Pancyclicity on Generalized Recursive Circulant Graphs *

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Abstract

The generalized recursive circulant graph *(GRCG for short) is a generalization of the recursive circulant graph. It provides a new topology for interconnection networks. A graph G with n vertices is called* pancyclic *if G contains cycles of every length k, $3 \leqslant k \leqslant n$. In this paper, we shall prove that a GRCG with two or more dimensions contains all cycles of even lengths, and odd length cycles which are larger than a specific length.*

Keywords: *Generalized recursive circulant graphs; recursive circulant graphs; pancyclicity; bipancyclicity.*

1 Introduction

The *circulant graph* class is a famous network topology due to its regular and symmetric connection property. Let $C(n; c_1, c_2, \ldots, c_m)$ be a circulant graph. There are n vertices in C, and vertices u and v are adjacent if and only if $u \equiv v \pm c_i$ (mod n) for $1 \leqslant i \leqslant m$ and $u, v \in \{1, 2, \ldots, n\}$ [7, 8]. For example, $C(24; 1, 3, 12)$ and $C(24; 1, 4, 8)$ are shown in Figures 1(a) and 1(b), respectively. Circulant graphs, which are vertex-symmetric, form a subclass of Cayley graphs [1, 4].

The *recursive circulant graph* (RCG for short) is a subclass of circulant graphs with recursive structure. The RCG has been widely studied, such as pancyclicity [2], edge-pancyclicity [3], parallel routing algorithm [9], strong hamiltonicity [15], Hamiltonian properties in faulty condition [17], super-connected property [18], and independent spanning trees problem on RCGs [19, 20]. Embeddings of hypercubes and meshes [14], trees [11], full ternary trees [10], and disjoint Hamiltonian cycles [6, 12] were also studied.

Let $R(cd^m, d)$ be an RCG. Then, the graph has $N = cd^m$ vertices, and recursively consists of d subgraphs of $R(cd^{m-1}, d)$. Note that the conditions of parameters $0 < c < d$ and $m > 0$ should be satisfied. Suppose all vertices in $R(cd^m, d)$ are labeled from 0 to $cd^m - 1$. The adjacent vertices of vertex u are vertices labeled with $u \pm d^k$ (mod N), for $k = 0, 1, 2, \ldots, \lceil \log_d N \rceil - 1$. Further, all edges of $(u, u + 1 \pmod N)$ form a Hamiltonian cycle on $R(cd^m, d)$, called *basic cycle*, which links d subgraphs of $R(cd^{m-1}, d)$ together [6]. From the viewpoint of circulant graphs, $R(cd^m, d)$ can also be denoted by $C(N; d^0, d^1, d^2, \ldots, d^{\lceil log_d N \rceil - 1})$.

Due to the restriction $0 < c < d$ and $m > 0$ on the parameters of RCGs, $C(12; 1, 4)$ is an RCG, but $C(12; 1, 3)$ is not. In order to extend the RCG class, Tang and Wang propose a general class of graphs including RCGs [16]. They use a multidimensional vertex labeling in the definition to simplify the design of algorithms.

The *generalized recursive circulant graph* (GRCG for short), denoted by $GR(m_h, m_{h-1}, \ldots, m_1)$, has $N = \Pi_{i=1}^{h} m_i$ vertices, where $m_i \geqslant 2$ is the *size* of dimension i for $i = 1, 2, \ldots, h$. Sometimes, we call it the h-dimensional GRCG. The label of each vertex is represented as a vector, i.e., $(x_h, x_{h-1}, \ldots, x_1)$, where $0 \leqslant x_i \leqslant m_i - 1$ for dimension i. The adjacency of a vertex in a GRCG, they define the term jump as follows. Let i^+ and i^- be the two *jumps* of dimension i. Then, vertex $(x_h, \ldots, x_{i+1}, x_i, \ldots, x_1)$ takes jump i^+ or i^- to reach vertex $(x_h, \ldots, x_{i+1}, x_i + 1, \ldots, x_1)$ or

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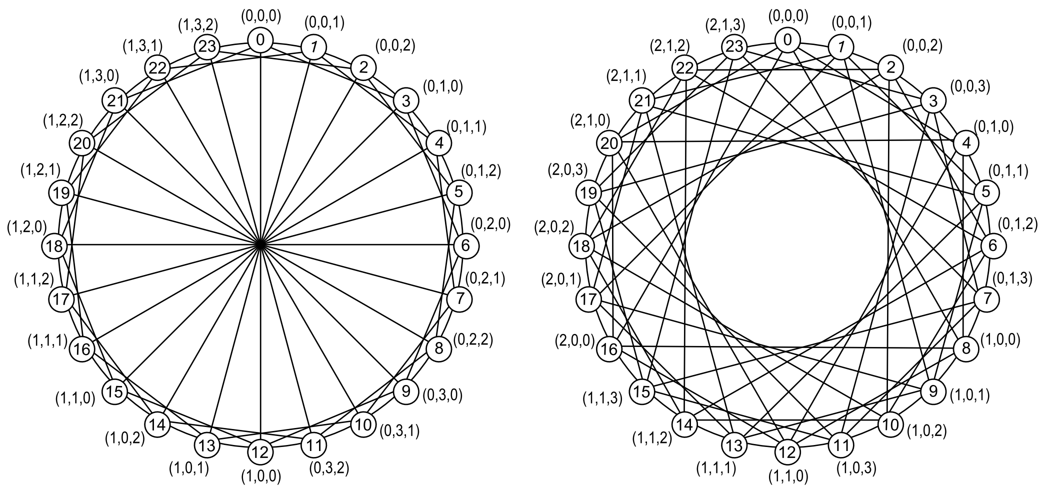

(a) (b)

Figure 1: Two GRCGs: (a) GR(2,4,3) and (b) GR(3,2,4).

$(x_h, \ldots, x_{i+1}, x_i - 1, \ldots, x_1)$, respectively. Notice that we increase x_{i+1} by 1 and set $x_i = 0$ (called *carrying*) when $x_i + 1 = m_i$; meanwhile, we decrease x_{i+1} by 1 and set $x_i = m_i - 1$ (called *borrowing*) when $x_i - 1 = -1$. Further, the carrying and borrowing operation may occur subsequently till dimension h is encountered. Taking GR(2,4,3) in Figure 1(a) as an example. Vertex (1,3,0) reaches vertices (0,3,0), (0,0,0), (1,2,0), (1,3,1) and (1,2,2) by jumps 3^-, 2^+, 2^-, 1^+, and 1^-, respectively. In Figure 1(b), GR(3,2,4) have m_1=4, m_2=2, and m_3=3. Vertex (0,1,2) takes jumps 3^+, 3^-, 2^+, 2^-, 1^+, and 1^- to reach (1,1,2), (2,1,2), (1,0,2), (0,0,2), (0,1,3), and (0,1,1), respectively.

We adapt the meaning of jump to RCGs. It turns out that an $RC(cd^m, d)$ is isomorphic to an $(m+1)$-dimensional $GR(c, d, d, \ldots, d)$ (in case that $c > 1$) or an m-dimensional $GR(d, d, \ldots, d)$ (in case that $c = 1$). Therefore, RCGs are a subclass of GRCGs. Since each vertex of a GRCG has the same jump set, GRCGs are also a subclass of circulant graphs. That is, $GR(m_h, m_{h-1}, \ldots, m_2, m_1)$ is isomorphic to $C(N; 1, m_1, m_1m_2, \ldots, \prod_{k=1}^{h-1} m_k)$.

GRCGs still have the recursive structure as RCGs. Following the definition of GRCGs, we define GR(2) as K_2 and GR(m) as a cycle with m vertices (or C_m) when $m \geqslant 3$. At first, it is obvious that GR(m_2, m_1) contains m_1 copies of GR(m_2). Further, we can figure out by removing the basic cycle that GR$(m_h, \ldots, m_2, m_1)$ contains m_1 number of GR$(m_h, m_{h-1}, \ldots, m_2)$ as induced subgraphs.

In case of $m_h = 2$, jumps h^+ and h^- reach the same vertex and thus are viewed as one single jump h^-.

A graph G with n vertices is *pancyclic* if and only if G contains cycles of every length k for $3 \leqslant k \leqslant n$. Particularly, if G contains cycles of every even length, then G is *bipancyclic*. In [2], the authors have studied the *pancyclicity* of RCGs. In the following section, we shall investigate the pancyclicity of GRCGs.

2 The pancyclicity of generalized recursive circulant graphs

We firstly examine the bipancyclicity of GRCGs, and then investigate the existence of every odd cycles in GRCGs. A one dimensional GRCG is a cycle such that its pancyclicity can be recognized intuitively. Thus, we consider hereafter only GRCGs with higher dimensions.

For proving the bipancyclicity of GRCGs, we shall show that an $r \times c$ grid (or mesh, i.e., a product graph of two paths) can be embedded in $GR(m_h, m_{h-1}, \ldots, m_2, m_1)$, since it is obvious bipancyclic when c and r are greater than or equal to 2. Let $c = m_1$ be the number of columns and $r = \prod_{i=2}^{h} m_i$ be the number of rows of the grid.

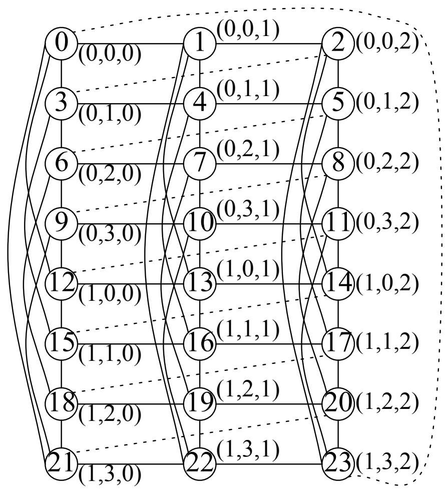

Figure 2: The embedding of an 8×3 grid on GR(2,4,3).

Then, every vertex in the grid is labeled as (a, b) where $0 \leqslant a \leqslant r-1$ and $0 \leqslant b \leqslant c-1$. The embedding is achieved by mapping vertex (a, b) of the grid on vertex $(x_h, x_{h-1}, \ldots, x_1)$ of the GRCG, where $x_1 = b$, $x_2 = a - \sum_{k=3}^{h}(x_k \cdot \prod_{j=2}^{k-1} m_j)$, $x_h = a \,/\, \prod_{j=2}^{h-1} m_j$, and

$$x_i = \left\lfloor (a - \sum_{k=i+1}^{h} (x_k \cdot \prod_{j=2}^{k-1} m_j)) \,/\, \prod_{j=2}^{i-1} m_j \right\rfloor,$$

for i=3,4,...,h-1.

Conversely, a vertex $(x_h, x_{h-1}, \ldots, x_1)$ in the host graph can also be relabeled as (a, b) by setting

$$a = \sum_{k=2}^{h}(x_k \cdot \prod_{j=2}^{k-1} m_j) \text{ and } b = x_1.$$

For example, see Figure 2. Given vertex (7,0) in the 8×3 grid, its corresponding vertex in GR(2,4,3) is (1,3,0) since $x_3 = \lfloor a/m_2 \rfloor = \lfloor 7/4 \rfloor = 1$, $x_2 = a - x_3 \cdot m_2 = 7 - 1 \cdot 4 = 3$. Conversely, given vertex (1,3,0) in GR(2,4,3), its corresponding grid vertex is (a, b) where $a = x_2 + x_3 \cdot m_2 = 3 + 1 \cdot 4 = 7$ and $b = x_1 = 0$.

For a vertex (a, b) in the grid, its neighbors are reached by taking jumps 1^+, 1^-, 2^+ and 2^- based on the multidimensional label of the embedded GRCG. In addition, jump 2^- is absent if $a = 0$. If $a = r-1$, then jump 2^+ is absent. Jump 1^- is absent if $b = 0$. If $b = c-1$, then jump 1^+ is absent.

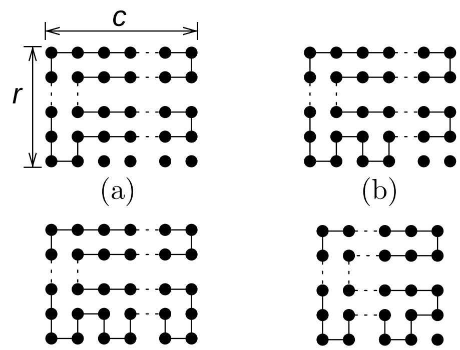

Figure 3: The construction of even length cycles: (a) the $(N-c+2)$-cycle of Case 2, (b) the $(N-c+4)$-cycle of Case 2, (c) the N-cycle of Case 2, and (d) the $(N-1)$-cycle of Case 3.

Corollary 1. *$GR(m_h, m_{h-1}, \ldots, m_1)$ with $h \geqslant 2$ contains every even length cycle.*

In the following, we will discuss the existence of odd cycles in GRCGs. Since the smallest odd cycle may have length greater than three, or might not exist, we denote the length of the smallest odd cycle in $GR(m_h, m_{h-1}, \ldots, m_1)$ by s. In order to prove that there exist all odd cycles of lengths greater than or equal to s, we give the following two propositions.

Proposition 2. *There exist m_h-cycle for $m_h > 2$ and $(m_i + 1)$-cycle for $1 \leqslant i \leqslant h-1$ in $GR(m_h, m_{h-1}, \ldots, m_1)$.*

Let $j^{x,k}$ denote the repetition of jump j^x with k times where $x \in \{+, -\}$. For the m_h-cycle, it starts from an arbitrary vertex u, then to take jumps h^{+,m_h}, finally goes back to u. As to the (m_i+1)-cycle in a GRCG, a vertex takes jumps i^{+,m_i} and then takes jump $(i+1)^-$ to reach itself again.

Proposition 3. *Let $s' = \min_{1 \leqslant i \leqslant h-1}\{m_i + 1 \mid m_i$ is even$\}$. For $GR(m_h, m_{h-1}, ..., m_1)$, the length s of the shortest odd cycle is equal to s' if m_h is even. If m_h is odd, then $s = min\{s', m_h\}$.*

By Proposition 2, for odd m_h and even m_i, there exist odd cycles if $m_h > 2$ and $1 \leqslant i \leqslant h-1$. Therefore, we choose the minimum length among them as s. The case that s does not exist occurs only when m_h is even and all m_i are odd. In this case. Consequently, the corresponding GRCG must be a bipartite graph.

Lemma 4. *If a GRCG has a cycle with the minimum odd length s, then it also contains cycles of every odd length greater than s and less than or equal to N.*

Proof. By definition, $GR(m_h, m_{h-1}, \ldots, m_1)$ contains m_1 copies of $GR(m_h, m_{h-1}, \ldots, m_2)$ as induced subgraphs and an $r \times c$ grid can be embedded into it where $c = m_1$ and $r = \prod_{i=2}^{h} m_i$. We prove this lemma by mathematical induction on h.

For the basis step, i.e., $h = 1$, a GRCG is itself an m_1-cycle if $m_1 > 2$ or K_2 if $m_1 = 2$. Thus, there is exactly one odd cycle with length m_1 if m_1 is odd; otherwise, s does not exist. Therefore, the base holds.

By the mathematical induction hypothesis, the statement is true for $h = z$. Note that $GR(m_{z+1}, m_z, \ldots, m_1)$ (GR_{z+1} for short) contains m_1 copies of $GR(m_{z+1}, m_z, \ldots, m_2)$ (GR_z for short) as its induced subgraphs. We use l and t to denote the lengths of the smallest odd cycle and the largest odd cycle, respectively, in GR_z. There are five cases taken into consideration as follows.

- Case 1: Both r and c are odd.

  In this case, $m_1 = c$ is odd and $m_1 + 1$ is even. Thus, m_1 has nothing to do with the minimum length of odd cycles and $s = l$. Since GR_z contains odd cycles of every length from l to t. Note that $t = r$. We have to show the existence of odd cycles with length j in GR_{z+1} for $t + 2 \leqslant j \leqslant N$. For $t + 2 \leqslant j \leqslant N - c + 1$, the j-cycle can be constructed regularly. We choose any vertex in one of subgraphs GR_z and extend the t-cycle to $(t+2)$-cycle, $(t+4)$-cycle, and so forth. This extension is achieved by the repetition of jump sequence $1^+, 2^+, 1^-$ and delete an edge. (see Figures 4(a) to 4(c) for an illustration). For $N - c + 3 \leqslant j \leqslant N$, the cycle is extended by replacing jump 1^+ with jumps 2^+, 1^+, and 2^- (see Figures 4(d) and 4(e)).

- Case 2: r is odd, c is even, and $c + 1 \geqslant l$.

  Since $s = l$, the odd cycles are extended similarly as Case 1. The only difference is that N is even in this case. The length of the largest odd cycle is $N - 1$. Figure 4(f) shows the situation.

- Case 3: c is even and $c + 1 < l$.

  In this case, $s = c + 1$, we have to find all odd cycles with length j for $s \leqslant j \leqslant N - 1$. First,

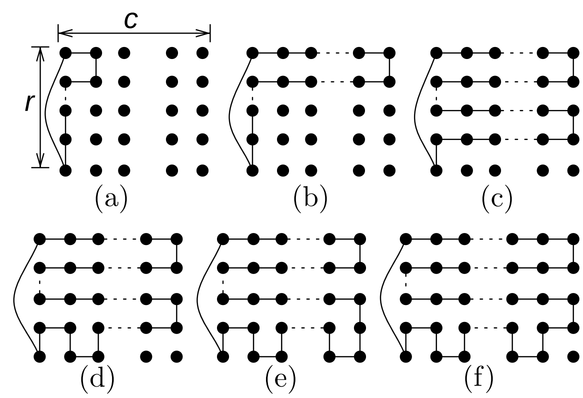

Figure 4: Odd cycles of lengths from $t + 2$ to N in Case 1. (a) $(t + 2)$-cycle, (b) $(t + 2c - 2)$-cycle, (c) $(N - c + 1)$-cycle, (d) $(N - c + 3)$-cycle, (e) N-cycle, and (f) $(N - 1)$-cycle for even N.

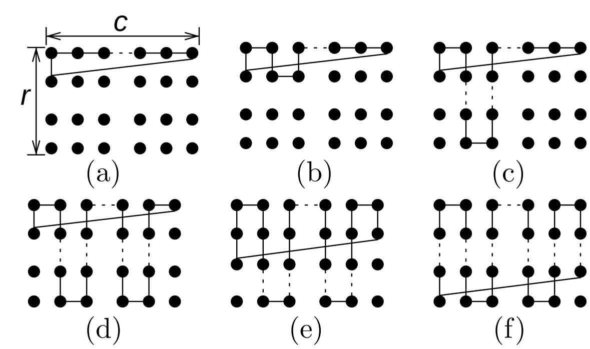

Figure 5: Odd cycles of lengths from s to $N - 1$ in Case 3: (a) s-cycle, (b) $(s+2)$-cycle, (c) $(s+2(r-1))$-cycle, (d) $(s+2r)$-cycle, (e) $(N-2r+5)$-cycle, and (f) $(N - 1)$-cycle.

  the s-cycle starts from an arbitrary vertex u is constructed by taking jumps $1^{+,c}$ and 2^- (see Figure 5(a)). For $s + 2 \leqslant j \leqslant N - 1$, the cycle can be constructed by continuously replacing one 1^+ with three jumps 2^+, 1^+ and 2^-. See Figures 5(b) to 5(f) for illustrating the extension of odd cycles.

- Case 4: r is even and c is odd.

  In this case, odd c cannot reduce s, and thus $s = l$. If GR_z is not a bipartite graph, then all odd cycles of lengths from s to t exist. Note that $r = t + 1$. We have to show the existence of odd cycles with lengths from $t+2$ to $N-1$. See Figures 6 for illustrating the extension of these cycles.

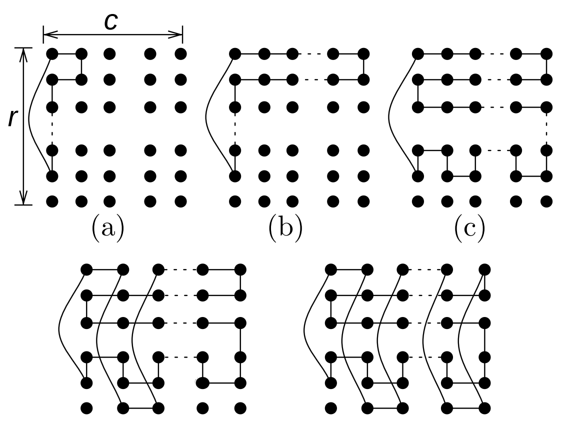

Figure 6: The odd cycles of Case 4: (a) $(t+2)$-cycle, (b) $(t+2c-2)$-cycle, (c) $(N-c)$-cycle, (d) $(N-c+2)$-cycle, and (e) $(N-1)$-cycle.

- Case 5: Both r and c are even and $c+1 \geqslant l$. In this case, $s = l$. Odd cycles of lengths from s to $r-1$ exist. If $c-1 > r-1$, then odd cycles of lengths from $r+1$ to $c-1$ can be formed by means of jump extension as used in Case 1. Notice that $c-r < c$ and the first two rows contain $2c$ vertices. Thus, the jump extension is bounded in the first two rows of the grid. For odd cycles with lengths from $c+1$ to $N-1$, we apply the same approach mentioned in Case 3 to construct them. Therefore, odd cycles of lengths from s to $N-1$ exist. This establishes the lemma. □

By Corollary 1 and 4, we obtain the following theorem.

Theorem 5. *A generalized recursive circulant graph is pancyclic if and only if 3-cycle exists.*

3 Concluding remarks

In this paper, we study the pancyclic property of a new class of circulant graphs GRCGs which is a generalization of RCGs. A good labeling of the vertices in a graph class might result in simple algorithms for solving problems in graphs. Our proof is also based on the vertex labeling of GRCGs. Moreover, since the structure of GRCGs is similar to multidimensional torus networks [5], the topology of GRCGs provides an alternative for designing parallel computers. Many network properties, as well as combinatorial problems, are worth to study on GRCGs.

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