Mutually Independent Hamiltonian Cycle of Burnt Pancake Graphs

Yung-Ling Lai, Da-Chung Yu
Department of Computer Science and Information Engineering
National Chiayi University
{yllai, s0970400}@mail.ncyu.edu.tw

Abstract

Let $G=(V,E)$ be a graph of order n. A Hamiltonian cycle of G is a cycle contains every vertex in G. Two Hamiltonian cycles $C_1=\langle u_1,u_2,u_3,...u_n,u_1\rangle$ and $C_2=\langle v_1,v_2,v_3,...v_n,v_1\rangle$ of G are independent if $u_1=v_1$ and $u_i\neq v_i$ for $2\leq i\leq n$. A set of Hamiltonian cycles $\{C_1,C_2,C_3,...C_k\}$ of G is mutually independent if its elements are pairwise independent. The mutually independent hamiltonicity $IHC(G)$ of a graph G is the maximum integer k such that for any vertex u of G there are k mutually independent Hamiltonian cycles of G starting at u. For the n-dimensional burnt pancake graph B_n, this paper proved that $IHC(B_2)=1$ and $IHC(B_n)=n$ for $n\geq 3$

1 Introduction

The interconnection network, one crucial step on designing a massively multiprocessor system, is an important application of graph theory. Since Cayley graphs possess many favorable properties, such as node symmetry, recursive construction, and fault tolerance [1], they are suitable serving as the network topologies of massively multiprocessor systems. Let G be a finite group and H be its generating set such that $H=H^{-1}$ and $E=\{(v,vh)|v\in G, h\in H\}$, then a Cayley graph (G,E) is denoted as $Cay(G,H)$. The n-dimensional burnt pancake graph B_n, proposed by [4], is also a member of Cayley graphs. A burnt pancake graph, based on the signed permutations, can provide interconnection networks that consist of different number of nodes from others such as a star graph, a pancake graph, a rotator graph, and so on. Because the degree and diameter are smaller than a hypercube with similar number of vertices, the topology of the burnt pancake graphs is especially suitable for massively parallel systems. In particular, the burnt pancake graph can be used for the genome analysis [5]. Recently, many interesting properties of the burnt pancake graphs are studied (see [2], [3], [9], [10]). There are many mutually conflicting requirements in designing the topology for computer networks. Many studies on mutually independent Hamiltonian cycles on hypercube and other interconnection networks have been published (see [6], [7], [8], [11], [12], [13], [14], [15], [16], [17], [18], [19]). This paper proved that the n-dimensional burnt pancake graph B_n has n mutually independent Hamiltonian cycles when $n\geq 3$.

2 Preliminaries

In this section, we introduce signed permutations, prefix reversal operations, burnt pancake graphs, and mutually independent Hamiltonian cycles as well as other requisite definitions.

Let $\langle n\rangle$ denote the positive integer set $\{1,2,3,...n\}$. For the negative integers, we put the negative sign on the top of an expression. Thus, $-u=\bar{u}$ for some $u\in N$. Let $[n]$ denote the integer set $\langle n\rangle\cup\{\bar{i}|i\in\langle n\rangle\}$. A *signed permutation* of $\langle n\rangle$ is an n-permutation $u_1u_2...u_n$ of $[n]$ such that the set of absolute value of each element $\{|u_1|,|u_2|,...,|u_n|\}=\langle n\rangle$. For example, $12\bar{4}5\bar{3}$ is a signed permutation of $\langle 5\rangle$. For a signed permutation $u=u_1u_2...u_n$ of $\langle n\rangle$, the i-th *prefix reversal* of u, denoted by $(u)^i$, is $\overline{u_iu_{i-1}\cdots u_2u_1}u_{i+1}\cdots u_n$. For example, $(1\bar{4}3\bar{5}2)^3$ $=\bar{3}4\bar{1}\bar{5}2$.

The *n-dimensional burnt pancake graph* B_n is an n-regular graph with $2^n n!$ vertices each of which has a unique label from the signed permutation of $\langle n\rangle$. Two vertices u and v are adjacent in B_n if and only if $(u)^i=v$ for some $1\leq i\leq n$. We use $(u)_i$ to denote the i-th digit u_i of u and B_n^i to denote the i-th subgraph of B_n induced by those vertices u with $(u)_n=i$. A B_n can be decomposed into $2n$ disjoint subgraphs B_n^i for every $i \in [n]$ such that each B_n^i is isomorphic to B_{n-1}. For $1 \le i, j \le n$ and $i \ne j$, we use $E^{i,j}$ to denote the set of edges connecting B_n^i and B_n^j. Figure 2 shows B_i for $i \le 3$. Observe that in B_n^n, each vertex is labeled by appending digit n to the last digit of B_{n-1}. The subgraph indicated by the dotted circle in Figure 2 is B_3^3.

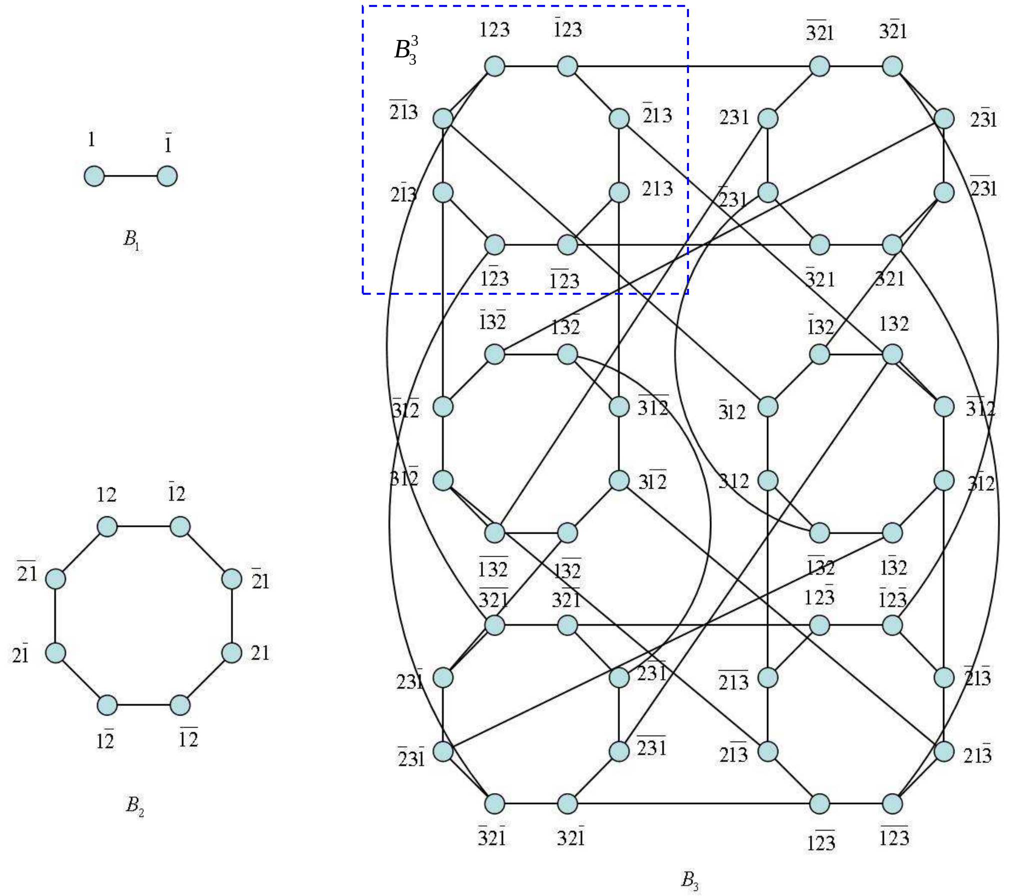


Figure 2: B_i for $i \le 3$

The *Hamiltonian path* of a graph $G = (V, E)$ is a path that traverses all vertices in G once. For any arbitrary pair of vertices in a graph *G*, if there is a Hamiltonian path between them, then *G* is called *Hamiltonian connected*. The *Hamiltonian cycle* of a graph $G = (V, E)$ is a cycle that traverses all vertices in G once. Without considering the edge bandwidth of G, one may be able to find some Hamiltonian cycles, which have the same origin, but each of the cycle does not arrive the same vertex at the same step. Two Hamiltonian cycles $C_1 = \{u_1, u_2, u_3, \ldots u_n, u_1\}$ and $C_2 = \{v_1, v_2, v_3, \ldots v_n, v_1\}$ of graph G are *independent* if that $u_1 = v_1$ and $u_i \ne v_i$ for $i \in \{2, 3, \ldots, n\}$. A set of Hamiltonian cycles $\{C_1, C_2, C_3, \ldots C_k\}$ of G is *mutually independent* if its elements are pairwise independent. The maximum number of mutually independent Hamiltonian cycles of G is denoted by $IHC(G)$.

3 Construction of Mutually Independent Hamiltonian Cycles

The following lemmas are very useful in our construction of the mutually independent Hamiltonian cycles of burnt pancake graphs.

For $H \subseteq [n]$, let B_n^H denote the subgraph of B_n induced by $\bigcup_{i \in H} V(B_n^i)$.

Lemma 1. (from [3]) *Let* $n \ge 4$ *and* $i_1, i_2, i_3, \ldots, i_m$ *be an m-permutation of* $[n]$ *such that* $i_k \ne \overline{i_{k+1}}$ *for* $1 \le k < m$. *Let H denote the set* $\{i_1, i_2, i_3, \ldots, i_m\}$. *Then there is a Hamiltonian* *path of* B_n^H *joining any vertex* $u \in V\left(B_n^{i_1}\right)$ *to any other vertex* $v \in V\left(B_n^{i_m}\right)$.

Let F be the set of the faulty elements, that is, the faulty nodes and the faulty edges in B_n. Lemma 2 is from [10]

Lemma 2. *Let* $n \geq 3$. *If* $|F| \leq n-2$, *then* $B_n - F$ *has a Hamiltonian cycle; if* $|F| \leq n-3$, $B_n - F$ *is Hamiltonian connected.*

Theorem 1. *Let* B_i *be a burnt pancake graph for* $2 \leq i \leq 3$, *then* $IHC(B_2)=1$, $IHC(B_3)=3$.

Proof Since B_2 is isomorphic to an even cycle C_8, $IHC(B_2)=1$. Since B_3 is vertex symmetric, without lose of generality, we may assume that the Hamiltonian cycles begin at the vertex (123). The three cycles C_1, C_2 and C_3 beginning at (123) are constructed specifically in Appendix A. Hence $IHC(B_3)=3$. ■

Theorem 2. *There are four mutually independent Hamiltonian cycles in burnt pancake* B_4. *That is* $IHC(B_4)=4$.

Proof. Using the mutually independent Hamiltonian cycles in B_3, we can construct four mutually independent Hamiltonian cycles in B_4. Since $B_4^4 \cong B_3$, by Theorem 1, there are three mutually independent Hamiltonian cycles in B_4^4. Let P_i for $1 \leq i \leq 3$ be the Hamilton path from $r=1234$ follow the same route as C_i in Theorem 1 to the vertices $\overline{321}4$, 2134 and $\overline{1}324$ respectively, which are shown in appendix B. Appendix C shows a Hamiltonian path P_4 of B_4^4 to make sure our new Hamiltonian cycle won't have collision in the last two vertices. Figure 3 shows the 4 mutually Hamiltonian cycles on B_4 we constructed by using Lemma 2 and P_i for $1 \leq i \leq 4$. In order to make the figure clearer, we use dark block to represent B_n^i for positive i. ■

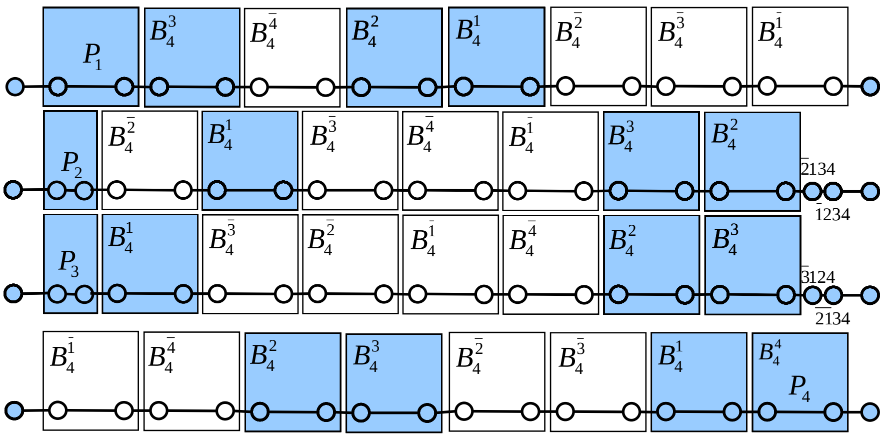


Figure 3. Four mutually independent Hamiltonian cycles in B_4

Theorem 3. *There are five mutually independent Hamiltonian cycles in burnt pancake* B_5. *That is* $IHC(B_5)=5$.

Proof Since $B_5^5 \cong B_4$, by Theorem 2, there are four mutually independent Hamiltonian cycles C_i, $1 \leq i \leq 4$ in B_5^5. Let P_i be the Hamiltonian path using C_i from $r=12345$, for $1 \leq i \leq 4$, to $\overline{4321}5$, $\overline{431}25$, $\overline{421}35$, and $\overline{13}245$, respectively. Note that because we use the previous cycles to construct the new set of mutually independent Hamiltonian cycles, when they arrived to $\overline{4321}5$, $\overline{431}25$ and $\overline{421}35$, they will enter the same subgraph $B_5^{\bar{4}}$, hence we have to find three mutually independent Hamiltonian paths in $B_5^{\bar{4}}$ connected to subgraphs B_5^2, B_5^3 and $B_5^{\bar{2}}$ respectively. The three mutually independent Hamiltonian paths in $B_5^{\bar{4}}$, denoted as Q_i, $1 \leq i \leq 3$, are shown in appendix D. By using P_i for $1 \leq i \leq 4$ with the Q_i for $1 \leq i \leq 3$, we can construct 5 mutually independent Hamiltonian cycles in B_5 as shown in figure 4. In order to avoid the collision of the last two cycles in $B_5^{\bar{1}}$, we have them follow two mutually independent paths in $B_5^{\bar{1}}$ as shown in appendix E. ■

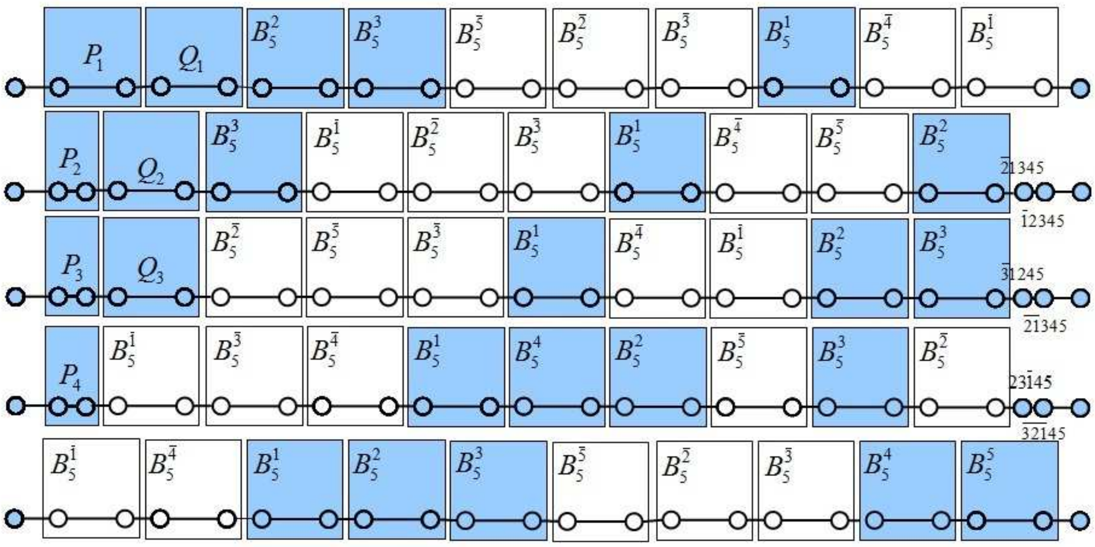
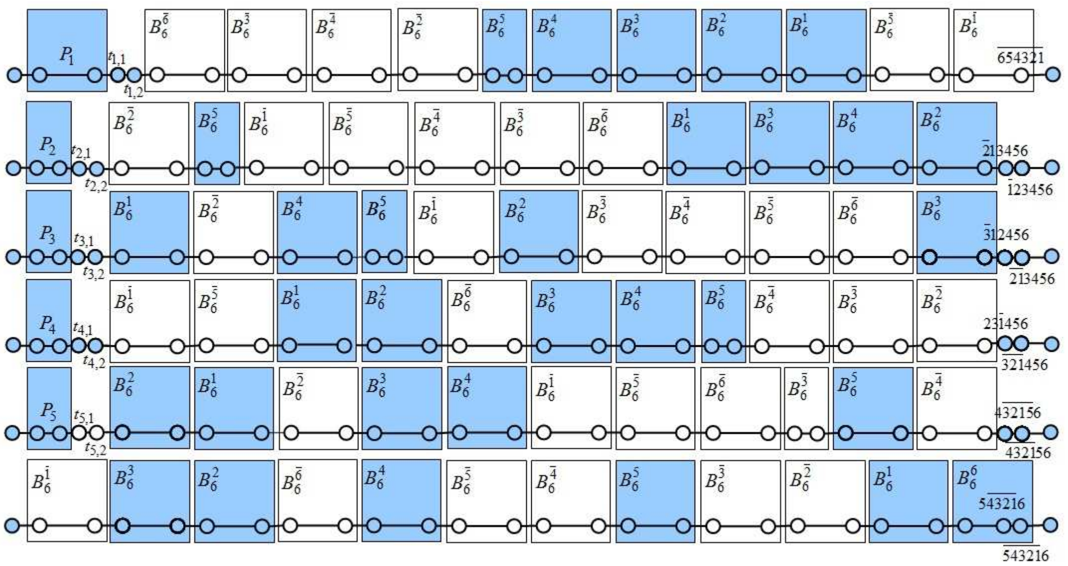


Figure 4: Five mutually independent Hamiltonian cycles in B_5

Theorem 4. *There are six mutually independent Hamiltonian cycles in burnt pancake burnt pancake* B_6. *That is* $IHC(B_6)=6$

Proof. Since $B_6^6 \cong B_5$, by Theorem 3, there are five mutually independent Hamiltonian cycles C_i, $1 \le i \le 5$ in B_6^6. Let P_i be a Hamiltonian path using C_i from $r=123456$, for $1 \le i \le 5$, to vertices $\overline{5432}16$, $\overline{5431}26$, $\overline{5421}36$, $\overline{54}\overline{13}26$ and $34\overline{2}156$ respectively. Similar to Theorem 3, when constructing mutually independent Hamiltonian cycles in B_6 passing vertices $\overline{5431}26$, $\overline{54321}6$, $\overline{5431}26$ and $\overline{54}1326$, they will enter the same subgraph B_6^5. Then for each path, we may use two vertices ($t_{i,1}$ and $t_{i,2}$ for $1 \le i \le 4$ as indicated in figure 5) in B_6^5 to connect to different subgraphs $B_6^{\bar 6}, B_6^{\bar 2}, B_6^{\bar 1}$, and $B_6^{\bar 1}$ respectively. For P_5 which enter $B_6^{\bar 3}$ first, let the path go through $t_{5,1}$ and $t_{5,2}$ as in figure 5 to B_6^2. Then the six mutually independent Hamiltonian cycles in B_6 may be constructed as in shown in Figure 5. ■



$t_{1,1}=\bar{6}12345$ $t_{2,1}=\overline{62}1345$ $t_{3,1}=\overline{63}1245$ $t_{4,1}=\bar{6}23\bar{1}45$ $t_{5,1}=\overline{65}12\overline{43}$

$t_{1,2}=612345$ $t_{2,2}=261345$ $t_{3,2}=\bar{1}36245$ $t_{4,2}=1\overline{32}645$ $t_{5,2}=\overline{21}56\overline{43}$

Figure 5: Six mutually independent Hamiltonian cycles in B_6

Theorem 5. *For $n \geq 7$, the burnt pancake graph B_n has n mutually independent Hamiltonian cycles. That is $IHC(B_n) = n$.*

Proof Let $F \subseteq V\left(B_n^t\right)$, $|F| \leq 3$, for some $t \in [n]$, by Lemma 2, for any two vertices $u, v \in B_n^t - F$, there is a Hamiltonian path from u to v. Let $r = 123...n$. The first Hamiltonian cycle C_1 of B_n may be constructed as follows:

Let x, y be two vertices of $B_n^n - \left\{(r),(r)^1\right\}$ such that $(x)_1 = \overline{n-1}$, $(y)_1 = 1$. By Lemma 2, there is a $x-y$ Hamiltonian path H_1 in $B_n^n - \left\{(r),(r)^1\right\}$. Let z be a vertex of $B_n^{\bar{1}} - \left\{(r)^n,((r)^n)^1\right\}$ such that $(z)_1 = 2$. Since $(y)^n$ is the vertex of $B_n^{\bar{1}}$, by Lemma 2 there is a $(y)^n - z$ Hamiltonian path H_2 in $B_n^{\bar{1}} - \left\{(r)^n,((r)^n)^1\right\}$. Note that $((r)^1)^n$ is the vertex of B_n^1 and $(x)^n$ is the vertex of B_n^{n-1}, by Lemma 1 there is a Hamiltonian path P_1 of $\bigcup_{s=1}^{n-1} B_n^s$ joining vertex $((r)^1)^n$ to vertex $(x)^n$. Since $(z)^n$ is the vertex of $B_n^{\bar{2}}$ and $((r^n)^1)^n$ is the vertex of $B_n^{\bar{n}}$, by Lemma 1, there is a Hamiltonian path Q_1 of $\bigcup_{t=\bar{2}}^{\bar{n}} B_n^t$ joining vertex $(z)^n$ to vertex $((r^n)^1)^n$. Then $C_1 = r, r^1, P_1, H_1, H_2, Q_1, (r^n)^1, r^n, r$ is a Hamiltonian cycle of B_n.

Next we construct the Hamiltonian cycles C_2 to C_{n-1} as follows. For each $2 \leq k \leq n-1$, let x, y be two vertices of $B_n^n - \left\{r, r^k, r^{k-1}\right\}$, such that $(x)_1 = \overline{n-1}$ and $(y)_1 = 1$. By Lemma 2, there is a $x-y$ Hamiltonian path H_k in $B_n^n - \left\{r, r^k, r^{k-1}\right\}$. Since that $(x)^n$ is a vertex of B_n^{n-1} and $(r^k)^n$ is a vertex of B_n^k, by Lemma 1 there is a Hamiltonian path P_k of $\bigcup_{s=k}^{n-1} B_n^s$ joining vertex $(r^k)^n$ to vertex $(x)^n$. Let w be a vertex of $B_n^{\bar{n}}$, such that $(w)_1 = \bar{1}$. Note that $(y)^n$ is a vertex of $B_n^{\bar{1}}$, by Lemma 1 there is a Hamiltonian path Q_k of $\bigcup_{t=\bar{n}}^{\bar{1}} B_n^t$ joining vertex $(y)^n$ to vertex w. Since that $(w)^n$ is a vertex of B_n^1, by Lemma 2 there is Hamiltonian path R_k of $\bigcup_{v=1}^{n-2} B_n^v$ joining vertex $(w)^n$ to vertex $(r)^{n-2}$. Then $C_k = r, r^k, P_k, H_k, Q_k, R_k, r$ is a Hamiltonian cycle of B_n

For the last Hamiltonian cycle C_n, we constructed as follow. Let m be a vertex of $B_n^{\bar{n}}$, such that $(m)_1 = \bar{1}$. Since that $(r)^n$ is a vertex of $B_n^{\bar{1}}$, by Lemma 1 there is a Hamiltonian path P_n of $\bigcup_{t=\bar{n}}^{\bar{1}} B_n^t$ joining vertex $(r)^n$ to vertex m. Note that $(m)^n$ is a vertex of B_n^1 and $(r)^{n-1}$ is a vertex of B_n^n, by Lemma 1 there is a Hamiltonian path Q_n of $\bigcup_{s=1}^{n} B_n^s$ joining vertex $(m)^n$ to vertex $(r)^{n-1}$. Then $C_k = r, P_n, Q_n, r$ is a Hamiltonian cycle of B_n. The k Hamiltonian cycles of B_n are shown in figure 5. ■

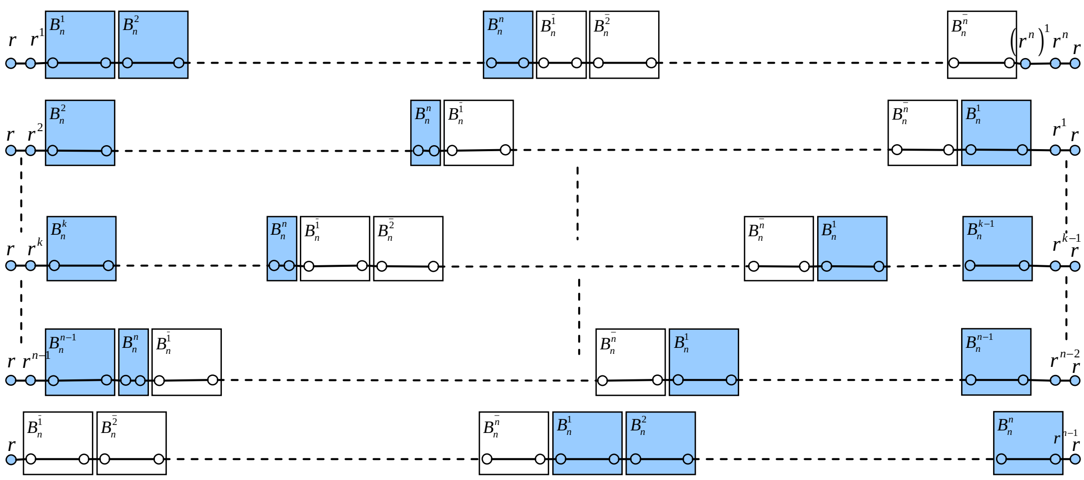


Figure 5. n mutually independent Hamiltonian cycles in B_n, $n \geq 7$.

4 Conclusion

In this paper, we proved that the Burnt Pancake B_n has n mutually independent Hamiltonian cycles for any integer $n \geq 3$

References

[1] Sheldon B. Akers and Balakrishnan Krishnamurthy, “A group theoretic model for symmetric interconnection networks,” *IEEE Trans, Comput*, Vol.38(4), pp.555-566, 1989.

[2] David S. Cohen and Manuel Blum, “On the problem of sorting burnt pancakes”, *Discrete Applied Mathematics*, Vol.61(2), pp.105-120, 1995.

[3] Chereng Chin, Tien-Hsiung Weng, Lih-Hsing Hsu and Sheng-Chia Chiou. “The Spanning Connectivity of the Burnt Pancake Graphs” *ICICE Trans. Inf. & Syst.*, Vol.E92-D(3), pp.389-400, 2009.

[4] William H. Gates and Christos H. Papadimitriou, “Bounds for sorting by prefix reversal,” *Discrete Mathematics*, Vol.27, pp. 47-57, 1979.

[5] Qian-Ping Gu, Shietung Peng, and Ivan Hal Sudborough, “A 2-approximation algorithm for genome rearrangements by reversals and transpositions,” *Theoretical Computer Science*, Vol.210(2), pp.327–339, 1999.

[6] Mohammad Hossain Heydari and Ivan Hal Sudborough, “On the diameter of the pancake network”, *Journal of Algorithms*, Vol.25(1) , pp.67-94, 1997.

[7] Chun-Nan Hung, Hong-Chun Hsu, Kao-Yung Liang and Lih-Hsing Hsu, “Ring embedding in faulty pancake graphs”, *Information Processing Letters*, Vol.86(5), pp.271–275, 2003.

[8] Chun-Nan Hung, Kao-Yung Liang and Lih-Hsing Hsu, “Embedding Hamiltonian paths and Hamiltonian Cycles in Faulty Pancake Graphs”, *International Symposium on Parallel Architectures, Algorithms and Networks*, pp.179-184, 2002.

[9] Keiichi Kaneko, “An Algorithm for Node-to-Set Disjoint Paths Problem in Burnt Pancake Graphs” *IEICE Trans. Inf & Syst.*, Vol.E86-D(12), 2003.

[10] Keiichi Kaneko, ”Hamiltonian cycles and Hamiltonian paths in faulty brunt pancake graphs”, *IEICE Trans. Inf. & Syst.*, Vol.E90-D(4), pp.716-721, 2007.

[11] Keiichi Kaneko, Shietung Peng, “Disjoint Paths Routing in Pancake Graphs”, *Seventh International Conference on Parallel and Distributed Computing, Applications and Technologies*, pp.254-259, 2006.

[12] Tz-Liang Kueng, Tyne Liang and Lih-Hsing Hsu, “Mutually independent Hamiltonian cycles of binary wrapped butterfly graphs” , Mathemetical and Computer Modeling, Vol48(11-12), pp.1814-1825, 2008

[13] Cheng-Kuan Lin, Hua-Min Huang, Lih-Hsing Hsu and Sheng Bau, ”Mutually Independent Hamiltonian Paths in Star Networks” *Networks*, Vol.46(2), pp.110-117, 2005.

[14] Cheng-Kuan Lin, Hua-Min Huang, Jimmy J. M. Tan and Lih-Hsing Hsu. “Mutually Independent Hamiltonian Cycles of Pancake Networks” *23rd Workshop on Combinatorial Mathematics and Computation Theory*, pp.46-52, 2006.

[15] Cheng-Kuan Lin, Yuan-Kang Shih, Jimmy J. M. Tan, Lih-Hsing Hsu, “Mutually independent Hamiltonian cycles in some graphs” , *Ars Combinatoria, accepted*, 2009.

[16] Cheng-Kuan Lin, Jimmy J.M. Tan, Hua-Min Huang, D. Frank Hsu and Lih-Hsing Hsu,” Mutually independent hamiltonian cycles for the pancake graphs and the star graphs” *Discrete Mathematics*, Vol.309(17), pp. 5474-5483, 2009.

[17] Naoki Sawada, Keiichi Kaneko and Shietung Peng, ”Pairwise disjoint paths in pancake graphs”, *Eighth International Conference on Parallel and Distributed Computing, Applications and Technologies*, pp.376-382, 2007.

[18] Chao-Ming Sun, Cheng-Kuan Lin, Hua-Min Huang, Lih-Hsing Hsu. “Mutually independent Hamiltonian paths and cycles in hypercubes”, *Journal of Interconnection Networks*, Vol.7(2), pp.235 -255 , 2006.

[19] Pi-Hsiang Wang, *Mutually independent hamiltonian cycles in k-ary n-cube when k is odd*, Master Thesis, Chung Yuan Christian University, 2009.

Appendix A: The three required cycles of Theorem 1

$C_1 = 123, \bar{1}23, \bar{2}13, 213, \overline{12}3, 1\bar{2}3, 2\overline{13}, \overline{21}3, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2}, \overline{312}, 3\overline{12}, 1\overline{32}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, \overline{231}, 2\overline{31},$ $3\overline{21}, 12\bar{3}, \bar{1}2\bar{3}, \bar{2}1\bar{3}, 21\bar{3}, \overline{123}, 1\overline{23}, 2\overline{13}, \overline{213}, \overline{13}2, 312, \bar{2}31, \bar{3}21, 321, \overline{23}1, 2\bar{3}1, 3\bar{2}1, \overline{32}1, 231,$ $\overline{132}, 31\bar{2}, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2}, \overline{312}, 3\overline{12}, 1\overline{32}, 23\bar{1}, \overline{321}, 123$
$C_2 = 123, \overline{21}3, 2\bar{1}3, 1\bar{2}3, \bar{3}2\bar{1}, \bar{2}3\bar{1}, \bar{2}3\bar{1}, \overline{321}, 3\overline{21}, 2\overline{31}, \overline{231}, 32\bar{1}, 1\overline{23}, 2\overline{13}, 31\bar{2}, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2}, \overline{312},$ $3\overline{12}, 1\overline{32}, \overline{132}$ $231, \overline{32}1, 3\bar{2}1, 2\bar{3}1, \overline{23}1, 321, \overline{123}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, \overline{213}, 312, \bar{3}12, \bar{1}32, 132,$ $\overline{31}2$ $3\bar{1}2, 1\bar{3}2, \overline{13}2, \bar{2}31, \bar{3}21, \overline{12}3, 213, \bar{2}13, \bar{1}23, 123$
$C_3 = 123, \overline{321}, 3\overline{21}, 2\overline{31}, \overline{231}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1}, 1\overline{32}, 31\bar{2}, \overline{312}, 13\bar{2}, \bar{1}3\bar{2}, \bar{3}1\bar{2}, 2\bar{1}3, 1\bar{2}3, \overline{12}3, 213,$ $\bar{2}13, \bar{1}23, \overline{32}1, 3\bar{2}1, 2\bar{3}1, \overline{23}1, 321, \bar{3}21, \bar{2}31, 231, \overline{132}, 1\overline{32}, 2\overline{13}, 1\overline{23}, \overline{123}, 21\bar{3}, , \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3},$ $\overline{213}$ $312, \overline{13}2, 1\bar{3}2, 3\bar{1}2, \overline{31}2, 132, \bar{1}32, \bar{3}12, \overline{21}3, 123$

Appendix B: The three required paths of Theorem 2

$P_1 = 1234, \bar{1}234, \bar{2}134, 2134, \overline{12}34, 1\bar{2}34, 2\overline{13}4, \overline{21}34, \bar{3}1\bar{2}4, \bar{1}3\bar{2}4, 13\bar{2}4, \overline{312}4, 3\overline{12}4, 1\overline{32}4, \bar{2}3\bar{1}4, \bar{3}2\bar{1}4, 32\bar{1}4,$ $\overline{231}4, 2\overline{31}4, 3\overline{21}4, 12\bar{3}4, \bar{1}2\bar{3}4, \bar{2}1\bar{3}4, 21\bar{3}4, \overline{123}4, 1\overline{23}4, 2\overline{13}4, \overline{213}4, \overline{13}24, 3124, \bar{2}314, \bar{3}214, 3214,$ $\overline{23}14, 2\bar{3}14, 3\bar{2}14, \overline{32}14, 2314$ $\overline{132}4, 31\bar{2}4, \bar{3}1\bar{2}4, \bar{1}3\bar{2}4, 13\bar{2}4, \overline{312}4, 3\overline{12}4, 1\overline{32}4, 23\bar{1}4, \overline{321}4$
$P_2 = 1234, \overline{21}34, 2\bar{1}34, 1\bar{2}34, \bar{3}2\bar{1}4, \bar{2}3\bar{1}4, \bar{2}3\bar{1}4, \overline{321}4, 3\overline{21}4, 2\overline{31}4, \overline{231}4, 32\bar{1}4, 1\overline{23}4, 2\overline{13}4, 31\bar{2}4, \bar{3}1\bar{2}4, \bar{1}3\bar{2}4,$ $13\bar{2}4, \overline{312}4$ $3\overline{12}4, 1\overline{32}4, \overline{132}4$ $2314, \overline{32}14, 3\bar{2}14, 2\bar{3}14, \overline{23}14, 3214, \overline{123}4, 21\bar{3}4, \bar{2}1\bar{3}4, \bar{1}2\bar{3}4, 12\bar{3}4, \overline{213}4,$ $3124, \bar{3}124, \bar{1}324, 1324, \overline{31}24$ $3\bar{1}24, 1\bar{3}24, \overline{13}24, \bar{2}314, \bar{3}214, \overline{12}34, 2134$
$P_3 = 1234, \overline{321}4, 3\overline{21}4, 2\overline{314}, \overline{231}4, 32\bar{1}4, \bar{3}2\bar{1}4, \bar{2}3\bar{1}4, 23\bar{1}4, 1\overline{32}4, 31\bar{2}4, \overline{312}4, 1\overline{32}4, \bar{1}3\bar{2}4, \bar{3}1\bar{2}4, 2\bar{1}34, 1\bar{2}34,$ $\bar{2}134, \bar{1}234, \overline{32}14, 3\bar{2}14, 2\bar{3}14, \overline{23}14, 3214, \bar{3}214, \bar{2}314, 2314, \overline{132}4, 1\overline{32}4, 2\overline{13}4, 1\overline{23}4, \overline{123}4,$ $21\bar{3}4, \bar{2}1\bar{3}4, \bar{1}2\bar{3}4, 12\bar{3}4, \overline{213}4, 3124, \overline{13}24, 1\bar{3}24, 3\bar{1}24, \overline{31}24, 1324, \bar{1}324$

Appendix C: A required path in $B_4^4 - 1234$ of Theorem 2

$P_4 = \bar{1}234, \bar{2}134, 2134, \overline{12}34, 1\bar{2}34, 2\overline{13}4, \overline{21}34, \bar{3}1\bar{2}4, \bar{1}3\bar{2}4, 13\bar{2}4, \overline{312}4, 3\overline{12}4, 1\overline{32}4, \bar{2}3\bar{1}4, \bar{3}2\bar{1}4, 32\bar{1}4,$ $\overline{231}4, 2\overline{31}4, 3\overline{21}4, 12\bar{3}4, \bar{1}2\bar{3}4, \bar{2}1\bar{3}4, 21\bar{3}4, \overline{123}4, 1\overline{23}4, 2\overline{13}4, \overline{213}4, \overline{13}24, 3124, \bar{2}314, \bar{3}214, 3214,$ $\overline{132}4, 31\bar{2}4, \bar{3}1\bar{2}4, \bar{1}3\bar{2}4, 13\bar{2}4, \overline{312}4, 3\overline{12}4, 1\overline{32}4, 23\bar{1}4, \overline{321}4$

Appendix D: The three required paths of Theorem 3

We use S_i to denote B_5^4's subgraph for $i \in [1,2,3,5]$. Three mutually independent paths used in Theorem 3 are as follows:

$Q_1 = S_3, S_{\bar{2}}, S_5, S_2, S_{\bar{3}}, S_1, S_{\bar{5}}, S_{\bar{1}}, \bar{2}35\bar{1}4$
$Q_2 = S_3, S_2, S_{\bar{5}}, S_{\bar{2}}, S_5, S_{\bar{1}}, S_{\bar{3}}, S_1, \bar{3}2514$
$Q_3 = S_2, S_1, S_3, S_{\bar{1}}, S_{\bar{5}}, S_{\bar{2}}, S_5, S_3, 21\bar{5}34$

Appendix E: Two required subgraph sequences in $B_5^{\bar{1}}$ of Theorem 3

We use S_i to denote $B_5^{\bar{1}}$'s subgraph for $i \in [2,3,4,5]$. The two subgraph sequences are as follows:

$S_3, S_4, S_5, S_{\bar{2}}, S_{\bar{3}}, S_{\bar{4}}, S_{\bar{5}}, S_2$
$S_{\bar{2}}, S_{\bar{3}}, S_{\bar{4}}, S_{\bar{5}}, S_2, S_3, S_4, S_5$