (n!2 + \sum_{3 \leq a \leq k} n! / i \cdot i)\), where
the $g_2$’s of $FAS_n$ is the $g_2$’s of $v$ for $s_1 < s_2$. Because $\left\{ \left\{ v \right\} \mid s_1 < s_2 \right\} = 1/2$. $\left\{ \forall g_2 \in FAS_n \right\} = n/2$. The $g_2$’s of $FAS_n$ is the $g_2$’s of $v$ for $s_1 < s_2 < s_3$. Because $\left\{ \left\{ v \right\} \mid s_1 < s_2 < s_3 \right\} = 1/6$. $\left\{ \forall g_2 \in FAS_n \right\} = n/6$. Therefore, $\left\{ FAS_n \right\} = (n/6 + \sum_{i=1}^{n} n/i)$. As proof in Theorem 3, there are $n!/6$ arcs disjoint directed cycles of arc $g_i$ in the $n$-rotator graph for $3 \leq i \leq n$. Then, the $n$-rotator graph contains $n!/3$, $n!/4$, ..., and $n!/n$ arcs disjoint directed cycles of length 3, 4, ..., and $n$, respectively. To break these directed cycles, we have to remove at least one arc from each disjoint directed cycle. Thus, we have to remove at least $n!/3 + n!/4 + ... + n!/n = \sum_{i=1}^{n} n/i$ arcs. Subsequently, we discuss the least number of arcs $g_2$’s which have to be removed from the $n$-rotator graph. The smallest directed cycle discussed in theorem 4 is the directed cycle $\pi(v, g_2 g_3 g_2 g_3)$. There are three directed cycles $\pi(v, g_2 g_3 g_2 g_3)$’s in a 3-rotator graph, and they are $C_1 = 123 (2) \rightarrow 312 (3) \rightarrow 123$ and $C_2 = 123 (2) \rightarrow 312 (3) \rightarrow 123, C_3 = 123 (2) \rightarrow 312 (3) \rightarrow 123$ and $C_4 = 123 (2) \rightarrow 321 (3) \rightarrow 123$ and $C_5 = 123 (2) \rightarrow 321 (3) \rightarrow 123$, respectively. All arcs $g_3$ in the three directed cycles $C_1, C_2$ and $C_3$ are disjoint. Furthermore, there are two disjoint directed cycles of arc $g_3$ also in the 3-rotator graph, and they are $C_1 = 123 (2) \rightarrow 312 (3) \rightarrow 123$ and $C_2 = 123 (2) \rightarrow 321 (3) \rightarrow 213$, respectively. We have to remove one arc $g_3$ from each disjoint directed cycle of arc $g_3$ in the 3-rotator graph. Since two arcs $g_3$’s have to be removed from the 3-rotator graph and all arcs $g_3$ in the three directed cycles $C_1, C_2$ and $C_3$ are disjoint, at most two of the three directed cycles $C_1, C_2$ and $C_3$ are disconnected by these two arcs $g_3$’s removed. In other words, at least one of the three directed cycles $C_1, C_2$ and $C_3$ is still connected. For example, the two arcs (123, $g_3$) and (132, $g_3$) are removed from the directed cycles $C_1$ and $C_2$, respectively. Because these two arcs (123, $g_3$) and (132, $g_3$) are also in the directed cycles $C_2$ and $C_3$, respectively, no arc $g_2$ has to be removed from these two directed cycles. Thus, only one arc $g_2$ has to be removed from the directed cycle $C_1$, and at least one arc $g_2$ has to be removed from a 3-rotator graph. There are $n!/6$ disjoint 3-rotator graphs in an $n$-rotator graph, and each disjoint 3-rotator graph is symmetric, so at least $n!/6$ arcs $g_2$’s have to be removed from the $n$-rotator graph. Consequently, the minimum size of the FAS for the $n$-rotator graph is $(n!/6 + \sum_{i=1}^{n} n/i)$, which is equivalent to our result.

**Theorem 8.** For an $n$-rotator graph with $n!$ nodes, Algorithm FAS-Rotator-I / Algorithm FAS-Rotator-II can be done in $O(n!)$ time.

**Proof:** Since there are $n!$ nodes in an $n$-rotator graph and Algorithm FAS-Rotator-I / Algorithm FAS-Rotator-II checks $(n - 1)$ arcs at most for each node, thus the time complexity is $O(n!)$.

4 Conclusions

This paper presents two efficient algorithms for finding a minimum FVS in an $n$-rotator graph in $O(n!)$ time. In other words, these algorithms derive the optimal result with linear time complexity in terms of the number of arcs in the rotator graph.

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**References**


