On Fault Tolerance Of \( D_{3,2,1} \)-Domination*

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Abstract

Distance two domination has practical usage in resource sharing problem. \( D_{3,2,1} \)-domination problem is a variable of distance-two domination. For a graph \( G = (V, E) \), let \( D \subseteq V \). Assume that every vertex \( v \) in \( D \) can get resource completely and share partial resource with its neighbors and offer little resource to the vertices that are distance two from it. Then the weight function of a vertex \( v \in V \) may be defined as \( w_D(v) = 3|v| \cap D| + 2|N_i(v) \cap D| + |N_2(v) \cap D| \) respect to \( D \) where \( N_i(v) \) denoted distance \( i \) neighborhood of \( v \). \( D \subseteq V \) is a \( D_{3,2,1} \)-dominating set of \( G \) if and only if \( w_D(v) \geq 3 \) for every vertex \( v \in V \). The \( D_{3,2,1} \)-domination number \( \gamma_{3,2,1}(G) \) of a graph \( G \) is the minimum cardinality of a \( D_{3,2,1} \)-dominating set of \( G \). The \( D_{3,2,1} \)-domination problem is the problem of finding the \( D_{3,2,1} \)-domination number of graphs.

This paper corrected the \( D_{3,2,1} \)-domination number of double loop graphs in some cases. Due to the possible connection fault, this paper also discussed the edge-fault-tolerance problem respect to \( D_{3,2,1} \)-domination on a number of graphs whose \( \gamma_{3,2,1}(G) \) are known.

1 Introduction

The domination can be used to model many location problems in operation research. In a graph \( G = (V, E) \), a dominating set is a subset \( D \) of \( V \) such that every vertex in \( V - D \) is adjacent to at least one vertex in \( D \). The related distance \( k \) domination may be used to solve the resource sharing problem. A distance-\( k \) dominating set \( D \) is a set of vertices such that for every vertex \( v \) in \( V - D \), there is a vertex at distance at most \( k \) from it. \( D_{3,2,1} \)-domination is a variation of distance-two domination. For any subset \( D \subseteq V \), define the weight-function \( w_D : V \to \mathbb{N} \) as \( w_D(v) = 3|v| \cap D| + 2|N_i(v) \cap D| + |N_2(v) \cap D| \) where \( N_i(v) \) denoted distance \( i \) neighborhood of \( v \), then \( D \) is a \( D_{3,2,1} \)-dominating set of \( G \) if and only if \( w_D(v) \geq 3 \) for every vertex \( v \in V \). The \( D_{3,2,1} \)-dominating number of \( G \), denoted as \( \gamma_{3,2,1}(G) \), is the minimum cardinality of a \( D_{3,2,1} \)-dominating set of \( G \). A \( D_{3,2,1} \)-dominating set \( D \) of \( G \) is optimal if and only if \( |D| = \gamma_{3,2,1}(G) \). Besides resource sharing, \( D_{3,2,1} \)-domination can also be applied to many other problems related to social relationship, including classroom order management, election running and so on. Distance domination has been studied for years and got many results (see [1], [2], [3], [4], [6] [7] [8] [9], [12] first considered the decreasing effect on distance in 2006. They discussed the \( D_{3,2,1} \)-domination problem of paths, cycles and complete full binary trees. In 2008, [11] discussed the \( D_{3,2,1} \)-domination problem and its application of double loop networks \( DL(n;1,2) \) and \( DL(n;1,3) \). Later on, [5] established the \( D_{3,2,1} \)-domination number for the composition graphs of a path with a path and a path with a cycle.

Fault tolerant ability has been discussed respect to many problems including domination [10]. A graph \( G = (V, E) \) is said to be \( k \)-edge-fault tolerance respect to \( D_{3,2,1} \)-domination if for every subset \( X \subseteq E \) such that \( |X| \leq k \) such that \( \gamma_{3,2,1}(G - X) = \gamma_{3,2,1}(G) \). A graph \( G \) is said to be critical respect to \( D_{3,2,1} \)-domination if and only if for all subset \( X \subseteq E \), \( X \neq \emptyset \), \( \gamma_{3,2,1}(G - X) > \gamma_{3,2,1}(G) \). Motivated by possible

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link errors or communication fault in the applications, this paper discussed the edge fault tolerant ability respect to $D_{3,2,1}$-domination on a number of graphs including paths, cycles, complete full binary trees, and double loop graphs whose $D_{3,2,1}$-domination number are known and corrected some errors in previous paper dealt with $D_{3,2,1}$-domination number of double loop graphs.

2 Paths and Cycles

According to [12], the $D_{3,2,1}$-domination number of paths and cycles are listed below.

**Lemma 1.** [12] The $D_{3,2,1}$-domination number of path $P_n$ for $n \geq 2$ is $\gamma_{3,2,1}(P_n) = \left\lceil (n+4)/3 \right\rceil$.

**Lemma 2.** [12] The $D_{3,2,1}$-domination number of path $C_n$ for $n \geq 3$ can be expressed as

$$\gamma_{3,2,1}(C_n) = \begin{cases} 
2, & \text{if } n = 3; \\
\left\lceil n/3 \right\rceil, & \text{if } n > 3.
\end{cases}$$

Since removing any edge of a path will result two subpaths. Let $P_t$ and $P_k$ such that $t+k=n$ be two subpaths for some $t,k \geq 1$ obtained by removing an edge from path $P_n$. Theorem 1 shows that $P_n$ is critical respect to $D_{3,2,1}$-domination except in the case that both $t$ and $k$ are 1 module 3.

**Theorem 1.** Let $P_n = P_t \cup P_k \cup \{e\}$ for $t+k=n$ and $t,k \geq 1$. Then $P_n$ is one-fault tolerance respect to $D_{3,2,1}$-domination if and only if both $t$ and $k$ are 1 module 3.

**Proof:** Since each faulty-edge will divide the path into subpaths $P_t$ and $P_k$ such that $t+k=n$ for some $t,k \geq 1$, we consider the following cases.

Case 1. One of $t$ and $k$ is 0 module 3, without loss of generality, let $t = 0 \mod 3$. Then by lemma 1, we have

$$\gamma_{3,2,1}(P_t) + \gamma_{3,2,1}(P_k) = \left\lceil (t+4)/3 \right\rceil + \left\lceil (k+4)/3 \right\rceil \\
= \frac{t}{3}+1+\left\lceil \frac{k+4}{3} \right\rceil \\
= 1+\left\lceil \frac{n+4}{3} \right\rceil \\
> \gamma_{3,2,1}(P_n)$$

Hence in this case $P_n$ is critical.

Case 2. One of $t$ and $k$ is 2 module 3, without loss of generality, let $t = 2 \mod 3$. Then by lemma 1, we have

$$\gamma_{3,2,1}(P_t) + \gamma_{3,2,1}(P_k) = \left\lceil (t+4)/3 \right\rceil + \left\lceil (k+4)/3 \right\rceil \\
= (t-2)/3+2+\left\lceil (k+4)/3 \right\rceil$$

Since $(t-2)/3$ is an integer, we have

$$\gamma_{3,2,1}(P_t) + \gamma_{3,2,1}(P_k) = 2+\left\lceil (n+2)/3 \right\rceil \\
= 1+\left\lceil (n+5)/3 \right\rceil \\
> \left\lceil (n+4)/3 \right\rceil = \gamma_{3,2,1}(P_n)$$

Hence in this case $P_n$ is critical.

Case 3. Both $t$ and $k$ are 1 module 3. In this case, $n = 2 \mod 3$. Then by lemma 1, we have

$$\gamma_{3,2,1}(P_t) + \gamma_{3,2,1}(P_k) = \left\lceil (n+4)/3 \right\rceil = \left\lceil (t+k+4)/3 \right\rceil \\
= (t-1)/3+(k-1)/3+2 \\
= \left\lceil t/3 \right\rceil + \left\lceil k/3 \right\rceil + 2 \\
= \left\lceil t/3+1 \right\rceil + \left\lceil k/3 +1 \right\rceil \\
= \left\lceil (t+4)/3 \right\rceil + \left\lceil (k+4)/3 \right\rceil \\
= \gamma_{3,2,1}(P_t) + \gamma_{3,2,1}(P_k)$$

Hence $P_n$ is one-fault tolerance in this case.

Since removing any edge of a cycle $C_n$ will result a path $P_n$, Theorem 2 established that $C_n$ is critical except when $n = 3k+1$ for some integer $k$ in which case $C_n$ is one-fault tolerance.

**Theorem 2.** $C_n$ is one-fault tolerance if and only if $n = 3k+1$ for some integer $k$.

**Proof:** Consider the following cases.

Case 1. $n = 3k$ for some integer $k$. Then by lemma 1 and lemma 2, we have

$$\gamma_{3,2,1}(P_n) = \left\lceil (n+4)/3 \right\rceil = k+1 \\
> k = \left\lceil n/3 \right\rceil = \gamma_{3,2,1}(C_n)$$

Hence $C_n$ is critical in this case.

Case 2. $n = 3k+2$ for some integer $k$. Then by lemma 1 and lemma 2, we have

$$\gamma_{3,2,1}(P_n) = \left\lceil (n+4)/3 \right\rceil = k+2 \\
> k+1 = \left\lceil n/3 \right\rceil = \gamma_{3,2,1}(C_n)$$
Hence \( C_n \) is critical in this case.

Case 3. \( n = 3k + 1 \) for some integer \( k \). Then by lemma 1 and lemma 2, we have

\[
\gamma_{3,2,1}(P_n) = \left\lfloor \frac{(n+4)/3} \right\rfloor = k + 1
\]

\[
= \left\lfloor \frac{n}{3} \right\rfloor = \gamma_{3,2,1}(C_n)
\]

Hence \( C_n \) is one-fault tolerance in this case. \( \square \)

3 Complete full binary trees

A complete full binary tree \( B_n \) is a rooted tree with height \( n \) and each internal vertex has exactly two children. The \( D_{3,2,1} \)-domination number of \( B_n \) is given by [12] is listed as lemma 3.

**Lemma 3.** [12] The \( D_{3,2,1} \)-domination number of a complete full binary tree \( B_n \) is

\[
\gamma_{3,2,1}(B_n) = \begin{cases} 
1 + \sum_{j=2}^{i} 2^j, & n = 5i, i \geq 0; \\
2 + \sum_{j=3}^{i} 2^j, & n = 1 + 5i, i \geq 0; \\
\sum_{j=3}^{i} 2^j, & \text{otherwise}.
\end{cases}
\]

where \( S = \{n-2-5t, n-1-5t\} \) for \( 0 \leq t \leq \left\lfloor \frac{(n+4)/5} \right\rfloor - 1 \), and \( S' = \{n-2-5t, n-1-5t\} \) for \( 0 \leq t \leq \left\lfloor \frac{n+4}{5} \right\rfloor - 2 \).

According to lemma 3, the \( D_{3,2,1} \)-dominating set of \( B_n \) must contain all of the vertices in level \( j \) where \( j \in S' \) for \( n = 5i+1 \) and \( j \in S \) otherwise. Next theorem shows that although \( B_n \) is zero-fault tolerance in general but \( B_n \) is not critical with respect to \( D_{3,2,1} \)-domination. For the convenience, let \( l(v) \), \( r(v) \) and \( p(v) \) denote the left-child, right-child and the parent of vertex \( v \) respectively. Let \( T(v) \) denote the tree rooted by \( v \). \( T_l(v) \) and \( T_r(v) \) denote the left subtree and right subtree rooted by \( l(v) \) and \( r(v) \) respectively. Figure 1 shows the relation of a \( B_n \) with these notations.

![Diagram](image)

**Figure 1:** Relation of notation in \( B_n \)

**Lemma 4.** Let \( D \) be an optimal \( D_{3,2,1} \)-dominating set of a complete full binary tree \( B_n \) for \( n \geq 2 \). Then \( D \) does not contain any leaf vertex of \( B_n \).

**Proof:** Without lose of generality, assume to the contrary that there is a leaf vertex \( l(v) \in D \) and \( v \) is a leaf-child of its parent. Since another leaf vertex \( r(v) \) has only two distance-two neighbors, either \( v \) or \( r(v) \) has to be in \( D \). Hence \( \{p(v)\} \cap D \geq 2 \). Since the leaves in \( T_r(p(v)) \) does not get any weight from \( T_l(p(v)) \), they request at least two vertices of \( T_r(p(v)) \cup p(v) \) in \( D \). Hence \( \{p(v)\} \cap D \geq 4 \). Now consider \( D' = (D - T(p(v))) \cup \{p(v), v, r(p(v))\} \), notice that \( |D'| < |D| = \gamma_{3,2,1}(B_n) \). Since every leaves of \( T(p(v)) \) are distance two from \( p(v) \) and distance one from either \( v \) or \( r(p(v)) \), \( D' \) is a \( D_{3,2,1} \)-dominating set of \( B_n \) which contradict to \( D \) is optimal. \( \square \)

**Theorem 3.** The complete full binary tree \( B_n \) with height \( n \geq 2 \) has zero-fault tolerance and \( B_n \) is critical respect to \( D_{3,2,1} \)-domination if and only if \( n = 2 \).

**Proof.** We first show that \( B_n \) has zero-fault tolerance. Let \( D \) be an optimal \( D_{3,2,1} \)-dominating set of \( B_n \). By lemma 4, \( D \) does not contain any leaf. Let \( l(v) \) be a leaf of \( B_n \) and \( e_l = (v, l(v)) \). Then \( \gamma_{3,2,1}(B_n - e_l) = \)
number of $DL(n;1,3)$ in [6] is for any $n$, \[ \gamma_{3,2,1}(DL(n;1,3)) = \left\lceil \frac{n}{5} \right\rceil , \] which we corrected as lemma 6.

**Lemma 6.** Let $G = DL(n;1,3)$ be a double loop graph of order $n \geq 6$, then the $D_{3,2,1}$-domination number of $G$ is

\[ \gamma_{3,2,1}(G) = \left\lceil \frac{n}{5} \right\rceil +1 , \text{ if } n = 9 \text{ or } 14; \]
\[ \left\lceil \frac{n}{5} \right\rceil , \text{ otherwise.} \]

**Proof:** Let $G_1 = DL(9;1,3)$ and $G_2 = DL(14;1,3)$. Then we will show that $\gamma_{3,2,1}(G_1) = 3$ and $\gamma_{3,2,1}(G_2) = 4$. Since it is easy to see that $\{0,5,7\}$ and $\{0,5,7,9\}$ are $D_{3,2,1}$-dominating set of $G_1$ and $G_2$ respectively, which implies $\gamma_{3,2,1}(G_1) \leq 3$ and $\gamma_{3,2,1}(G_2) \leq 4$. To see the lower bound, we consider following two cases:

Case 1: $n = 9$. Since $G_1$ is a 4-regular graph, every vertex has four distance-two neighbors which form a path $P_4$. Since every vertex in $G_1$ is adjacent to at most three of these four vertices, hence $\gamma_{3,2,1}(G_1) > 2$.

Case 2: $n = 14$. Let $D$ be a $D_{3,2,1}$-dominating set of $G_2$. Without lose of generality, let vertex $0 \in D$. There are six possible subcases which we state as follows.

Case 2.1: either vertex $1 \in D$ or vertex $3 \in D$. In this case there are six vertices have only get weight one from $\{0,1\}$ or $\{0,3\}$. Since $G_2$ is a 4-regular graph, there is no vertex is within distance one of these six vertices. Hence we have $|D| > 3$.

Case 2.2: vertex $2 \in D$. In this case, there are two vertices do not get any weight from $\{0,2\}$ which requires at least two more vertices in $D$ to

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**4 Double Loop Graphs**

A double loop graph denoted as $DL(n; a, b)$ in this paper is an underlining graph of a directed graph with $n$ vertices $0, 1, 2, \ldots, n-1$ and $2n$ directed edges of the form $i \rightarrow i + a \mod n$ and $i \rightarrow i + b \mod n$, referred to as $a$-links and $b$-links. In 2008, [6] established the $D_{3,2,1}$-domination number of $DL(n;1,2)$ which we state as lemma 5. They also provided the $D_{3,2,1}$-domination number of $DL(n;1,3)$, however, there are some small errors which we corrected it as lemma 6.

**Lemma 5.** [11] Let $G = DL(n;1,2)$ be a double loop graph of order $n \geq 4$, then the $D_{3,2,1}$-domination number of $G$ is

\[ \gamma_{3,2,1}(G) = \begin{cases} 2, & \text{if } n = 4 \text{ or } 5; \\ \left\lceil \frac{n}{5} \right\rceil , & \text{if } n > 5. \end{cases} \]

The original theorem for the $D_{3,2,1}$-domination number of $DL(n;1,3)$ in [6] is for any $n$, \[ \gamma_{3,2,1}(DL(n;1,3)) = \left\lceil \frac{n}{5} \right\rceil , \] which we corrected as lemma 6.

Figure 2: The four vertices get weight 1 from $v$
$D_{3,2,1}$-dominate those two vertices. Hence we have $|D| > 3$.

Case 2.3: vertex $4 \in D$. Since vertex 9 is distance three from both vertices 0 and 4, that implies vertex 9 $\in D$. Consider $D' = \{0, 4, 9\} \subseteq D$. Since vertex 2 is distance two from both vertices 0 and 4, and distance three from vertex 9, we have $w_{D'}(2) = 2$. Since $D$ is a $D_{3,2,1}$-dominating set of $G_2$ and $D' \subseteq D$ is not a $D_{3,2,1}$-dominating set of $G_2$, we have $|D| > 3$.

Case 2.4: vertex $5 \in D$. In this case, the vertices 7, 9, 10, and 12 only get weight 1 from $\{0, 5\}$, and there is no vertex in $G_2$ are within distance one from all four of them. Hence we have $|D| > 3$.

Case 2.5: vertex $6 \in D$. In this case, every vertex gets weight 2 from $\{0, 6\}$. Since each vertex in $G_2$ has only four distance-one neighbors and six distance-two neighbors, a vertex can offer some weight to at most 11 vertices (containing itself). Since there are 12 vertices still have one short for weight, we have $|D| > 3$.

Case 2.6: vertex $7 \in D$. In this case, the four vertices getting only one weight from $\{0, 7\}$ are vertices 2, 5, 9, and 12. Since there is no vertex in $G_2$ are within distance one from all four of them, we have $|D| > 3$.

Next two theorems show the fault-tolerant ability of the graph $DL(n;1,2)$ and $DL(n;1,3)$.

**Theorem 4** A double loop graph $DL(n;1,2)$ is one-fault tolerance.

**Proof.** To show that $DL(n;1,2)$ is one-fault tolerance, we consider following two cases according to the type faulty edge.

Case 1: The faulty edge is a 1-link. Without loss of generality, let $(1,2)$ be the faulty edge of $DL(n;1,2)$. Then $D = \{S_i | 0 \leq i \leq (n-1)/2\}$ is a $D_{3,2,1}$-dominating set of $DL(n;1,2)$ and $|D| = \gamma_{3,2,1}(DL(n;1,2))$ which implies that $DL(n;1,2)$ is one-fault tolerance in this case.

Case 2: The faulty edge is 2-link. Without loss of generality, let $(1,3)$ be the faulty edge of $DL(n;1,2)$. Then $D = \{S_i | 0 \leq i \leq (n-1)/5\}$ is a $D_{3,2,1}$-dominating set of $DL(n;1,2) – (1,3)$ and $|D| = \gamma_{3,2,1}(DL(n;1,2))$ which implies that $DL(n;1,2)$ is one-fault tolerance in this case. □

In fact, $DL(n;1,2)$ is not only one-fault tolerance, but has conditional two-fault tolerance. As long as the faulty edges are not adjacent to the same vertex in the same direction, that is, every vertex has at least one good edge from each side of the cycle, we then are able to use the same number of vertices as $\gamma_{3,2,1}(DL(n;1,2))$ to $D_{3,2,1}$-dominate the resulting graph.

**Corollary 1.** A double loop graph $DL(n;1,2)$ has two-fault tolerance with condition that the faulty-edges are not joined to the same vertex in the same direction.

**Theorem 5** A double loop graph $DL(n;1,3)$ is one-fault tolerance.

**Proof:** To show that $DL(n;1,3)$ is one-fault tolerance, we consider following cases according to the order of graph.

Case 1: $n \not\equiv 5k +4$ for some integer $k \geq 3$. Since the faulty edge may be either a 1-link or a 3-link, without loss of generality, let $(1,2)$ or $(1,4)$ be the faulty edge of $DL(n;1,3)$. Consider the optimal $D_{3,2,1}$-dominating set $D = \{S_i | 0 \leq i \leq [n/5] -1\}$ of $DL(n;1,3)$ for $n \equiv 5k + 4$. We show that $D$ is also a $D_{3,2,1}$-dominating set of $DL(n;1,3) – (1,2)$ and $DL(n;1,3) – (1,4)$. Notice that the vertices affected by the faulty edges are vertices $\{1,2,4\}$. Since without one of two edges $(1,2)$ and $(1,4)$, vertex 1 is distance one from vertex 0 and distance two from vertex 5. The distance between vertices 2 and 4 with vertices 0 and 5 is not changed. Hence $D$ is also a $D_{3,2,1}$-dominating set of $DL(n;1,3) – (1,2)$ and $DL(n;1,3) – (1,4)$, which implies that $DL(n;1,3)$ is one-fault tolerance in this case.

Case 2. $n = 5k +4$, and $k \geq 3$. Without lose of generality, let either $(9,10)$ or $(9,12)$ be the faulty edge of $DL(n;1,3)$. Consider the optimal $D_{3,2,1}$-dominating set $D = \{0, 2\} \cup \{S_i + 3 | 1 \leq i \leq [n/5] -1\}$ of $DL(n;1,3)$ for $n = 5k + 4$,
$k \geq 3$. We want to show that $D$ is also a $D_{3,2,1}$-dominating set of $DL(n;1,3)-(9,10)$ and $DL(n;1,3)-(9,12)$. Notice that the vertices affected by the faulty edges are vertices $\{9,10,12\}$. Since without one of two edges $(9,10)$ and $(9,12)$, vertex 9 is distance one from vertex 8 and distance two from vertex 13. The distance between vertices 10 and 12 with vertices 8 and 13 is not changed. Hence $D$ is also a $D_{3,2,1}$-dominating set of $DL(n;1,3)-(9,10)$ and $DL(n;1,3)-(9,12)$, which implies that $DL(n;1,3)$ is one-fault tolerance in this case.

Case 3. $n = 5k + 4$ and $1 \leq k \leq 2$. Without lose of generality, let either $(1, 2)$ or $(1, 4)$ be the faulty edge of $DL(n;1,3)$. Then consider the optimal $D_{3,2,1}$-dominating set $D = [n-2] \cup \{ i | 0 \leq i \leq [n/5]-1 \}$ of $DL(n;1,3)$, similar to case 1. $D$ is also a $D_{3,2,1}$-dominating set of $DL(n;1,3)-(1,2)$ and $DL(n;1,3)-(1,4)$. Hence $DL(n;1,3)$ is one-fault tolerance.

5 Conclusion

By giving different weight to each vertex according to the distance in the domination set, $D_{3,2,1}$-domination can not only represent the effect decreasing of the shared resource caused by distance but can also represent the decreasing of the influence on people by their close relationship. Since the edges represent the direct link or close relation between people, it is reasonable to consider the edge fault tolerance ability respect to $D_{3,2,1}$-domination. This paper shows that paths, and cycles have one-fault tolerance under certain condition and are critical for other cases. For complete full binary trees $B_n$ with height $n$, they all have zero-fault tolerance but only $B_2$ is critical. For double loop graphs, we corrected the $D_{3,2,1}$-dominating number given by [11] for $DL(9,1,3)$ and $DL(14,1,3)$ and proved that both $DL(n;1,2)$ and $DL(n;1,3)$ have one-fault tolerance respect to $D_{3,2,1}$-domination.

References