Acyclic chromatic index of Cartesian product of graphs

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Abstract

A proper edge colouring of a graph \( G \) is said to be acyclic if every cycle of \( G \) receives at least three colors. The acyclic chromatic index, denoted \( a'(G) \), is the least number of colors required for an acyclic edge color of \( G \). This paper shows an upper bound of the acyclic chromatic index of a class of graphs which can be expressed as the Cartesian product of some graphs. We also give exact values for some known classes of graphs like cylinders, tori, \( n \)-dimensional hypercubes and \( n \)-dimensional grids.

1 Introduction

Graphs considered in this paper are undirected, finite, and simple. We use \( \Delta(G) \) to denote the maximum degree of a graph \( G \) throughout the paper. A proper \( k \)–vertex colouring of a graph \( G \) is a labeling \( c : V(G) \rightarrow \{1,2,\ldots,k\} \) such that adjacent vertices in \( G \) have different labels. The chromatic number \( \chi(G) \) is the least \( k \) such that \( G \) has a proper \( k \)–vertex colouring. Analogous to vertex colouring of a graph \( G \), an edge colouring is said to be proper if adjacent edges receive distinct colors. The least number of colors needed for a proper edge colouring of \( G \) is called the chromatic index of \( G \) and denoted \( \chi'(G) \). A proper edge colouring is said to be acyclic if any cycle is colored with at least 3 colors. The acyclic chromatic index, denoted \( a'(G) \), is the least number of colors required for an acyclic edge colouring of \( G \). The concept of acyclic vertex colouring of a graph was introduced by among others. The acyclic chromatic index and its vertex analogue can be used to bound other parameters like oriented chromatic number.

two decades in several works, \([1, 2, 4, 7, 8, 9]\) and star chromatic number of a graph, both of which have many practical applications such as in wavelength routing in optical networks \([5, 14]\).

By Vizing’s theorem, we have \( \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \) (see \([11]\) for proof). Since any acyclic edge colouring is also proper, we have \( \Delta(G) \leq \chi'(G) \leq a'(G) \). An outstanding conjecture was independently posed by Fiamčík \([12]\) and Alon, Sudakov and Zaks \([3]\) that \( a'(G) \leq \Delta(G) + 2 \) for any \( G \). Using probabilistic arguments, Alon, McDiarmid and Reed \([2]\) proved that the edges of any graph \( G \) can be acyclically colored using at most \( 64\Delta(G) \) colors. The best known result up to now for arbitrary graph, was by Molloy and Reed \([16]\) who showed that \( a'(G) \leq 16\Delta(G) \). Alon et al. \([3]\) claimed that the constant 16 can be further improved. Muthu, Narayanan and Subramanian \([17]\) proved that \( a'(G) \leq 4.52\Delta(G) \) for graphs \( G \) of girth at least 220 (Girth is the length of a shortest cycle in a graph). Though the best known upper bound for general case is far from the acyclic edge colouring conjecture (AECC) \( \Delta(G) + 2 \), the conjecture has been shown to be true for some special classes of graphs. Alon, Sudakov, and Zaks \([3]\) proved that there exists a constant \( k \) such that AECC holds for any graph \( G \) whose girth is at least \( k\Delta(G) \log \Delta(G) \).

They also proved that AECC holds for almost all \( \Delta(G) \)–regular graphs. This result has been improved by Nešetřil and Wormald \([21]\) to \( a'(G) \leq \Delta(G) + 1 \) for a random \( \Delta(G) \)–regular graph. Muthu, Narayanan, and Subramanian \([18, 19]\) proved that \( a'(G) \leq \Delta(G) + 1 \) for partial 2-trees, outerplanar graphs, and a partial tori. From Burnštein’s \([10]\) result it follows that the conjecture is true for subcubic graphs. The only known graphs \( G \) for which \( a'(G) > \Delta(G) + 1 \) are the subgraphs of \( K_{2n} \) that have at least \( 2n^2 - 2n + 2 \) edges \([3]\). A graph \( G \) is called

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2-degenerate if each subgraph \( H \) of \( G \) contains a vertex of degree at most 2 in \( H \). Recently, Basavaraju and Sunil Chandran [6] have obtained that if \( G \) is a 2-degenerate graph, then \( a'(G) \leq \Delta(G) + 1 \). In [15], Lai and Lih extended a 2-degenerate graph by attaching a particular kind of cycle to it so that the acyclic edge chromatic number remains bounded by \( \Delta(G) + 1 \). They then applied this extension result to show that \( a'(G) \leq \Delta(G) + 1 \) when \( G \) is a subdivision of a Halin graph.

Determining \( a'(G) \) is a hard problem from both theoretical and algorithmic points of view. Even for the simple and highly structured class of complete graphs, the value of \( a'(G) \) is still not determined exactly. It has also been shown by Alon and Zaks [4] that determining whether \( a'(G) \leq 3 \) is an NP-complete problem for an arbitrary graph \( G \).

There has been very little algorithmic study of acyclic edge colouring except for the following works. Molloy and Reed [16] provided a general framework that can be used to develop algorithms for applications of the Lovász Local Lemma. Customization of this general framework to acyclic colouring led to a polynomial time algorithm to construct \( \beta \)-frugal colouring [16]. They also remarked that this method can be applied to find an acyclic edge colouring of a graph with maximum degree \( \Delta(G) \) using at most \( 20 \Delta(G) \) colors. Subramanian proposed a simple greedy algorithm with polynomial time that uses at most \( 5 \Delta(G)(\log \Delta(G) + 2) \) colors to find an acyclic edge colouring of an arbitrary graph [22]. Burnštěin [10] showed that acyclic vertex chromatic number \( a(G) \) of \( G \) is at most 5 if \( \Delta(G) = 4 \). Since any acyclic vertex colouring of the line graph \( L(G) \) is an acyclic edge colouring of \( G \) and vice-versa, this implies that \( a'(G) = a(L(G)) \leq 5 \) if \( \Delta(G) = 3 \). Alon et al. [3] claimed that they have another proof for this case, which yields a polynomial algorithm to acyclically edge color a subcubic graph by using 5 colors. San Skulrattanakulchai [21] presented a first linear time algorithm to acyclically edge color a subcubic graph by using at most 5 colors.

In view of the discussion relating acyclic edge colouring to perfect 1-factorization conjecture, it may be inferred that finding the exact values of \( a'(K_n) \) for every \( n \) seems hard. However, Alon et al. [2] designed an algorithm that can acyclically edge color \( K_p \). Through this work, they constructively showed that \( a'(K_p) = p \).

This corresponds to the known construction proving that \( K_p \) has a perfect near 1-factorization [25]. They also gave another algorithm that can acyclically edge color a complete bipartite graph \( K_{p-1,p-1} \) and uses \( p \) colors. They further showed that \( a'(K_{p-1,p-1}) = p \). The algorithm that acyclically edge colors a complete graph on prime number of vertices may be used to acyclically edge color a complete graph on an arbitrary number, \( n \), of vertices by taking \( p \) to be the least prime greater than or equal to \( n \). By the known results about the distribution of primes, it may be inferred that the resulting colouring requires \( n + O(n^{\frac{1}{2}}) \) colors [2]. Similarly, the number of colors required for a complete bipartite graph \( K_{p,n} \), \( n \) arbitrary, using the above mentioned algorithm of Alon et al. is \( n + O(n^{\frac{1}{2}}) \) [2]. In [23], a simple linear time algorithm to acyclically edge color \( K_n \), for any \( n \), using at most \( 2n - 3 \) colors was presented.

So far, it is hard to find an article investigating the acyclic edge colouring on graph operations. In this article, we intend to study this problem on a special operation, namely Cartesian product, of graphs.

## 2 Preliminaries

The Cartesian product of graphs \( G \) and \( H \), written \( G \times H \) (or \( G \Box H \)), is a graph with vertex set \( V(G) \times V(H) \) such that \((x, y) \) is adjacent to \((x', y') \) in \( G \times H \) if and only if

- \( x = x' \) with \( yy' \in E(H) \)
- or

- \( y = y' \) with \( xx' \in E(G) \).

For convenience, in case of the Cartesian product, suppose

\[
V(G) = \{x_i \mid 1 \leq i \leq |V(G)|\}
\]

\[
E(G) = \{e_s \mid 1 \leq s \leq |E(G)|\}
\]

and

\[
V(H) = \{y_j \mid 1 \leq j \leq |V(H)|\}
\]

\[
E(H) = \{f_t \mid 1 \leq t \leq |E(H)|\}
\]

We may write \((x_i, y_j) = v_{i,j}\).

Let

\[
R_i = \{v_{i,j} \mid 1 \leq j \leq |V(H)|\}
\]

represent the \( i \)th row (a copy of \( H \)) and

\[
C_j = \{v_{i,j} \mid 1 \leq i \leq |V(G)|\}
\]

the \( j \)th column (a copy of \( G \)).

\( e_{i,j} \) means the edge \( e_i \) in the \( j \)th copy of \( G \) (or the \( j \)th column in \( G \times H \)), and \( f_{i,j} \) means the edge \( f_i \) in the \( j \)th copy of \( H \) (or the \( i \)th row in \( G \times H \)).

In the following, the notation \([n]\) is used to denote the set \([1, 2, 3, \ldots, n]\).
Lemma 1. (König [1916]) If \( G \) is a bipartite graph, then \( \chi'(G) = \Delta(G) \).

Lemma 2. (Brooks [1941]) Let \( G \) be a connected graph. If \( G \) is neither an odd cycle nor a complete graph, then \( \chi(G) \leq \Delta(G) \).

Lemma 3. If a graph \( G \) has a \( \Delta(G) \)-regular connected subgraph with \( \Delta(G) \geq 2 \), then \( a'(G) \) is at least \( \Delta(G) + 1 \).

Proof. Suppose that \( \tilde{G} \) is the \( \Delta(G) \)-regular connected subgraph with \( a'(\tilde{G}) = \Delta(G) \), since in any proper edge colouring of \( \tilde{G} \) with \( \Delta(G) \) colors, for each vertex all colors are used on all edges incident with this vertex. We start from any vertex \( u \) in \( \tilde{G} \), and choose any pair of distinct colors \( c_1 \) and \( c_2 \). Claim that there is a unique cycle in \( \tilde{G} \) going through \( u \) such that the edges in this cycle are colored with \( c_1 \) and \( c_2 \) alternatively. If not, there exists a longest path beginning from \( u \) and ending at \( v \) with this property. W.L.O.G., assume the last edge in the path is colored with \( c_1 \), since all the other edges incident with \( v \) are in \( u,v \)-path, there exists an edge \( vv' \) colored with \( c_2 \) and \( v' \) must be one of the vertices but \( u,v \) in \( u,v \)-path. Whatever \( v' \) be, there are two edges incident with \( v' \) colored with the same color. This leads a contradiction! And hence we know it is impossible to use \( \Delta(G) \) colors to make a proper acyclic edge colouring of \( \tilde{G} \). That is to say, \( a'(\tilde{G}) \geq \Delta(G) + 1 \). By \( a'(\tilde{G}) \geq a'(G) \), we have this lower bound.

Now, we are ready to move on to our consequences.

3 Main Results

Theorem 4. Let \( G \) and \( H \) be two connected graphs, then \( a'(G \times H) \leq a'(G) + a'(H) \) if \( G \times H \neq C_4 \).

Proof. Since \( C_4 = P_2 \times P_2 \), without loss of generality we then assume that \( G \neq P_2 \) and \( \chi(G) \geq \chi(H) \geq 2 \).

Let \( c_G : E(G) \rightarrow [a'(G)] \) be an optimal acyclic edge colouring of \( G \),

\[
c_G : E(H) \rightarrow [a'(H)]
\]

be an optimal acyclic edge colouring of \( H \), and let

\[
c^{(k)}_G(e_i) = \begin{cases} c_G(e_i) + k - 1, & \text{if } c_G(e_i) \leq a'(G) - k + 1 \\ c_G(e_i) + k - a'(G) - 1, & \text{if } c_G(e_i) \geq a'(G) - k + 2 \end{cases}
\]

for \( 1 \leq k \leq a'(G) \), it is trivial that all \( c^{(k)}_G \)'s are also optimal acyclic edge colourings of \( G \).

Now we define

\[
c : E(G \times H) \rightarrow [a'(G) + a'(H)]
\]

to be a function with the following conditions:

1. \( c(f_{i,j}) = c_H(f_j) \), for \( 1 \leq i \leq |V(G)| \),
2. \( f_i \in E(H) \),
3. For each \( v_j \in V(H) \), \( 1 \leq j \leq |V(H)| \), there is a \( c^{(k)}_G(e_i) \), \( k \in [a'(G)] \) such that \( c(e_{i,j}) = c^{(k)}_G(e_i) + a'(H) \),
4. For \( v_{s,p} \in E(H) \), \( 1 \leq s \leq |E(G)| \), \( c(e_{s,p}) \neq c(e_{i,j}) \).

Claim that \( a'(G) \geq \chi(G) \). If \( G \) is neither an odd cycle nor a complete graph, then \( \chi(G) \leq \Delta(G) \leq \chi'(G) \leq a'(G) \) by Lemma 2. If \( G = K_n \), then by Lemma 3 \( a'(G) \geq (n-1) + 1 = n = \chi(G) \). If \( G \) is an odd cycle, then \( a'(G) = 3 = \chi(G) \). In a word, \( a'(G) \geq \chi(G) \). And hence by assumption, \( a'(G) \geq \chi(G) \geq \chi(H) \). Since the condition (3) is equivalent to that \( H \) has a \( a'(G) \)-vertex colouring, by the fact \( \chi(H) \leq a'(G) \), it is clear that there is really such a function \( c \) which satisfies the three conditions. Claim that \( c \) is an acyclic edge colouring of \( G \times H \). Obviously, \( c \) is an edge colouring. Next, we need to show that \( c \) is acyclic. Given a cycle \( \mathcal{C} \) in \( G \times H \), if all edges of \( \mathcal{C} \) are in one copy of \( G \) or \( H \) of \( G \times H \), then by definition of \( c \), the cycle \( \mathcal{C} \) is colored with at least 3 colors. If it is not the case above, then some of \( \mathcal{C} \) are in at least two copies of \( G \) of \( G \times H \) and the others are in at least two copies of \( H \) of \( G \times H \).

Let \( G_j (1 \leq j \leq |V(H)|) \) and \( H_j (1 \leq i \leq |V(G)|) \) be the copies of \( G \) and \( H \) of \( G \times H \), respectively.

We use the following notations:

Let \( E_i^k (\mathcal{C}) = E(\mathcal{C}) \cap E(H_i) \).

Let \( E^i_j (\mathcal{C}) = E(\mathcal{C}) \cap E(H_j) \).

Let \( P^k_i (\mathcal{C}) = \{ f_i \mid j \in \mathcal{C}, f_{i,j} \in E^i_j (\mathcal{C}) \} \).

Let \( P^i_j (\mathcal{C}) = \{ e_{i,j} \mid j \in \mathcal{C}, e_{i,j} \in E^i_j (\mathcal{C}) \} \).

Then, we consider the following three cases.
**Case 1:** $P^h(\mathcal{E})$ forms a cycle in $H$.

In this case, as the cycle $P^h(\mathcal{E})$ is colored by $c_H$ with at least 3 colors, by the condition (1) and (2) in the definition of $\mathcal{E}$, the cycle $\mathcal{E}$ is colored with at least 4 colors.

**Case 2:** $P^h(\mathcal{E})$ forms a forest but $P_2$ in $H$.

In this case, as $P^h(\mathcal{E})$ is colored by $c_H$ with at least 2 colors, by the condition (2) in the definition of $\mathcal{E}$, the cycle $\mathcal{E}$ is colored with at least 3 colors.

**Case 3:** $P^h(\mathcal{E})$ has just a $P_2$ in $H$.

**Case 3-1:** $P^h(\mathcal{E})$ forms a cycle or a forest (In fact, it only could possibly be a path) but $P_2$ in $G$.

In this subcase, as there are two incident edges in $\mathcal{E}$ such that they are in the same copy of $G$, they are colored by some $c^1_{(o)}$ with 2 colors. By the condition (1) and (2) in the definition of $\mathcal{E}$, the cycle $\mathcal{E}$ is colored with at least 3 colors. Therefore $\mathcal{E}$ is an acyclic edge colouring of $G \times H$, and the theorem then follows.

**Case 3-2:** $P^h(\mathcal{E})$ has just a $P_2$ in $G$.

In this subcase, $\mathcal{E}$ is a 4-cycle. Two non-incident edges of it are in distinct copies of $G$, they are colored with 2 colors by the condition (3) in the definition of $\mathcal{E}$. And the other two non-incident edges of it are in distinct copies of $H$, they are colored with the same color by the condition (1) in the definition of $\mathcal{E}$. Those together imply that the cycle $\mathcal{E}$ is colored with 3 colors.

**Theorem 6.** Let $T_i, i \in [n]$ be trees and there are at least one of them with maximum degree not less than 2, then

$$a'(T_1 \times T_2 \times \cdots \times T_n) = \sum_{i=1}^{n} \Delta(T_i).$$

**Proof.** Because

$$\Delta(T_1 \times T_2 \times \cdots \times T_n) = \sum_{i=1}^{n} \Delta(T_i),$$

trivially

$$a'(T_1 \times T_2 \cdots \times T_n) \geq \sum_{i=1}^{n} \Delta(T_i).$$

Since the operation $\times$ is commutative, we may suppose that $T_1$ is the tree with maximum degree not less than 2. As $a'(T_1) = \Delta(T_1) \geq 2$, by Corollary 5 we get

$$a'(T_1 \times T_2) \leq a'(T_1) + \Delta(T_2).$$

By induction on $n$, since

$$a'(T_1 \times T_2 \cdots \times T_n) \geq \sum_{i=1}^{n} \Delta(T_i) \geq 2,$$

applying Corollary 5 again,

$$a'(T_1 \times T_2 \cdots \times T_n) \leq \sum_{i=1}^{n} \Delta(T_i)$$

can be obtained. Combining the lower bound and the upper bound, it implies the equality.

$P_{i_1} \times P_{i_2} \times \cdots \times P_{i_k}$ is called a hypergrid (or $n$—dimensional grid). From Theorem 6, we may easily conclude the following corollary.

**Corollary 7.** $a'(P_{i_1} \times P_{i_2} \times \cdots \times P_{i_k}) = 2n$ if $i_k \geq 3$ for $1 \leq k \leq n$. That is to say, under this condition, the acyclic chromatic index of a $n$—dimensional grid must be $2n$.

With regard to $\Delta(T_i) = 1, i \in [n]$, that is $T_i = P_2$. What is $a'(T_1 \times T_2 \cdots T_n)$? The answer is in the following corollary. We use $Q_n$ to denote the $n$—fold Cartesian product of $P_2$ with itself and call it the $n$—dimensional hypercube.

**Corollary 8.** $a'(Q_n) = n+1$ for $n \geq 2$.

**Proof.** Because $Q_n$ is $n$—regular, $a'(Q_n) \geq n+1$ by Lemma 3. We have known $a'(Q_n) = a'(C_4) = 3 \leq 2+1$, using hypothesis $a'(Q_{n-1}) \leq (n-1)+1 = n$ and by induction on $n$, it implies $a'(Q_n) = a'(Q_{n-1} \times P_2) \leq a'(Q_{n-1}) + 1 \leq n + 1$ from Theorem 4.

By Theorem 4, it should imply $a'(G \times C_n) \leq a'(G) + 3$. In fact, we can lower its upper bound to $a'(G) + 2$ under some conditions.
Theorem 9. \( a'(G \times C_n) \leq a'(G) + 2 \) if \( a'(G) \geq 4 \).

Proof. W.L.O.G., we assume that
\[
V(C_n) = \{y_j | j \in [n]\}
\]
and
\[
E(C_n) = \{(y_j, y_j') | j = 1, i, j \in [n]\},
\]
where \(|i - j| = \min(|i - j|, n - |i - j|)\).
Suppose \( c_G : E(G) \rightarrow [a'(G)] \) is an optimal acyclic edge colouring of \( G \), and let
\[
c_{C_n} : E(C_n) \rightarrow [a'(C_n)]
\]
be an optimal acyclic edge colouring of \( C_n \), which is defined by
\[
c_{C_n}(f_j) = \begin{cases} 
\frac{3 + (-1)^j}{2}, & \text{for } j \in [n - 1], \\
3, & \text{for } j = n
\end{cases},
\]
where \( f_j = y_j, y_{j+1}, j \in [n] \). Besides, we let
\[
c^{(0)}_G(e_s) = \begin{cases} 
0, & \text{if } c_G(e_s) = 1 \\
2, & \text{if } c_G(e_s) = 2
\end{cases},
\]
\[
c^{(1)}_G(e_s) = \begin{cases} 
\frac{c_G(e_s) + 7 + (-1)^{j-1}}{2}, & \text{if } c_G(e_s) \leq a'(G) - \frac{3 + (-1)^{j-1}}{2}, \\
\frac{c_G(e_s) - a'(G) + \frac{7 + (-1)^{j-1}}{2}}{2}, & \text{if } c_G(e_s) \geq a'(G) - \frac{1 + (-1)^{j-1}}{2}
\end{cases},
\]
for \( 2 \leq j \leq n - 1 \),
\[
c^{(n)}_G(e_s) = \begin{cases} 
\frac{c_G(e_s) + 5}{2}, & \text{if } c_G(e_s) \leq a'(G) - 3 \\
\frac{3 + (-1)^n}{2}, & \text{if } c_G(e_s) = a'(G) - 2 \\
c_G(e_s) - a'(G) + 5, & \text{if otherwise (o.w.)}
\end{cases},
\]
Clearly, they all are also optimal acyclic edge colourings of \( G \).

Now we define \( c : E(G \times C_n) \rightarrow [a'(G) + 2] \) by
\[
c(f_{i,s}) = c_{C_n}(f_j), \text{ for } i \in [V(G)], j \in [n];
\]
and
\[
c(e_{s,j}) = c_{C_n}^{(j)}(e_s), \text{ for } s \in [E(G)], j \in [n].
\]

Trivially, \( c \) is a proper colouring of \( G \times C_n \).

Next, we show that \( c \) is acyclic. Given a cycle \( \mathcal{E} \) in \( G \times C_n \). If all edges of \( \mathcal{E} \) are in one copy of \( G \) or \( C_n \) of \( G \times C_n \), by definition of \( c \), the cycle \( \mathcal{E} \) is colored with at least 3 colors. If it is not the case above, then some of \( \mathcal{E} \) are in at least two copies of \( G \) of \( G \times C_n \) and the others are in at least two copies of \( C_n \) of \( G \times C_n \).

Let \( G_j (1 \leq j \leq \lceil V(C_n) \rceil) \) and \( C_n(i)(1 \leq i \leq V(G)) \) be the copies of \( G \) and \( C_n \) of \( G \times C_n \), respectively.

We also use the same notations in Theorem 4:
Let \( E_i^h(\mathcal{E}) = E(\mathcal{E}) \cap E(C_n(i)) \).
Let \( E_i^j(\mathcal{E}) = E(\mathcal{E}) \cap E(G_j) \).
Let \( P^h(\mathcal{E}) = \{f_j | j \in [n], f_{i,j} \in E_i^h(\mathcal{E})\} \).
Let \( P^v(\mathcal{E}) = \{e_s | j \in [n], e_s,j \in E_i^j(\mathcal{E})\} \).

Then, we discuss the following cases.

Case 1: \( P^h(\mathcal{E}) \) is just \( C_n \).
In this case, from the definition of \( c \), there is no doubt that the cycle \( \mathcal{E} \) is colored with at least 3 colors.

Case 2: \( P^h(\mathcal{E}) \) forms a path in \( C_n \).

Case 2-1: \( y_1 y_n \in P^h(\mathcal{E}) \)
If the length of this path is at least 3, trivially the cycle \( \mathcal{E} \) is colored with at least 3 colors. If the length of this path is just 2, then the path is \( y_1 y_{n-1} \) or \( y_2 y_n \). In the former case, since the range of colors of \( c \) on \( G_i \) is \( [a'(G) + 2] - [1,3] \) and \( E(\mathcal{E}) \cap E(G_i) \neq \emptyset \), the cycle \( \mathcal{E} \) is colored with at least 3 colors. In the latter case, since the range of colors of \( c \) on \( G_{n-i} \) is \( [a'(G) + 2] - [1,3] \) (or \( [a'(G) + 2] - [2,3] \)) and \( E(\mathcal{E}) \cap E(G_{n-i}) \neq \emptyset \), the cycle \( \mathcal{E} \) is also colored with at least 3 colors. If the length of this path is only 1, then \( E(\mathcal{E}) \cap E_j^j(\mathcal{E}) \neq \emptyset \) for \( 2 \leq j \leq n - 1 \). If either there exist incident edges \( e_{a,b}, e_{b,a} \) in \( E_i^j(\mathcal{E}) \) or \( e_{a,b}, e_{b,a} \) in \( E_j^j(\mathcal{E}) \) or there exist \( e_{a,b} \) in \( E_i^j(\mathcal{E}) \) and \( e_{a,b} \) in \( E_j^j(\mathcal{E}) \), then we can derive at least two other colors by \( c \) from the definition of it. And so the cycle \( \mathcal{E} \) is colored with at least 3 colors. Otherwise, the edges in \( E_i^j(\mathcal{E}) \) and edges in \( E_j^j(\mathcal{E}) \) are mutually not incident, respectively, and then \( E^v(\mathcal{E}) \) forms a cycle. If the edges in \( E_i^v(\mathcal{E}) \cup E_j^v(\mathcal{E}) \) are colored with the same
color, then $P^r(\mathcal{C})$ is forced to be a 2-colored cycle in $G$ from the definition of $c$, $c_G^{(1)}$ and $c_G^{(n)}$. This contradicts to the assumption that $c$ is an acyclic edge colouring. Thus we confirm that the cycle $\mathcal{C}$ is colored with at least 3 colors in this case.

**Case 2-2:** $y_jy_n \not\in P^b(\mathcal{C})$

If the length of this path is at least 2 and there is a nonempty $E_v^j(\mathcal{C})$ for $2 \leq j \leq n-1$, then there exist edges colored with other colors which are different from color 1 and color 2. If $E_v^j(\mathcal{C})$ is empty for $2 \leq j \leq n-1$, then this path is of the length $n-1$. If either there exist incident edges $e_{a,1}, e_{b,1}$ in $E_v^1(\mathcal{C})$ or $e_{a,n}, e_{b,n}$ in $E_v^n(\mathcal{C})$, or there exist $e_{a,1}$ in $E_v^1(\mathcal{C})$ and $e_{a,n}$ in $E_v^n(\mathcal{C})$, then we can derive other colors by $c$ from the definition of $c_G^{(1)}$ and $c_G^{(n)}$. Therefore the cycle $\mathcal{C}$ is colored with at least 3 colors. Otherwise, the edges in $E_v^1(\mathcal{C})$ and edges in $E_v^n(\mathcal{C})$ are mutually not incident, respectively, and then $P^r(\mathcal{C})$ forms a cycle. If the edges in $E_v^1(\mathcal{C})$ are colored with the same color and so are the edges in $E_v^n(\mathcal{C})$, then $P^r(\mathcal{C})$ is forced to be an at most 2-colored cycle in $G$ from the definition of $c$, $c_G^{(1)}$ and $c_G^{(n)}$. This contradicts to the assumption that $c$ is an acyclic edge colouring. Hence in this case we ensure that the cycle $\mathcal{C}$ is colored with at least 3 colors. All the remains we need to consider is the case that the length of this path is only 1. The argument in this case is similar to the above. We omit it.

Therefore $c$ is an acyclic edge colouring of $G \times C_n$, and we then complete the proof. ■

In fact, combining the similar criterion $c_G^{(k)}$ for $1 \leq k \leq a'(G)$ in Theorem 4 and the similar criterion $c_G^{(1)}$ and $c_G^{(n)}$ in Theorem 9, we can the argument in the above theorem to imitate justify a better result than Theorem 9 in the following.

**Theorem 10.** $a'(G \times C_n) \leq a'(G) + 2$ if $a'(G) \geq 3$.

Applying the above theorem, since $a'(T) = \Delta(T)$ for a tree $T$, we have immediately the following corollary.

**Corollary 11.** Let $T$ be a tree with $\Delta(T) \geq 3$, then $a'(T \times C_n) = \Delta(T) + 2$.

In the following, we consider two examples of $G \times C_n$ with $a'(G) \leq 3$. The first we consider is $P_n \times C_n$ which is so-called a cylinder and show its acyclic chromatic index as follow.

**Theorem 12.** $a'(P_n \times C_n) = 4$ if $n$ is even.

**Proof.** As $\Delta(P_n \times C_n) = 4$, $a'(P_n \times C_n) = 4$ is obvious. If we could give a proper acyclic edge colouring of $P_n \times C_n$, then it was done. W.L.O.G, we assume that $V(P_n) = \{x_i | i \in [m]\}$ and $E(P_n) = \{(x_i-x_j) | i,j \in [m]\}$.

Similarly, we assume that $V(C_n) = \{y_i | i \in [n]\}$ and $E(C_n) = \{(y_i-y_j) | i,j \in [n]\}$, where $|i-j| = \min\{|i-j|, n-|i-j|\}$.

Now, define $c$ to be a colouring of $P_n \times C_n$ by

$$c(y_i, y_{i+1}) = \begin{cases} \frac{3+(-1)^i}{2}, & \text{for } i \in [m], j \in [n-1] \\ 3, & \text{for } i \in [m], j = n \end{cases}$$

and

$$c(x_i, x_{i+1}) = \begin{cases} \frac{7+(-1)^i}{2}, & \text{for } i \in [m-1], j \in [n-1]-|i| \\ \frac{3+(-1)^i}{2}, & \text{for } i \in [m-1], j \in [1, n] \end{cases}$$

It is not hard to check that there is no 2-colored cycle, and hence $c$ is a proper acyclic edge colouring of $P_n \times C_n$ with the range $\{1,2,3,4\}$. We omit the detailed proof here. Then we have this result. ■

See Fig. 1 for an optimal colouring of $P_7 \times C_6$.

![Fig. 1. An optimal colouring of $P_7 \times C_6$.](image-url)
Theorem 13. \( a'(C_m \times C_n) = 5 \).

Proof. For the reason that \( C_m \times C_n \) is 4-regular graph, \( a'(C_m \times C_n) \geq 5 \) by Lemma 3. Next, we give a proper acyclic edge colouring of \( C_m \times C_n \) to obtain the upper bound 5.

W.L.O.G., we assume that
\[
V(C_m) = \{x, y \in [m]\} \quad \text{and} \quad E(C_m) = \{(x, y) \mid y - j \in [m]\},
\]
where \( |y - j| = \min \{y - j, n - |y - j|\} \).

Similarly, we assume for \( C_n \).

Now, according to the conditions on \( m, n \), define \( c \) to be a colouring of \( C_m \times C_n \) in the following cases.

Case 1: At least one of \( m, n \) is even. (W.L.O.G., let \( n \) be even.)

\[
c(v_{i,j}, v_{i+1,j}) = \begin{cases} 
17 + (-1)^i + (-1)^j - 3(-1)^{i+j} & \text{for } i \in [m-1], j \in [n-1] - \{1\} \\
5 + (-1)^i & \text{for } i \in [m-1], j = 1 \\
2 & \text{for } i \in [m-1], j = n \\
3 + (-1)^{i+1} & \text{for } i \in [m], j = n \\
7 + (-1)^{i+1} & \text{for } i = m, j \in [n] \\
\end{cases}
\]

and

\[
c(v_{i,j}, v_{i,j+1}) = \begin{cases} 
3 + (-1)^i & \text{for } i \in [m], j \in [n-1] \\
5, & \text{for } i \in [m], j = n \\
\end{cases}
\]

Case 2: Both of \( m, n \) are odd.

\[
c(v_{i,j}, v_{i+1,j}) = \begin{cases} 
7 + (-1)^{i+j} & \text{for } i \in [m-1], j \in [n-1] \\
2, & \text{for } i \in [m-2], j = n \\
7 + (-1)^{i+1} & \text{for } i \in [m-1], j = n \\
2, & \text{for } i = m, j \in [n] \\
\end{cases}
\]

It is not hard to check that there is no 2-colored cycle, and then \( c \) is a proper acyclic edge colouring of \( C_m \times C_n \) with the range \([1, 2, 3, 4, 5]\). We omit the detailed proof here. Hence this proof is completed. \( \blacksquare \)

See Fig. 2 for an optimal colouring of \( C_7 \times C_6 \) and see Fig. 3 for an optimal colouring of \( C_9 \times C_7 \).
4 Conclusions and future work

This article shows that $a'(G \times H)$ is bounded above by $a'(G) + a'(H)$ for $G \times H \neq C_4$. Utilizing this theorem, we can easily derive the exact acyclic chromatic index of the Cartesian product of several trees if at least one of them with maximum degree not less than 2 and the acyclic chromatic index of hypercubes, respectively. Moreover, we obtain the exact acyclic chromatic indices of hypergrids, cylinders and tori, respectively. Each of these exact indices above is $\Delta(G) + 1$. Thus we verify that AECC holds for these special classes of graphs in this paper.

In the future, we will try to change the condition of Theorem 4 to get better upper bound or try to depict the relation of the graphs $G$ and $H$ which satisfy $a'(G \times H) = a'(G) + a'(H)$. We regret that we can not give the result for the acyclic chromatic index of cylinders when $C_n$ is an odd cycle in this paper, hope to finish it soon. We are also interested to find the exact acyclic chromatic indices of graphs such as hypercubes, cylinders and tori, etc. Additionally, we think the density of regularity of a graph $G$ should play an important role on determining the lower bound of $a'(G)$. We will make efforts to confirm it.

References