An Upper Bound on F-domination Number in Grids †

Fu–Hsing Wang
Department of Information Management
Chinese Culture University, Taipei, Taiwan
cysir@pchome.com.tw

Abstract

A graph G is 2-stratified if its vertex set is colored into two nonempty classes, where one class of vertices colored red and the other color class blue. Let F be a 2-stratified graph rooted at one fixed blue vertex v. The F-domination number of G is the minimum number of red vertices of G in a red-blue coloring of the vertices of G such that for every blue vertex v of G, there is a copy of F in G rooted at v. In this paper, we explore an upper bound on the F-domination number for a specified 2-stratified graphs F in grids, where F is a path P_3 rooted at a blue vertex that is adjacent to a blue vertex and with the remaining vertex colored red. As far as we know, no such an F-domination number was known for grids.

1 Introduction

A graph is *2-stratified* if its vertex set is colored into two nonempty classes, where one class of vertices colored red and the other color class blue. Let F be a 2-stratified graph rooted at one fixed blue vertex v. An F-coloring of a graph G is defined to be a red-blue coloring of the vertices of G such that every blue vertex v of G belongs to a copy of F rooted at v. Each blue vertex in an F-coloring is *F-dominated* by a red vertex. The *F-domination number* $\gamma_F(G)$ of G is the minimum number of red vertices of G in a red-blue coloring of the vertices of G such that for every blue vertex v of G, there is a copy of F in G rooted at v and the set of red vertices is called *F-dominating set*. In [3], an F-coloring of G that colors $\gamma_F(G)$ vertices red is called a γ_F-coloring of G and the F-dominating set is called γ_F-dominating set. If G has order n and G has no copy of F, then each vertex is certainly red and $\gamma_F(G) = n$. Figure 1 depicts a 2-stratified graph F and a γ_F-coloring of a graph G that colors $\gamma_F(G) = 6$ vertices red, where the red vertices are darkened.

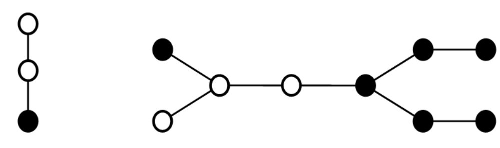

Figure 1: (a) A 2-stratified graph F (b) A γ_F-coloring of a graph G with $\gamma_F(G) = 6$.

If F is a path P_2 rooted at a blue vertex that is adjacent to a red vertex, then the F-dominating set of G is indeed a minimum dominating set of G. A *dominating set* of G is a set $S \subseteq V(G)$, where every vertex not in S is adjacent to a vertex in S. When F is a 2-stratified path P_3 on three vertices rooted at a blue vertex, the five possible choices, F_i, for $i = 1, 2, \ldots, 5$, for the graph F are shown in Figure 2, where F_3 is a 2-stratified path P_3 rooted at a blue vertex that is adjacent to a red vertex and with the remaining vertex colored red. The red vertices in Figure 2 are darkened. Chartrand et al. showed that for $i \in \{1, 2, 4, 5\}$, the parameters $\gamma_{F_i}(G)$ are well known domination type parameters [3]. In [7], Henning and Martiz showed the results for paths and trees. In this paper, we discuss $\gamma_{F_3}(G)$ for grids. For convenience, we use $\gamma_F(G)$ to represent $\gamma_{F_3}(G)$ in the remaining discussion.

Stratified domination was first introduced by Chartrand et al. in 2003 [3]. In [8], Rashidi studied a number of problems involving stratified graphs; while distance in stratified graphs was investigated in [1, 2, 4]. The most studied types of domination in graphs can be defined in terms of an appropriately chosen rooted 2-stratified graph. The study of stratified domination as a graph-

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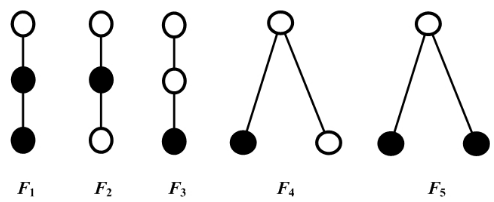

Figure 2: The five 2-stratified graphs P_3.

theoretic concept has recently attracted a great deal of attention [6]. By definition, stratified domination intuitively combines stratification and domination.

We consider the problem for k-dimensional grid networks. The topological structure of a k-dimensional grid $G_{m_1,m_2,\ldots,m_k}$ is defined as the Cartesian product $P_{m_1} \times P_{m_2} \times \cdots \times P_{m_k}$ of k paths. A 2-dimensional grid is also called a mesh. The grid networks are an important class of topological structures of interconnection networks which suited for parallel processing of data. In this paper, we first present an upper bound for F-domination number on meshes, where F is a 2-stratified graph path P_3 and actually the algorithm is also applicable to solve the problem on k-dimensional grids, where $k \geq 3$.

The remaining part of this paper is organized as follows. In Section 2, we give the definition of grids and introduce some basic terminology and notation. We also demonstrate previous results which are helpful to determine the F-domination number on grids. In Section 3, an upper bound on the F-domination number for n-dimensional grids is proposed. Finally, some concluding remarks and future research are given in the last section.

2 Preliminaries and Previous Results

All graphs considered in this paper are finite and simple (i.e., without loops and multiple edges). Let $G = (V, E)$ be a graph with vertex set V and edge set E, where $E \subseteq V \times V$. For any set $S \subseteq V$, the *induced subgraph* of $< S >$ is the maximal subgraph with vertices set S.

A k-dimensional grid $G_{m_1,m_2,\ldots,m_k}$ has vertex set $V(G_{m_1,m_2,\ldots,m_k}) = \{\,(a_1, a_2, \ldots, a_k) \,|\, 0 \leq a_i \leq m_i - 1, 1 \leq i \leq k\}$ and vertices $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ are connected by an edge if

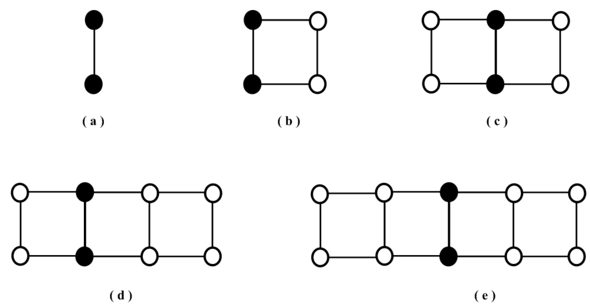

Figure 3: Grids $G_{2,1}, G_{2,2}, G_{2,3}, G_{2,4}$ and $G_{2,5}$. The red vertices are darkened.

and only if $\sum_{1 \leq i \leq k} |a_i - b_i| = 1$. We label the vertex v of a mesh as (x_v, y_v), where x_v and y_v is the row number and column number, respectively. The set of vertices with the same row number x (respectively, column number y) is called *row x* (respectively, *column y*).

The following previous results are helpful to clarify our proofs.

Lemma 1. [5] *$\gamma_F(G) = 1$ if and only if G contains a vertex u such that $N(u)$ is a total dominating set of G.*

Lemma 2. [7] *For $n \geq 1$, $\gamma_F(P_n) = \lfloor \frac{n+7}{3} \rfloor + \lfloor \frac{n}{3} \rfloor - \lceil \frac{n}{3} \rceil$.*

3 Our Results

In a k-dimensional grid G, each vertex of G has an open neighborhood consisting of pairwisely nonadjacent vertices. So, by Lemma 1, $\gamma_F(G) \geq 2$. In the following discussion, we assume $\gamma_F(G) \geq 2$ for a grid G.

Theorem 3. *$\gamma_F(G_{2,n}) = \lceil \frac{n}{5} \rceil \cdot 2$, for $n \geq 1$.*

Proof. Figure 3 shows F-colorings for $G_{2,n}$, where $1 \leq i \leq 5$, that colors two vertices red. Consider $G_{2,n}$, where $i \geq 6$. Every vertex in $G_{2,n}$ has at most four neighbors of distance exactly two. This implies that each red vertex belongs to at most four copies of F in $G_{2,n}$. So, $\gamma_F(G_{2,n}) \geq \lceil \frac{2n}{5} \rceil$. We next proceed by induction on the order n of a mesh $G_{2,n}$. The theorem holds for $n \leq 5$. Assume then that the result is true for $n < k$ and $k \geq 6$. When $n = k$. We show first that there exists a γ_F-coloring of $G_{2,k}$, denoted $\mathcal{F}$, that colors vertices $(0, 2)$ and $(1, 2)$ red, while the columns 0,1,3 and 4 be colored blue. The region R_1 induced by columns 0,1 and 2 is indeed a $G_{2,3}$ and then, by basis, we need at least 2 vertices of R_1 to be colored red. Let S be the subgraph induced by columns 0 to 4. S is indeed a $G_{2,5}$. Furthermore, since $\gamma_F(G_{2,5}) = 2$, we assume that there are exactly two red vertices in S. It is clear that each of the two red vertices of S must be on four copies of F in S. Therefore $\mathcal{F}$ is the desire γ_F-coloring of $G_{2,k}$.

Let G' be the subgraph induced by columns 5 to k and $\mathcal{F}'$ be a γ_F-coloring of S'. Note that G' is indeed a grid $G_{2,k-5}$. Then the restriction of $\mathcal{F}'$ to the grid G' is an γ_F-coloring of G' that colors $\gamma_F(G_{2,k}) - 2$ vertices red. Hence, $\gamma_F(G') \leq \gamma_F(G_{2,k}) - 2$. On the other hand, any γ_F-coloring of G' can be extended to an γ_F-coloring of $G_{2,k}$. Thus, $\gamma_F(G_{2,k}) \leq \gamma_F(G') + 2$. Consequently, $\gamma_F(G_{2,k}) = \gamma_F(G') + 2$. The result follows by applying the inductive hypothesis to G'. □

For meshes $G_{3,n}$, where $3 \leq n \leq 5$, we use Figure 4 to indicate a F-dominating set of $G_{3,n}$ and get the following result.

Lemma 4. *$\gamma_F(G_{3,n}) = 3$, for each $n = 1, 2, \ldots, 5$.*

Proof. Figure 4 gives F-colorings of $G_{3,n}$, where $1 \leq n \leq 5$, that colors three vertices red. Thus, an upper bound $\gamma_F(G_{3,n}) \leq 3$ is established. We only need to find lower bounds to $\gamma_F(G_{3,n})$. The result follows from Lemma 2 and Theorem 3 for $n = 1$ and 2, respectively. We immediately consider $n \geq 3$. We first show that $\gamma_F(G_{3,3}) \geq 3$. Suppose that $\gamma_F(G_{3,3}) \leq 2$. Consider the vertex $(1,1)$. If we color $(1,1)$ red in any γ_F-coloring of $G_{3,3}$, where $\gamma_F(G_{3,3}) = 2$, then let, by vertex symmetry of $G_{3,3}$, $(1,0)$ be colored blue as $\gamma_F(G_{3,3}) \leq 2$ and $(1,1)$ has four open neighbors. Then, either $(0,1)$ or $(2,1)$ is colored red to F-dominate $(1,0)$. By vertex symmetry property of $G_{3,3}$, we let $(0,1)$ be such a vertex that F-dominate $(1,0)$. It can be seen that $(2,1)$ is then colored blue and not F-dominated by any red vertex. Therefore, $(1,1)$ should be colored blue and hence each red vertex belongs to at most three copies of F. It means $\gamma_F(G_{3,3}) \geq \lceil \frac{9}{4} \rceil = 3$. Furthermore, $\gamma_F(G_{3,3}) = 3$ because $\{(0,1),(1,1),(2,1)\}$ is clearly a γ_F-dominating set of $G_{3,3}$. Next, we show $\gamma_F(G_{3,n}) = 3$ for $n = 4, 5$. Since the subgraph induced by columns 0,1 and 2 of $G_{3,n}$, where $n = 4$ or 5 is indeed a $G_{3,3}$, $\gamma_F(G_{3,n}) \geq 3$. Actually, $\gamma_F(G_{3,n}) = 3$ because we have a γ_F-dominating set $\{(0,2),(1,2),(2,2)\}$. □

Lemma 5. *$\gamma_F(G_{4,n}) = 4$, for each $n = 4, 5$.*

Proof. Suppose, to the contrary, that $\gamma_F(G_{4,n}) \leq 3$. Let region $R_1 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$. We first claim that any γ_F-coloring colors at least one vertex in R_1 red. If all vertices of R_1 are colored blue, then $(2,0)$ must be colored red to F-dominate $(0,0)$. Furthermore, $(2,1)$ is the only one red vertex to F-dominate $(1,0)$. We now consider the vertices $(0,n-1)$ and $(1,n-1)$. The two vertices are both not on any copy of F containing the red vertices $(2,0)$ or $(2,1)$. Thus, we need another red vertex to F-dominate $(0,n-1)$ and $(1,n-1)$. In fact, there is no red vertex belongs to both a copy of F rooted at $(0,n-1)$ and a copy of F rooted at $(1,n-1)$. So, R_1 contains at least one red vertex. Actually, R_1 contains exactly one red vertex. Otherwise, $\gamma_F(G_{4,n}) \geq 4$, a contradiction. Moreover, we now show that the region R_2 consisting of rows 0 and 1 has at least two red vertices. When $n = 4$. Suppose R_2 has exactly one red vertex v. The four possible choices of v are $(0,1),(0,2),(1,1)$ and $(1,2)$ to reveal the fact that R_1 has exactly one red vertex.

Case 1: $v = (0,1)$.

Clearly, vertices $(0,0)$ and $(0,2)$ can only be F-dominated by $(2,0)$ and $(2,2)$, respectively. And we now have three red vertices. However, vertex $(3,0)$ is not F-dominated by any red vertex. This contradicts the fact that $\gamma_F(G_{4,n}) \leq 3$.

Case 2: $v = (0,2)$.

By vertex symmetry property of grids, the case is similar to *Case* 1 and we omit it.

Case 3: $v = (1,1)$.

Vertex $(0,1)$ can not be F-dominated by any red vertex of $G_{4,n}$, a contradiction.

Case 4: $v = (1,2)$.

The proof of this case is similar to *Case* 3.

Therefore, we get $\gamma_F(G_{4,n}) \geq 4$. Since $\{(0,2),(1,2),(2,2),(3,2\}$ is a γ_F-dominating set of $G_{4,4}$, $\gamma_F(G_{4,4}) = 4$. To determine $\gamma_F(G_{4,5})$, we find that the subgraph induced by the columns 0,1,2 and 3 is indeed a $G_{4,4}$. So, $\gamma_F(G_{4,5}) \geq 4$. Moreover, $\gamma_F(G_{4,n}) = 4$ by the reason that

$\{(0,2),(1,2),(2,2),(3,2\}$ is a γ_F-dominating set of $G_{4,5}$. □

Lemma 6. $\gamma_F(G_{5,5}) = 4$.

Proof. Suppose, to the contrary, that $\gamma_F(G_{5,5}) \leq 3$. Let region $R_1 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$. We first claim that any γ_F-coloring colors at least one vertex in R_1 red. If all vertices of R_1 are colored blue, then $(2,0)$ must be colored red to F-dominate $(0,0)$. Furthermore, $(2,1)$ is then the only one red vertex to F-dominate $(1,0)$. However, the vertices of column $n-1$ are not on any copy of F containing the red vertices $(2,0)$ or $(2,1)$. Thus, we need another red vertex to F-dominate $(0,n-1)$ and $(1,n-1)$. In fact, there is no red vertex belongs to both a copy of F rooted at $(0,n-1)$ and a copy of F rooted at $(1,n-1)$. So, R_1 contains at least one red vertex. Actually, R_1 contains exactly one red vertex. Otherwise, $\gamma_F(G_{5,5}) \geq 4$, a contradiction. Moreover, we now show that the region R_2 consisting of rows 0 and 1 has at least two red vertices. Suppose R_2 has exactly one red vertex v. Then v is either $(0,2)$ or $(1,2)$ to reveal the fact that R_1 has exactly one red vertex.

Case 1: $v = (0,2)$.

Clearly, vertices $(0,1)$ and $(0,3)$ can only be F-dominated by $(2,1)$ and $(2,3)$, respectively. And we now have three red vertices. However, vertices $(3,1)$ and $(3,3)$ are not F-dominated by any red vertex. This contradicts the fact that $\gamma_F(G_{5,5}) \leq 3$.

Case 2: $v = (1,2)$.

Vertex $(0,2)$ can not be F-dominated by any vertex of $G_{5,5}$, a contradiction.

We conclude that $\gamma_F(G_{5,5}) = 4$ by the fact that $\{(0,2),(1,2),(2,2),(3,2)\}$ is a γ_F-dominating set of $G_{5,5}$. □

Theorem 7. *Let* $m_i = 5 \cdot q_i + r_i$, *where* $0 \leq r_i \leq 4$. $\gamma_F(G_{m_1,m_2,\ldots,m_n}) \leq \prod_{i=1}^{n-1} m_i \cdot q_n + \sum_{i=2}^{n-1}(\prod_{j=1}^{n-i} m_j \cdot q_{n-i+1} \cdot \prod_{j=n-i+2}^{n} r_j) + q_1 \cdot \prod_{j=2}^{n} r_j + \gamma_F(G_{r_1,r_2,\ldots,r_n})$.

Proof. If we regard $G_{m_1,m_2,\ldots,m_n}$ as a tiling of subgrids with possibly different size, then $G_{m_1,m_2,\ldots,m_n}$ is a tiling of $(5 \cdot q_i + r_i)$ grids $G_{m_1,m_2,\ldots,m_{i-1},1,m_{i+1},\ldots,m_n}$. So, we get

$\gamma_F(G_{m_1,m_2,\ldots,m_n}) \leq q_n \cdot \gamma_F(G_{m_1,m_2,\ldots,m_{n-1},5}) + \gamma_F(G_{m_1,m_2,\ldots,m_{n-1},r_n})$.

Since, by Theorem 3 and Lemmas 4-6, we have $\gamma_F(G_{m_1,m_2,\ldots,m_{n-1},5}) \leq m_1 \cdot m_2 \cdots m_{n-1}$. Then $\gamma_F(G_{m_1,m_2,\ldots,m_n}) \leq q_n \cdot m_1 \cdot m_2 \cdots m_{n-1} + \gamma_F(G_{m_1,m_2,\ldots,m_{n-1},r_n})$. By the same way, we can find a tiling of $G_{m_1,m_2,\ldots,m_{n-1},r_n}$ and get

$$
\begin{aligned}
& \gamma_F(G_{m_1,m_2,\ldots,m_{n-1},r_n}) \\
\leq\ & q_{n-1} \cdot \gamma_F(G_{m_1,m_2,\ldots,m_{n-2},5,r_n}) + \\
& \gamma_F(G_{m_1,m_2,\ldots,m_{n-2},r_{n-1},r_n}) \\
\leq\ & q_{n-1} \cdot m_1 \cdot m_2 \cdots m_{n-2} \cdot r_n + \\
& \gamma_F(G_{m_1,m_2,\ldots,m_{n-2},r_n,r_{n-1}}).
\end{aligned}
$$

Finally, the theorem follows. □

4 Concluding Remarks

This paper considers F-dominating sets on a grid G and gives an upper bound on $\gamma_F(G)$. We believe that the upper bound value is probably the exact one. So, we naturally proceed to try show it. Future research directions could be solving the problem on another interconnection networks.

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