The two-equal-disjoint path cover problem of the hierarchical crossed cube *

Chih-Min Chien[1], Jheng-Cheng Chen[1], Pao-Lien Lai[1]†, Cheng-Hsuing Tsai[1], and Hong-Chun Hsu[2]
[1]Department of Computer Science and Information Engineering
National Dong Hwa University, Shoufeng, Hualien, Taiwan 97401, R.O.C.
[2]Department of Medical Informatics
Tzu Chi University, Hualien, Taiwan 970, R.O.C.

Abstract

To construct parallel paths among nodes in interconnection networks is an important issue concerned with efficient data transmission. Embedding of paths have attracted much attention in the parallel processing. A graph G is globally two-equal-disjoint path coverable ($GTEDPC$) if for any two distinct pairs of vertices a, b and c, d of G, there exist two disjoint paths P and Q satisfy that (1) *P (Q, respectively) joins a and b (c and d, respectively),* (2) *$|P| = |Q|$, and* (3) *$V(P \cup Q) = V(G)$. The hierarchical crossed cube is a new hierarchical interconnection network. In this paper, we study the globally two-equal-disjoint path cover property of $HCC(k, n)$ for $k \geq 1$ and $n \geq 5$.*

Keywords: hierarchical crossed cube, globally two-equal-disjoint path coverable, GTEDPC, Hamiltonian path.

1 Introduction

Recently, hierarchical interconnection networks have attracted lot of concern. There are many research about the hierarchical structure[7, 10, 12, 20, 21]. A new hierarchical interconnection network, the hierarchical crossed cube $HCC(k, n)$, was proposed in [8]. The hierarchical crossed cube draws upon constructions used within the well-known hypercube[18] and also the crossed cube (a variation of the hypercube as proposed by Efe [4, 5]). The hierarchical crossed cube has many advantages such as lower degree, smaller diameter, and maximum fault tolerance in same number of vertices. Usually, the interconnection network is represented as a graph whose vertices represent the nodes (i.e., processors) of the network and whose edges represent the communication links of the network. In this paper, we use standard terminology in graphs[1].

The graph-theoretic properties of interconnection networks have been investigated with their applications in parallel computing. However, finding parallel paths among nodes in interconnection networks is one of the important problems concerned with efficient data transmission. Parallel paths are usually studied in terms of disjoint paths in graphs. A set of paths in G is called *disjoint* if they do not share any vertices. To construct disjoint paths in interconnection networks is important since they can be used to increase the transmission rate and enhance the transmission reliability. Moreover, applications of disjoint paths have been researched in some fields such as multipath routing[17] and fault tolerance[6]. Depending on the number of source vertices or destination vertices, there are one-to-one[2, 13], one-to-many[3], and many-to-many disjoint path problems[14, 15, 16]. However, the many-to-many disjoint path problem is the most generalized one. The existence of the globally two-equal-disjoint path cover in a network implies there are two disjoint paths for any two source-destination pairs. Clearly, the globally two-equal-disjoint path cover problem is a specialized many-to-many disjoint path problem.

In this paper, we study the globally two-equal-disjoint path cover property of hierarchical crossed

*This work was supported in part by the National Science Council of Republic of China under NSC 97-2221-E-259-020.

†Correspondence to: Assistant Professor Pao-Lien Lai, Department of Computer Science and Information Engineering, National Dong Hwa University, Shoufeng, Hualien, Taiwan 97401, R.O.C. Fax: 886-3-8634053 e-mail: baolein@mail.ndhu.edu.tw.

cube. In next section, we give the definitions of two-equal-disjoint path cover property and hierarchical crossed cube. Then we discuss the main contribution in section 3. In the final section, we give the conclusion.

2 Preliminary

Let $G = (V, E)$ be an graph if V is a finite set and E is a subset of $\{(u, v)|(u, v)$ is an unordered pair of $V\}$. We say V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. A path in a graph is denoted by a sequence of distinct vertices $\langle v_0, v_1, v_2, ..., v_n\rangle$, where v_i and v_{i+1} are adjacent for $0 \leq i \leq n-1$. We use $\langle v_0, P, v_n\rangle$ or $P_{(v_0,v_n)}$ to represent a path $P = \langle v_0, v_1, v_2, ..., v_n\rangle$. For convenience, we use $\bar{a}$ to represent the complement of binary string a. The length of P denoted by $|P|$ is the number of edges in P. A path is a *Hamiltonian path* if its vertices are distinct and span V. A graph G is *Hamiltonian connected* if there exists a Hamiltonian path joining any two distinct vertices. A graph G is *Hamiltonian laceable* if G is bipartite and it has a Hamiltonian path $P_{(u,v)}$ for any pair of vertices u and v, where u belongs to one set of the bipartition and v to the other.

Definition 1 *Let graph G have $2k$ vertices, $k \geq 2$, and let a, b, c, and d be four distinct vertices of G. G is (a, b, c, d)-two-equal-disjoint path coverable if there are two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$ with $|P_{(a,b)}| = |Q_{(c,d)}| = k-1$.*

Definition 2 *A graph G is globally two-equal-disjoint path coverable ($GTEDPC$) if for any four distinct vertices a, b, c, and d, G is (a, b, c, d)-two-equal-disjoint path coverable.*

As we shall see, the construction of $HCC(k, n)$ is built around those of hypercubes Q_n and crossed cubes CQ_n. Now we define the hierarchical crossed cubes as follows:

Definition 3 *Fix k, $n \geq 1$. The hierarchical crossed cube has vertex set $\{0,1\}^{k+2n}$. Each vertex of $HCC(k, n)$ is written as (u, v, w), where $u \in \{0,1\}^k$ and v, $w \in \{0,1\}^n$. The set of edges of $HCC(k, n)$ is partitioned into 2 sets, E_{int} and E_{ext}. The set E_{int} is referred to as the set of internal edges, while the set E_{ext} is referred to as the set of external edges. In more detail,*
$E_{int} = \{((u, v, w), (u, v, w')) : (w, w')$ is an edge of CQ_n.$\}$ and
$E_{ext} = \{((u, v, w), (u', w, v)) : (u, u')$ is an edge of Q_n.$\}$

The hierarchical crossed cube $HCC(k, n)$ is formed by taking 2^{k+n} disjoint copies of CQ_n, with $CQ_n(u, v)$ denoting the copy of CQ_n on the set of vertices $\{(u, v, w) : w \in \{0,1\}^n\}$ (the edges of these copies of CQ_n form the internal edges). The vertices in these copies of CQ_n are then joined by additional edges (the external edges) whereby the vertices are partitioned into 2^{2n} sets of 2^k vertices, with each set of 2^k vertices joined by edges to form a copy of Q_k. Consequently, edges lie in the *internal layer* or the *external layer*. Clearly, $HCC(k, n)$ has 2^{k+2n} vertices, $n2^{k+2n-1}$ internal edges, and $k2^{k+2n-1}$ external edges, making $(n + k)2^{k+2n-1}$ edges in total.

For each vertex a of $HCC(k, n)$, we often label it as (u_a, v_a, w_a) in this paper. Moreover, for $k \geq 1$ and $i \in \{0, 1\}$, denoted by $H_{k-1}(i)$, the subgraph of $HCC(k, n)$ induced by the vertices of $\left\{(iu, v, w) : u \in \{0,1\}^{k-1}, v, w \in \{0,1\}^n\right\}$. Note that for $k \geq 2$, $H_{k-1}(0)$ and $H_{k-1}(1)$ are isomorphic to $HCC(k-1, n)$; for k=1, $H_0(0)$ and $H_0(1)$ are two disjoint sets of CQ_ns. For each vertex a in $H_k(i)$, $i \in \{0, 1\}$, we use a'' to represent the neighbor of a in $H_k(\bar{i})$ in this paper.

Consider each subgraph $CQ_n(u, v)$ as a super vertex, then we can view $HCC(1, n)$ as a complete bipartite graph $K_{2^n,2^n}$ of super vertices. Herein, the super vertex of $CQ_n(u, v)$ is labeled by uv. A Hamiltonian path consists of super vertices of $HCC(1, n)$ is called a *super Hamiltonian path*.

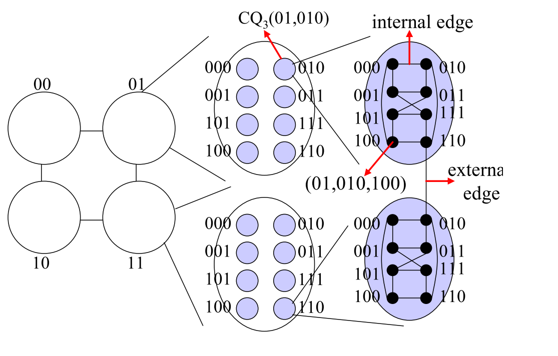

Figure 1: Structure of $HCC(2, 3)$.

3 $HCC(k,n)$ is globally two-equal-disjoint path coverable

In this section, we will first focus on the globally two-equal-disjoint path cover property of $HCC(1,n)$ as Theorem 1 for $n \geq 5$. Then we will use induction to prove $HCC(k,n)$ is globally two-equal-disjoint path coverable (GTEDPC) as Theorem 2 for $k \geq 1$ and $n \geq 5$. The following lemmas are very important to establish our base case, Theorem 1, of Theorem 2.

Lemma 1 [9] *CQ_n is $(n-3)$ fault Hamiltonian connected.*

Let $a = (u, v_a, w)$ and $b = (u, v_b, w)$ be two distinct vertices of $HCC(1,n)$ for $n \geq 3$. Since there exists a Hamiltonian path joining $(\bar{u}, w, v_a)$ and $(\bar{u}, w, v_b)$ by Lemma 1, we can obtain a path $P_{(a,b)}$ traverses every vertex of $V(CQ_n(\bar{u}, w)) \cup \{a, b\}$ exactly once and have the Observation 1 as follows. (See Fig. 2)

Observation 1 *For any two vertices $a = (u, v_a, w)$ and $b = (u, v_b, w)$ of $HCC(1,n)$ for $n \geq 3$, there exists a path $P_{(a,b)}$ traverses every vertex of $V(CQ_n(\bar{u}, w)) \cup \{a, b\}$ exactly once.*

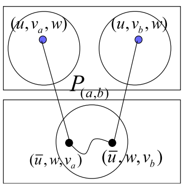

Figure 2: Illustration of Observation 1.

Lemma 2 [19] *A bipartite graph is Hamiltonian laceable if it is equitable and whenever x and y are two vertices from different partite set, there exists a Hamiltonian path P joining x and y.*

Lemma 3 [11] *The crossed cube CQ_n is globally two-equal-disjoint path coverable for $n \geq 5$.*

In the following discussion, we prefer using $CQ_n(u, p_i)$ and $CQ_n(\bar{u}, q_j)$ to denote two CQ_ns in $H_0(u)$ and $H_0(\bar{u})$, respectively, where $1 \leq i, j \leq t$ for some integer t, $p_i \in \{0,1\}^n$, $q_j \in \{0,1\}^n$, and $u \in \{0,1\}$. Moreover, assume that $p_i \neq p_j$ and $q_i \neq q_j$ if $i \neq j$.

Lemma 4 *Let $1 \leq t \leq 2^n$, $1 \leq i, j \leq t$ and let H be a subgraph of $HCC(1,n)$, $n \geq 5$, consists of $2 \times t \times 2^n$ vertices of $V(CQ_n(u, p_i)) \cup V(CQ_n(\bar{u}, q_j))$. For two arbitrary vertices $a \in CQ_n(u, v)$ and $b \in CQ_n(\bar{u}, v')$ in H, there exists a Hamiltonian path of H joining a and b.*

Proof. Consider each CQ_n of H as a super vertex, by Lemma 2, then there exists a super Hamiltonian path P' of H joining uv and $\bar{u}v'$. Without loss of generality, let $v = p_1$ and $v' = q_t$ and let $P' = \langle up_1, \bar{u}q_1, up_2, \bar{u}q_2, ..., up_t, \bar{u}q_t \rangle$ (See Fig. 3). Then we back to our work in terms of CQ_n. By Lemma 1, there exists a Hamiltonian path P_i of $CQ_n(u, p_i)$ and a Hamiltonian path Q_i of $CQ_n(\bar{u}, q_j)$ for $1 \leq i, j \leq t$. Therefore, we can obtain a Hamiltonian path P of H joining a and b that $P = \langle a, P_1, Q_1, P_2, Q_2, ..., P_t, Q_t, b \rangle$. $\square$

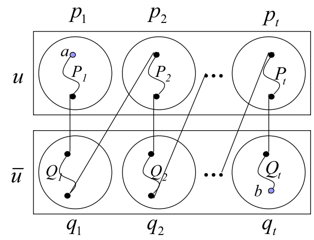

Figure 3: Illustration of Lemma 4.

Lemma 5 *Let $2 \leq t \leq 2^n$, $1 \leq i, j \leq t$ and let H be a subgraph of $HCC(1,n)$, $n \geq 5$, consists of $2 \times t \times 2^n$ vertices of $V(CQ_n(u, p_i)) \cup V(CQ_n(\bar{u}, q_j))$. For two arbitrary vertices $a, b \in CQ_n(u, v)$ in H, there exists a Hamiltonian path of H joining a and b.*

Proof. Without loss of generality, let $v = p_1$ and let x and y be two vertices in $CQ_n(u, p_1) - \{a, b\}$ with w_x, $w_y \in \{p_i, q_j\}$ for some i and j. Without loss of generality, let $w_x = q_1$ and $w_y = q_t$. By Lemma 3, there exist two-equal-disjoint paths P_1^1 joining a and x and P_1^2 joining b and y in $CQ_n(u, p_1)$ (See Fig. 4). Then let $z = (u, p_2, q_1)$, by Observation 1, there exists a path $P_{(x,z)}$ traverses every vertex of $V(CQ_n(\bar{u}, q_1)) \cup \{x, z\}$ exactly once. Then, let $H - CQ_n(u, p_1) \cup CQ_n(\bar{u}, q_1)$ be a subgraph of H called H'. By Lemma 4, there exists a Hamiltonian path P' of H' joining z and y''. As a result, we can obtain a Hamiltonian path P of H joining a and b as $P = \langle a, P_1^1, x, P_{(x,z)}, z, P', y'', y, P_1^2, b \rangle$. $\square$

Lemma 6 *Let $3 \leq t \leq 2^n$, $1 \leq i, j \leq t$, and let H be a subgraph of $HCC(1,n)$, $n \geq 5$, consists of $2 \times t \times 2^n$ vertices of $V(CQ_n(u, p_i)) \cup V(CQ_n(\bar{u}, q_j))$. For two arbitrary vertices $a \in$*

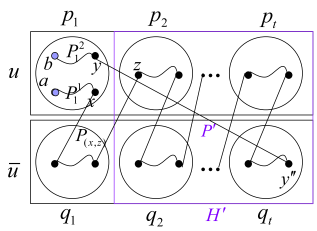

Figure 4: Illustration of Lemma 5.

$CQ_n(u,v)$ and $b \in CQ_n(u,v')$ in H, there exists a Hamiltonian path of H joining a and b.

Proof. Without loss of generality, let $v = p_1$ and $v' = p_3$. let x, y, and z be three vertices in $CQ_n(u,p_1) - a$ with w_x, w_y, $w_z \in \{p_i, q_j\}$ for some i and j. Without loss of generality, let $w_x = q_t$, $w_y = q_2$, and $w_z = q_1$. By Lemma 3, there exist two-equal-disjoint paths P_1^1 joining a and z and P_1^2 joining x and y in $CQ_n(u,p_1)$ (See Fig.5). Let $s = (u,p_2,q_1)$ and $w = (u,p_2,q_2)$ be two distinct vertices, then by Observation 1, there exist two paths $P_{(z,s)}$ traverses every vertex of $V(CQ_n(\bar{u},q_1)) \cup \{z,s\}$ exactly once and $P_{(w,y)}$ traverses every vertex of $V(CQ_n(\bar{u},q_2)) \cup \{w,y\}$ exactly once. By Lemma 1, there exists a Hamiltonian path $P_{(s,w)}$ of $CQ_n(u,p_2)$. Then, let $H - CQ_n(u,p_1) \cup CQ_n(u,p_2) \cup CQ_n(\bar{u},q_1) \cup CQ_n(\bar{u},q_2)$ be a subgraph of H called H'. By Lemma 4, we can obtain a Hamiltonian path P' of H' joining x'' and b. Therefore, there exists a Hamiltonian path P of H joining a and b with $P = \langle a, P_1^1, z, P_{(z,s)}, s, P_{(s,w)}, w, P_{(w,y)}, y, P_1^2, x, x'', P', b\rangle$. □

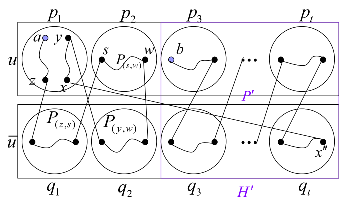

Figure 5: Illustration of Lemma 6.

Theorem 1 *$HCC(1,n)$ is globally two-equal-disjoint path coverable for $n \geq 5$.*

Proof. Let a, b and c, d be two distinct source-destination pairs of $HCC(1,n)$. We establish two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$. Furthermore, the two disjoint paths satisfy that $|P_{(a,b)}| = |Q_{(c,d)}|$ and $V(P_{(a,b)} \cup Q_{(c,d)}) = V(HCC(1,n))$. We consider the relative positions of a and c of $HCC(1,n)$ as the following two conditions:
(1) a and c are in different CQ_ns.
(2) a and c are in the same CQ_n.

Moreover, both condition 1 and condition 2 are divided into more subcases as Table 1. (See table 1)

Case1 a and c are in different CQ_ns.	**Subcase 1.1** a and b are both in $CQ_n(u_a,v_a)$; c and d are both in $CQ_n(u_c,v_c)$.	**Subcase 1.1.1** $u_a{=}u_c$ **Subcase 1.1.2** $u_a \neq u_c$
	Subcase 1.2 a and b are both in $CQ_n(u_a,v_a)$, c is in $CQ_n(u_c,v_c)$, and d is in $CQ_n(u_d,v_d)$.	**Subcase 1.2.1** $u_a{=}u_c{=}u_d$ **Subcase 1.2.2** $(u_a{=}u_c) \neq u_d$ **Subcase 1.2.3** $u_a \neq (u_c{=}u_d)$
	Subcase 1.3 The four vertices a, b, c, and d belong to four CQ_ns.	**Subcase 1.3.1** $u_a{=}u_b{=}u_c{=}u_d$ **Subcase 1.3.2** $(u_a{=}u_b{=}u_c) \neq u_d$ **Subcase 1.3.3** $(u_a{=}u_b) \neq (u_c{=}u_d)$ **Subcase 1.3.4** $(u_a{=}u_c) \neq (u_b{=}u_d)$
Case2 a and c are in the same CQ_n.	**Subcase 2.1** The four vertices a, b, c, and d belong to the same CQ_n.	
	Subcase 2.2 a, b, and c are in $CQ_n(u_a,v_a)$; d is in $CQ_n(u_d,v_d)$.	
	Subcase 2.3 a and c are both in $CQ_n(u_a,v_a)$; b and d are both in $CQ_n(u_b,v_b)$.	**Subcase 2.3.1** $u_a{=}u_b$ **Subcase 2.3.2** $u_a \neq u_b$
	Subcase 2.4 a and c are both in $CQ_n(u_a,v_a)$, b is in $CQ_n(u_b,v_b)$, and d is in $CQ_n(u_d,v_d)$.	

Table 1: All cases of Theorem 1.

Case 1 a and c are in different CQ_ns.
Subcase 1.1 a and b are both in $CQ_n(u_a,v_a)$; c and d are both in $CQ_n(u_c,v_c)$.
Subcase 1.1.1 $u_a = u_c$
Without loss of generality, let $v_a = p_1$ and $v_c = p_{2^n}$. Let $1 \leq i,j \leq 2^{n-1}$ and let H_1 be a subgraph of $HCC(1,n)$ consists of $2 \times 2^{n-1} \times 2^n$ vertices of $V(CQ_n(u,p_i)) \cup V(CQ_n(\bar{u},q_j))$ (See Fig. 6(a)). Let $HCC(1,n) - H_1$ be another subgraph called H_2. Clearly, c and d are in H_2. By Lemma 5, there exist two Hamiltonian paths $P_{(a,b)}$ of H_1 and $Q_{(c,d)}$ of H_2. Therefore, we can obtain two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$ with $|P_{(a,b)}| = |Q_{(c,d)}| = 2^{2n} - 1$.

Subcase 1.1.2 $u_a \neq u_c$
Without loss of generality, let $v_a = p_1$ and $v_c = q_{2^n}$. Let $1 \leq i,j \leq 2^{n-1}$ and let H_1 be a subgraph of $HCC(1,n)$ consists of $2 \times 2^{n-1} \times 2^n$ vertices of $V(CQ_n(u,p_i)) \cup V(CQ_n(\bar{u},q_j))$ (See Fig. 6(b)). Let $HCC(1,n) - H_1$ be another subgraph called H_2. Clearly, c, d are in H_2. By Lemma 5, there exist two Hamiltonian paths $P_{(a,b)}$ of H_1 and $Q_{(c,d)}$ of H_2. Therefore, we can obtain two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$ with $|P_{(a,b)}| = |Q_{(c,d)}| = 2^{2n} - 1$.

Subcase 1.2 a and b are both in $CQ_n(u_a,v_a)$, c is in $CQ_n(u_c,v_c)$, and d is in $CQ_n(u_d,v_d)$.
Subcase 1.2.1 $u_a = u_c = u_d$

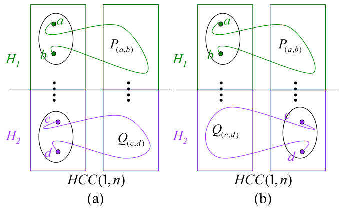

Figure 6: Illustration of Subcase 1.1.

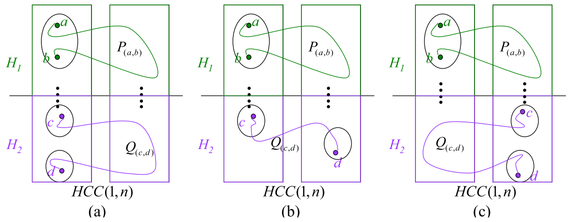

Figure 7: Illustration of Subcase 1.2.

Without loss of generality, let $v_a = p_1$, $v_c = p_{2^{n-1}}$, and $v_d = p_{2^n}$. Let $1 \le i, j \le 2^{n-1}$ and let H_1 be a subgraph of $HCC(1,n)$ consists of $2 \times 2^{n-1} \times 2^n$ vertices of $V(CQ_n(u,p_i)) \cup V(CQ_n(\bar{u},q_j))$ (See Fig. 7(a)). Let $HCC(1,n) - H_1$ be another subgraph called H_2. Clearly, c, d are in H_2. By Lemma 5, we can find a Hamiltonian path $P_{(a,b)}$ of H_1. Then by Lemma 6, there exists a Hamiltonian path $Q_{(c,d)}$ of H_2. Therefore, we can obtain two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$ with $|P_{(a,b)}| = |Q_{(c,d)}| = 2^{2n} - 1$.

Subcase 1.2.2 $(u_a = u_c) \ne u_d$
Without loss of generality, let $v_a = p_1$, $v_c = p_{2^n}$, and $v_d = q_{2^n}$. Let $1 \le i, j \le 2^{n-1}$ and let H_1 be a subgraph of $HCC(1,n)$ consists of $2 \times 2^{n-1} \times 2^n$ vertices of $V(CQ_n(u,p_i)) \cup V(CQ_n(\bar{u},q_j))$ (See Fig. 7(b)). Let $HCC(1,n) - H_1$ be another subgraph called H_2. Clearly, c, d are in H_2. By Lemma 5, we can find a Hamiltonian path $P_{(a,b)}$ of H_1. Then by Lemma 4, there exists a Hamiltonian path $Q_{(c,d)}$ of H_2. Therefore, we can obtain two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$ with $|P_{(a,b)}| = |Q_{(c,d)}| = 2^{2n} - 1$.

Subcase 1.2.3 $u_a \ne (u_c = u_d)$
Without loss of generality, let $v_a = p_1$, $v_c = q_{2^{n-1}}$, and $v_d = q_{2^n}$. Let $1 \le i, j \le 2^{n-1}$ and let H_1 be a subgraph of $HCC(1,n)$ consists of $2 \times 2^{n-1} \times 2^n$ vertices of $V(CQ_n(u,p_i)) \cup V(CQ_n(\bar{u},q_j))$ (See Fig. 7(c)). Let $HCC(1,n) - H_1$ be another subgraph called H_2. Clearly, c, d are in H_2. By Lemma 5, we can find a Hamiltonian path $P_{(a,b)}$ of H_1. Then by Lemma 6, there exists a Hamiltonian path $Q_{(c,d)}$ of H_2. Therefore, we can obtain two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$ with $|P_{(a,b)}| = |Q_{(c,d)}| = 2^{2n} - 1$.

Subcase 1.3 The four vertices a, b, c, and d belong to four CQ_ns.
Subcase 1.3.1 $u_a = u_b = u_c = u_d$
Without loss of generality, let $v_a = p_1$, $v_b = p_2$, $v_c = p_{2^{n-1}}$, and $v_d = p_{2^n}$. Let $1 \le i, j \le 2^{n-1}$ and let H_1 be a subgraph of $HCC(1,n)$ consists of $2 \times 2^{n-1} \times 2^n$ vertices of $V(CQ_n(u,p_i)) \cup V(CQ_n(\bar{u},q_j))$ (See Fig. 8(a)). Let $HCC(1,n) - H_1$ be another subgraph called H_2. Clearly, c, d are in H_2. By Lemma 6, there exist two Hamiltonian paths $P_{(a,b)}$ of H_1 and $Q_{(c,d)}$ of H_2. Therefore, we can obtain two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$ with $|P_{(a,b)}| = |Q_{(c,d)}| = 2^{2n} - 1$.

Subcase 1.3.2 $(u_a = u_b = u_c) \ne u_d$
Without loss of generality, let $v_a = p_1$, $v_b = p_2$, $v_c = p_{2^n}$, and $v_d = q_{2^n}$. Let $1 \le i, j \le 2^{n-1}$ and let H_1 be a subgraph of $HCC(1,n)$ consists of $2 \times 2^{n-1} \times 2^n$ vertices of $V(CQ_n(u,p_i)) \cup V(CQ_n(\bar{u},q_j))$ (See Fig. 8(b)). Let $HCC(1,n) - H_1$ be another subgraph called H_2. Clearly, c, d are in H_2. By Lemma 6, there exists a Hamiltonian path $P_{(a,b)}$ of H_1. Then by Lemma 4, there exists a Hamiltonian path $Q_{(c,d)}$ of H_2. Hence, we can obtain two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$ with $|P_{(a,b)}| = |Q_{(c,d)}| = 2^{2n} - 1$.

Subcase 1.3.3 $(u_a = u_b) \ne (u_c = u_d)$
Without loss of generality, let $v_a = p_1$, $v_b = p_2$, $v_c = q_{2^{n-1}}$, and $v_d = q_{2^n}$. Let $1 \le i, j \le 2^{n-1}$ and let H_1 be a subgraph of $HCC(1,n)$ consists of $2 \times 2^{n-1} \times 2^n$ vertices of $V(CQ_n(u,p_i)) \cup V(CQ_n(\bar{u},q_j))$ (See Fig. 8(c)). Let $HCC(1,n) - H_1$ be another subgraph called H_2. Clearly, c, d are in H_2. By Lemma 6, there exist two Hamiltonian paths $P_{(a,b)}$ of H_1 and $Q_{(c,d)}$ of H_2. Therefore, we can obtain two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$ with $|P_{(a,b)}| = |Q_{(c,d)}| = 2^{2n} - 1$.

Subcase 1.3.4 $(u_a = u_c) \ne (u_b = u_d)$
Without loss of generality, let $v_a = p_1$, $v_b = q_1$, $v_c = p_{2^n}$, and $v_d = q_{2^n}$. Let $1 \le i, j \le 2^{n-1}$ and let H_1 be a subgraph of $HCC(1,n)$ consists of $2 \times 2^{n-1} \times 2^n$ vertices of $V(CQ_n(u,p_i)) \cup V(CQ_n(\bar{u},q_j))$ (See Fig. 8(d)). Let $HCC(1,n) - H_1$ be another subgraph called H_2. Clearly, c, d are in H_2. By Lemma 4, there exist two Hamiltonian paths $P_{(a,b)}$ of H_1 and $Q_{(c,d)}$ of H_2.

Therefore, we can obtain two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$ with $|P_{(a,b)}| = |Q_{(c,d)}| = 2^{2n} - 1$.

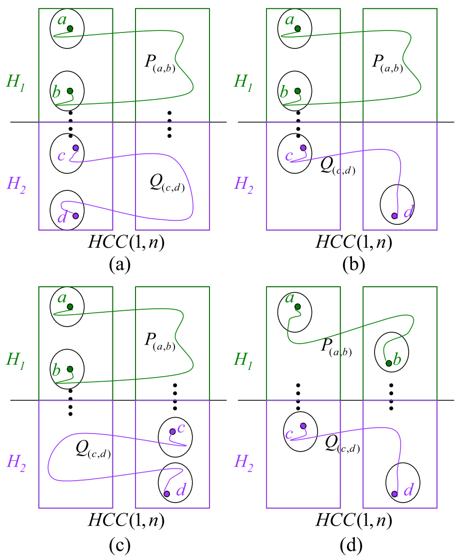

Figure 8: Illustration of Subcase 1.3.

Case 2 a and c are in the same CQ_n.

Subcase 2.1 The four vertices a, b, c, and d belong to the same CQ_n.

Without loss of generality, $v_a = p_1$. By Lemma 3, there exist two-equal-disjoint paths $P'_{(a,b)}$ and $Q'_{(c,d)}$ of $CQ_n(u, p_1)$. Without loss of generality, let $x = (u, p_1, w_x)$ be the neighbor of b on $P'_{(a,b)}$ and $y = (u, p_1, w_y)$ be the neighbor of d on $Q'_{(c,d)}$. Moreover, let $w_x = q_1$ and $w_y = q_2$. Then, let $P'_{(a,b)} = \langle a,\ P_{(a,x)},\ x,\ b\rangle$ and $Q'_{(c,d)} = \langle c,\ Q_{(c,y)},\ y,\ d\rangle$. Let H_1 be a subgraph of $HCC(1, n)$ consists of $V(CQ_n(u, p_i)) \cup V(CQ_n(\bar{u}, q_j))$ with $3 \le i, j \le 2^{n-1} + 1$ and let $HCC(1, n) - CQ_n(u, p_1) \cup CQ_n(u, p_2) \cup CQ_n(\bar{u}, q_1) \cup CQ_n(\bar{u}, q_2) \cup H_1$ be a subgraph called H_2. Without loss of generality, let b'' be in H_1 and d'' be in H_2. For two distinct vertices $x_1 = (u, p_2, q_1)$ and $y_1 = (u, p_2, q_2)$, by Observation 1, there exist two paths $P_{(x,x_1)}$ traverses every vertex of $V(CQ_n(\bar{u}, q_1)) \cup \{x, x_1\}$ and $Q_{(y,y_1)}$ traverses every vertex of $V(CQ_n(\bar{u}, q_2)) \cup \{y, y_1\}$ (See Fig. 9). Let x_2 and y_2 be two arbitrary vertices in $CQ_n(u, p_2)$ with $x_2,\ y_2 \notin \{x_1, y_1\}$, $x''_2 \in H_1$, and $y''_2 \in H_2$. By Lemma 3, there exist two-equal-disjoint paths $P_{(x_1,x_2)}$ and $Q_{(y_1,y_2)}$ of $CQ_n(u, p_2)$. Whether x''_2 and b'' are in the same CQ_n, by either Lemma 5 or Lemma 6, there exists a Hamiltonian path $P_{(x''_2,b'')}$ of H_1. Similarly, whether y''_2 and d'' are in the same CQ_n, by either Lemma 5 or Lemma 6, there exists a Hamiltonian path $Q_{(y''_2,d'')}$ of H_2. Therefore, we can obtain two-equal-disjoint paths $P_{(a,b)} = \langle a,\ P_{(a,x)},\ x,\ P_{(x,x_1)},\ x_1,\ P_{(x_1,x_2)},\ x_2,\ x''_2,\ P_{(x''_2,b'')},\ b'',\ b\rangle$ and $Q_{(c,d)} = \langle c,\ Q_{(c,y)},\ y,\ Q_{(y,y_1)},\ y_1,\ Q_{(y_1,y_2)},\ y_2,\ y''_2,\ Q_{(y''_2,d'')},\ d'',\ d\rangle$.

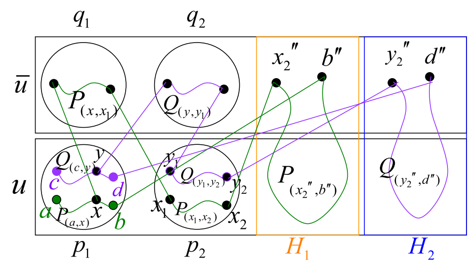

Figure 9: Illustration of Subcase 2.1.

Subcase 2.2 a, b, and c are in $CQ_n(u_a, v_a)$; d is in $CQ_n(u_d, v_d)$.

Without loss of generality, let $v_a = p_1$. Let $y = (u, p_1, w_y)$ be a vertex with $y \notin \{a, b, c\}$ and $w_y \neq v_d$. Without loss of generality, let $w_y = q_1$. By Lemma 3, there exist two-equal-disjoint paths $P'_{(a,b)}$ and $Q'_{(c,y)}$ of $CQ_n(u, p_1)$. Assume $x = (u, p_1, w_x)$ is the neighbor of a or b on $P'_{(a,b)}$ with $w_x \neq v_d$. Without loss of generality, let x be the neighbor of b on $P'_{(a,b)}$ and let $w_x = q_2$, then let $P'_{(a,b)} = \langle a,\ P_{(a,x)},\ x,\ b\rangle$. Let H_1 be a subgraph of $HCC(1, n)$ consists of $V(CQ_n(u, p_i)) \cup V(CQ_n(\bar{u}, q_j))$ with $3 \le i, j \le 2^{n-1} + 1$ and let $HCC(1, n) - CQ_n(u, p_1) \cup CQ_n(u, p_2) \cup CQ_n(\bar{u}, q_1) \cup CQ_n(\bar{u}, q_2) \cup H_1$ be a subgraph called H_2. Without loss of generality, let b'' be in H_1 and d be in H_2. For two vertices $x_1 = (u, p_2, q_2)$ and $y_1 = (u, p_2, q_1)$, by Observation 1, there exist two paths $P_{(x,x_1)}$ traverses every vertex of $V(CQ_n(\bar{u}, q_2)) \cup \{x, x_1\}$ and $Q_{(y,y_1)}$ traverses every vertex of $V(CQ_n(\bar{u}, q_1)) \cup \{y, y_1\}$ (See Fig. 10). Let x_2 and y_2 be two arbitrary vertices in $CQ_n(u, p_2)$ with $x_2,\ y_2 \notin \{x_1, y_1\}$, $x''_2 \in H_1$, and $y''_2 \in H_2$. By Lemma 3, there exist two-equal-disjoint paths $P_{(x_1,x_2)}$ and $Q_{(y_1,y_2)}$ of $CQ_n(u, p_2)$. Whether x''_2 and b'' are in the same CQ_n, by either Lemma 5 or Lemma 6, there exists a Hamiltonian path $P_{(x''_2,b'')}$ of H_1. Next we consider the position of d in two conditions: (1) $u_d = u_a$ and (2) $u_d \neq u_a$. By Lemma 4, there exists a Hamiltonian path $Q_{(y''_2,d)}$ of H_2 if $u_d = u_a$. If $u_d \neq u_a$, whether y''_2 is in $CQ_n(\bar{u}, v_d)$, there exists a Hamiltonian path $Q_{(y''_2,d)}$ of H_2 by either Lemma 5 or Lemma 6. Therefore, we can obtain two-equal-disjoint paths $P_{(a,b)} = \langle a,\ P_{(a,x)},\ x,\ P_{(x,x_1)},\ x_1,\ P_{(x_1,x_2)},\ x_2,\ x''_2,\ P_{(x''_2,b'')},\ b'',\ b\rangle$ and $Q_{(c,d)} = \langle c,\ Q_{(c,y)},\ y,\ Q_{(y,y_1)},\ y_1,\ Q_{(y_1,y_2)},\ y_2,\ y''_2,$

$Q_{(y_2'',d)}$, $d\rangle$.

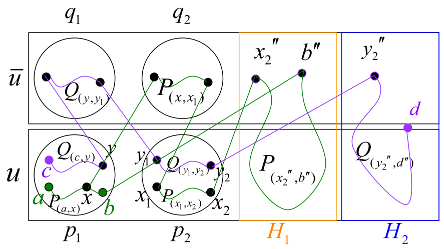

Figure 10: Illustration of Subcase 2.2.

Subcase 2.3 a and c are both in $CQ_n(u_a, v_a)$; b and d are both in $CQ_n(u_b, v_b)$.

Subcase 2.3.1 $u_a = u_b$

Without loss of generality, let $v_a = p_1$ and $v_b = p_2$ and let $x = (u, p_1, w_x)$ and $y = (u, p_1, w_y)$ be two vertices with x, $y \notin \{a, c\}$. Without loss of generality, let $w_x = q_1$ and $w_y = q_2$. By Lemma 3, there exist two-equal-disjoint paths $P_{(a,x)}$ and $Q_{(c,y)}$ of $CQ_n(u, p_1)$. Then, let H_1 be a subgraph of $HCC(1, n)$ consists of $V(CQ_n(u, p_i))$ $\cup$ $V(CQ_n(\bar{u}, q_j))$ with $3 \le i, j \le 2^{n-1} + 1$ and let $HCC(1, n)$ $-$ $CQ_n(u, p_1)$ $\cup$ $CQ_n(u, p_2)$ $\cup$ $CQ_n(\bar{u}, q_1)$ $\cup$ $CQ_n(\bar{u}, q_2)$ $\cup$ H_1 be a subgraph called H_2. For two vertices $x_1 = (u, p_3, q_1)$ and $y_1 = (u, p_{2^n}, q_2)$, by Observation 1, there exist two paths $P_{(x,x_1)}$ traverses every vertex of $V(CQ_n(\bar{u}, q_1))$ $\cup$ $\{x, x_1\}$ and $Q_{(y,y_1)}$ traverses every vertex of $V(CQ_n(\bar{u}, q_2)) \cup \{y, y_1\}$ (See Fig. 11). Let w and z be two vertices in $CQ_n(u, p_2)$ with w, $z \notin \{b, d\}$, $w'' \in H_1$, and $z'' \in H_2$. By Lemma 4, there exist two Hamiltonian paths $P_{(x_1,w'')}$ of H_1 and $Q_{(y_1,z'')}$ of H_2. Therefore, we can obtain two-equal-disjoint paths $P_{(a,b)} = \langle a$, $P_{(a,x)}$, x, $P_{(x,x_1)}$, x_1, $P_{(x_1,w'')}$, w'', w, $P_{(w,b)}$ $b\rangle$ and $Q_{(c,d)} = \langle c$, $Q_{(c,y)}$, y, $Q_{(y,y_1)}$, y_1, $Q_{(y_1,z'')}$, z'', z, $Q_{(z,d)}$, $d\rangle$.

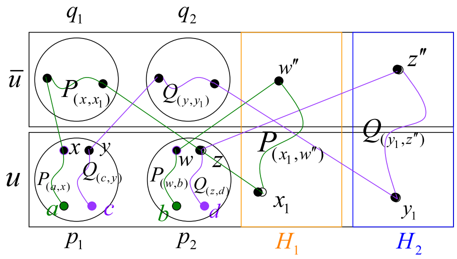

Figure 11: Illustration of Subcase 2.3.1.

Subcase 2.3.2 $u_a \neq u_b$

Without loss of generality, let $v_a = p_1$ and $v_b = q_{2^n}$. Let $x = (u, p_1, w_x)$ and $y = (u, p_1, w_y)$ be two distinct vertices with x, $y \notin \{a, c\}$ and w_x, $w_y \neq v_b$. Without loss of generality, let $w_x = q_1$ and $w_y = q_2$. By Lemma 3, there exist two-equal-disjoint paths $P_{(a,x)}$ and $Q_{(c,y)}$ of $CQ_n(u, p_1)$. For two distinct vertices $x_1 = (u, p_2, q_1)$ and $y_1 = (u, p_2, q_2)$, by Observation 1, there exist two paths $P_{(x,x_1)}$ traverses every vertex of $V(CQ_n(\bar{u}, q_1))$ $\cup$ $\{x, x_1\}$ and $Q_{(y,y_1)}$ traverses every vertex of $V(CQ_n(\bar{u}, q_2)) \cup \{y, y_1\}$ (See Fig. 12). Let $t = (\bar{u}, q_{2^n}, w_t)$ and $z = (\bar{u}, q_{2^n}, w_z)$ be two distinct vertices with t, $z \notin \{b, d\}$ and w_t, $w_z \neq v_a$. Without loss of generality, let $w_t = p_{2^n}$ and $w_z = p_{2^n-1}$. By Lemma 3, there exist two-equal-disjoint paths $P_{(t,b)}$ and $Q_{(z,d)}$ of $CQ_n(\bar{u}, q_{2^n})$. For two distinct vertices $t_1 = (\bar{u}, q_{2^n-1}, p_{2^n})$ and $z_1 = (\bar{u}, q_{2^n-1}, p_{2^n-1})$, by Observation 1, there exist two paths $P_{(t_1,t)}$ traverses every vertex of $V(CQ_n(u, p_{2^n}))$ $\cup$ $\{t_1, t\}$ and $Q_{(z_1,z)}$ traverses every vertex of $V(CQ_n(u, p_{2^n-1}))$ $\cup$ $\{z_1, z\}$. Then, let H_1 be a subgraph of $HCC(1, n)$ consists of $V(CQ_n(u, p_i))$ $\cup$ $V(CQ_n(\bar{u}, q_j))$ with $3 \le i, j \le 2^{n-1}$ and let $HCC(1, n)$ $-$ $CQ_n(u, p_1)$ $\cup$ $CQ_n(u, p_2)$ $\cup$ $CQ_n(\bar{u}, q_1)$ $\cup$ $CQ_n(\bar{u}, q_2)$ $\cup$ H_1 $\cup$ $CQ_n(u, p_{2^n-1})$ $\cup$ $CQ_n(u, p_{2^n})$ $\cup$ $CQ_n(\bar{u}, q_{2^n-1})$ $\cup$ $CQ_n(\bar{u}, q_{2^n})$ be a subgraph called H_2. Let x_2 and y_2 be two arbitrary vertices in $CQ_n(u, p_2)$ with x_2, $y_2 \notin \{x_1, y_1\}$, $x_2'' \in H_1$, and $y_2'' \in H_2$ and let t_2 and z_2 be two arbitrary vertices in $CQ_n(\bar{u}, q_{2^n-1})$ with t_2, $z_2 \notin \{t_1, z_1\}$, $t_2'' \in H_1$, and $z_2'' \in H_2$. By Lemma 3, there exist two-equal-disjoint paths $P_{(x_1,x_2)}$ and $Q_{(y_1,y_2)}$ of $CQ_n(u, p_2)$ and two-equal-disjoint paths $P_{(t_2,t_1)}$ and $Q_{(z_2,z_1)}$ of $CQ_n(\bar{u}, q_{2^n-1})$. By Lemma 4, there exist two Hamiltonian paths $P_{(x_2'',t_2'')}$ of H_1 and $Q_{(y_2'',z_2'')}$ of H_2. Therefore, we can obtain two-equal-disjoint paths $P_{(a,b)} = \langle a$, $P_{(a,x)}$, x, $P_{(x,x_1)}$, x_1, $P_{(x_1,x_2)}$, x_2, x_2'', $P_{(x_2'',t_2'')}$, t_2'', t_2, $P_{(t_2,t_1)}$, t_1, $P_{(t_1,t)}$, t, $P_{(t,b)}$, $b\rangle$ and $Q_{(c,d)} = \langle c$, $Q_{(c,y)}$, y, $Q_{(y,y_1)}$, y_1, $Q_{(y_1,y_2)}$, y_2, y_2'', $Q_{(y_2'',z_2'')}$, z_2'', z_2, $Q_{(z_2,z_1)}$, z_1, $Q_{(z_1,z)}$, z, $Q_{(z,d)}$, $d\rangle$.

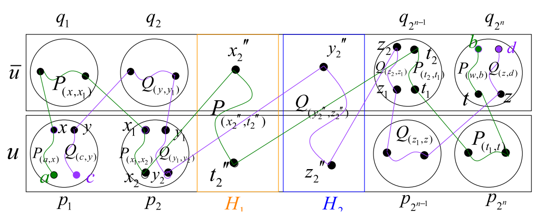

Figure 12: Illustration of Subcase 2.3.2.

Subcase 2.4 a and c are both in $CQ_n(u_a, v_a)$; b is in $CQ_n(u_b, v_b)$; d is in $CQ_n(u_d, v_d)$.

Without loss of generality, let $v_a = p_1$. Let $x = (u, p_1, w_x)$ and $y = (u, p_1, w_y)$ be two distinct vertices with x, $y \notin \{a, c\}$, w_x, $w_y \neq v_b$, and w_x, $w_y \neq v_d$. Without loss of generality, let $w_x = q_2$ and $w_y = q_1$. By Lemma 3, there exist two-equal-disjoint paths $P_{(a,x)}$ and $Q_{(c,y)}$ of $CQ_n(u, p_1)$. For two distinct vertices $x_1 = (u, p_2, q_2)$ and $y_1 = (u, p_2, q_1)$, by Observation 1, there exist two paths $P_{(x,x_1)}$ traverses every vertex of $V(CQ_n(\bar{u}, q_2)) \cup \{x, x_1\}$ and $Q_{(y,y_1)}$ traverses every vertex of $V(CQ_n(\bar{u}, q_1)) \cup \{y, y_1\}$ (See Fig. 13). Let H_1 be a subgraph of $HCC(1, n)$ consists of $V(CQ_n(u, p_i)) \cup V(CQ_n(\bar{u}, q_j))$ with $3 \leq i, j \leq 2^{n-1} + 1$ and let $HCC(1, n) - CQ_n(u, p_1) \cup CQ_n(u, p_2) \cup CQ_n(\bar{u}, q_1) \cup CQ_n(\bar{u}, q_2) \cup H_1$ be a subgraph called H_2. Without loss of generality, let b be in H_1 and d be in H_2. Let x_2 and y_2 be two distinct vertices with x_2, $y_2 \notin \{x_1, y_1\}$, $x_2'' \in H_1$, and $y_2'' \in H_2$. Next we consider the position of b and d in three conditions: (1) $u_b = u_d = u_a$, (2) $u_b = u_a$ and $u_d \neq u_a$, and (3) ($u_b = u_d$) $\neq u_a$. If $u_b = u_d = u_a$, we can obtain two Hamiltonian paths $P_{(x_2'',b)}$ of H_1 and $Q_{(y_2'',d)}$ of H_2 by Lemma 4. If $u_b = u_a$ and $u_d \neq u_a$, we can obtain a Hamiltonian path $P_{(x_2'',b)}$ of H_1 by Lemma 4 and whether y_2'' and d are in the same CQ_n, we can obtain a Hamiltonian path $Q_{(y_2'',d)}$ of H_2 by either Lemma 5 or Lemma 6. Moreover, if ($u_b = u_d$) $\neq u_a$, whether x_2'' and b are in the same CQ_n, we can obtain a Hamiltonian path $P_{(x_2'',b)}$ of H_1 by either Lemma 5 or Lemma 6. Similarly, whether y_2'' and d are in the same CQ_n, we can obtain a Hamiltonian path $Q_{(y_2'',d)}$ of H_2 by either Lemma 5 or Lemma 6. Therefore, there exist two-equal-disjoint paths $P_{(a,b)} = \langle a$, $P_{(a,x)}$, x, $P_{(x,x_1)}$, x_1, $P_{(x_1,x_2)}$, x_2, x_2'', $P_{(x_2'',b)}$, $b\rangle$ and $Q_{(c,d)} = \langle c$, $Q_{(c,y)}$, y, $Q_{(y,y_1)}$, y_1, $Q_{(y_1,y_2)}$, y_2, y_2'', $Q_{(y_2'',d)}$, $d\rangle$. □

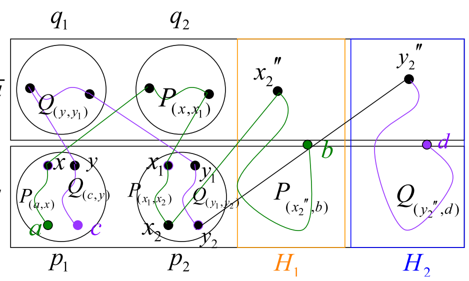

Figure 13: Illustration of Subcase 2.4.

After verifying Theorem 1, we ready to prove our main result. We present our main result as Theorem 2 that $HCC(k, n)$ is globally two-equal-disjoint path coverable for $k \geq 1$, $n \geq 5$.

Theorem 2 *For $k \geq 1$, $n \geq 5$, $HCC(k, n)$ is globally two-equal-disjoint path coverable.*

Proof. We verify this theorem by induction on k. Theorem 1 provides our base case of this theorem. By induction hypothesis, we can assume $HCC(k, n)$ is globally two-equal-disjoint path coverable. Next we have to verify $HCC(k+1, n)$ is globally two-equal-disjoint path coverable. Let a, b and c, d be two distinct source-destination pairs of $HCC(k+1, n)$. We establish two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$. Furthermore, the two disjoint paths must satisfy $|P_{(a,b)}| = |Q_{(c,d)}| = 2^{k+2n} - 1$. According to the relative positions of the four vertices, we divide the proof into the following four cases.

Case 1 a, b, c, and d are all in the same $HCC(k, n)$, say $H_k(i)$, $i \in \{0, 1\}$, of $HCC(k+1, n)$.

Without loss of generality, let a, b, c, and d are all in $H_k(0)$. By the hypothesis, there are two disjoint paths $\langle a, P_0, b\rangle$ and $\langle c, Q_0, d\rangle$ in $H_k(0)$, where $|P_0| = |Q_0| = 2^{k+2n-1} - 1$. Let (w, x) be an edge on P_0 and (y, z) be an edge on Q_0, and then let $P_0 = \langle a, P_0^1, w, x, P_0^2, b\rangle$ and $Q_0 = \langle c, Q_0^1, y, z, Q_0^2, d\rangle$. Besides, we also have two paths P_1 joining w'' and x'' and Q_1 joining y'' and z'' with length $2^{k+2n-1} - 1$ in $H_k(1)$. Let $P = \langle a, P_0^1, w, w'', P_1, x'', x, P_0^2, b\rangle$ and $Q = \langle c, Q_0^1, y, y'', Q_1, z'', z, Q_0^2, d\rangle$. Hence it's obvious that P and Q are two disjoint paths of length $2^{k+2n} - 1$. (See Fig. 14)

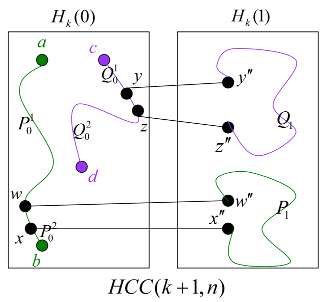

Figure 14: Illustration of Case 1 in Theorem 2.

Case 2 a, b, and c are in the same $HCC(k, n)$, say $H_k(i)$, $i \in \{0, 1\}$, of $HCC(k+1, n)$; d is in $H_k(\bar{i})$.

Without loss of generality, let a, b, and c be in $H_k(0)$ and d be in $H_k(1)$. Then let y be a vertex in $H_k(0)$ with $y \notin \{a, b, c, d''\}$. By the hypothesis, there are two disjoint paths $\langle a, P_0, b\rangle$ and $(c, Q_0, y\rangle$

in $H_k(0)$, where $|P_0| = |Q_0| = 2^{k+2n-1} - 1$. Let (w, x) be an edge on P_0 that $w \neq d''$ and $x \neq d''$, and then let $P_0 = \langle a, P_0^1, w, x, P_0^2, b\rangle$. Similarly, we also have two paths P_1 and Q_1 of length $2^{k+2n-1} - 1$ with end vertices w'', x'', y'' and d in $H_k(1)$. Note that $w'' \neq d$ and $x'' \neq d$. Let $P = \langle a, P_0^1, w, w'', P_1, x'', x, P_0^2, b\rangle$ and $Q = \langle c, Q_0, y, y'', Q_1, d\rangle$. Therefore, it's obvious that P and Q are two disjoint paths of length $2^{k+2n} - 1$. (See Fig. 15)

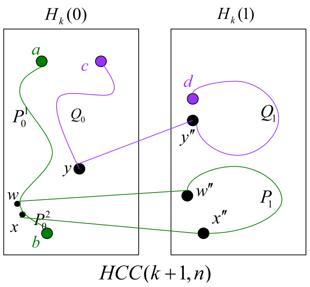

Figure 15: Illustration of Case 2 in Theorem 2.

Case 3 a and b are both in the same $HCC(k, n)$, say $H_k(i)$, $i \in \{0, 1\}$, of $HCC(k + 1, n)$; c and d are both in $H_k(\bar{i})$.
Without loss of generality, let a and b be in $H_k(0)$; and let c and d be in $H_k(1)$. There exist two disjoint paths $P_{(a,b)}$ and $Q_{(c,d)}$ with $|P_{(a,b)}| = |Q_{(c,d)}| = 2^{k+2n} - 1$. (See Fig. 16)

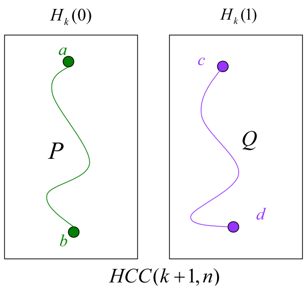

Figure 16: Illustration of Case 3 in Theorem 2.

Case 4 a and c are both in the same $HCC(k, n)$, say $H_k(i)$, $i \in \{0, 1\}$, of $HCC(k + 1, n)$; b and d are both in $H_k(\bar{i})$.
Without loss of generality, let a and c be in $H_k(0)$; and let b and d be in $H_k(1)$. Then let w and x be arbitrary two vertices in $H_k(0)$ except a and c and $w'' \notin \{b, d\}$, $x'' \notin \{b, d\}$. By the hypothesis, there are two disjoint paths (a, P_0, w) and (c, Q_0, x) in $H_k(0)$, where $|P_0| = |Q_0| = 2^{k+2n-1} - 1$. By the hypothesis again, there are also two paths P_1 and Q_1 of length $2^{k+2n-1} - 1$ with end vertices w'', b, x'' and d in $H_k(1)$. Let $P = \langle a, P_0, w, w'', P_1, b\rangle$ and $Q = \langle c, Q_0, x, xv, Q_1, d\rangle$. Accordingly, it's obvious that P and Q are two disjoint paths of length $2^{k+2n} - 1$. (See Fig. 17) $\square$

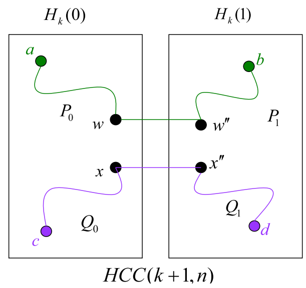

Figure 17: Illustration of Case 4 in Theorem 2.

4 Conclusion

The problem of many-to-many disjoint paths in networks is important and has received some attention because of its application in high performance and fault-tolerant routings. In this paper, we discussed the two-equal-disjoint path coverable problem and verify the hierarchical crossed cube $HCC(k, n)$ are globally two-equal-disjoint path coverable for $k \geq 1$ and $n \geq 5$.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North-Holland, New York, 1980.

[2] C. H. Chang, C. K. Lin, H. M. Huang, and L. H. Hsu, "The super laceability of the hypercubes," *Inf. Process. Lett.*, vol. 92, pp. 15-21, 2004.

[3] C. C. Chen and J. Chen, "Nearly Optimal One-to-Many Parallel Routing in Star Networks," *IEEE Trans. Parallel Distrib. Syst.*, vol. 8, pp. 1,196-1,202, 1997.

[4] K. Efe, "A Variation on the Hypercube with Lower Diameter," *IEEE Trans. Computers*, vol. 40, pp. 1,312-1,316, 1991.

[5] K. Efe, "The crossed cube architecture for parallel computation," *IEEE Trans. Parallel Distrib. Syst.*, vol. 3, pp. 513-524, 1992.

[6] J. S. Fu, "Longest fault-free paths in hypercubes with vertex faults," *Inf. Sci.*, vol. 176, pp. 759-771, 2006.

[7] S. Horiguchi and T. Ooki, "Hierarchical 3D-Torus Interconnection Network," *ISPAN*, pp. 50-56, 2000.

[8] H. C. Hsu, C. H. Tsai, C. H. Tsai, G. J. Wu, S. H. Liu, and P. L. Lai, "Hierarchical Crossed Cube: A New Interconnection Topology," *Proc. of the 25th Workshop on Combinatorial Mathematics and Computation Theory*, pp. 70-79, 2008.

[9] W. T. Huang, Y. C. Chuang, L. H. Hsu, and J. M. Tan, "On the fault-tolerant hamiltonicity of crossed cubes," *IEICE Trans. on Fundamentals*, vol. E85-A, pp. 1,359-1,371, 2002.

[10] J. M. Kumar and L. M. Patnaik, "Extended Hypercube: A Hierarchical Interconnection Network of Hypercubes," *IEEE Trans. Parallel Distrib. Syst.*, vol. 3, pp. 45-57, 1992.

[11] P. L. Lai and H. C. Hsu, "On the two-equal-disjoint path cover problem of crossed cubes," *JCIS*, pp. 603-606, 2006.

[12] Q. M. Malluhi and M. A. Bayoumi, "The hierarchical hypercube: a new interconnection topology for massively parallel systems," *IEEE Trans. Parallel Distrib. Syst.*, vol. 5, pp. 17-30, 1994.

[13] J. H. Park, "One-to-One Disjoint Path Covers in Recursive Circulants," *J. Korea Information Science Soc.*, vol. 30, pp. 691-698, 2003.

[14] J. H. Park, "Many-to-Many Disjoint Path Covers in Double Loop Networks," *J. Korea Information Science Soc.*, vol. 32, pp. 426-431, 2005.

[15] J. H. Park, "Unpaired Many-to-Many Disjoint Path Covers in Hypercube-Like Interconnection Networks," *J. Korea Information Science Soc.*, vol. 33, pp. 789-796, 2006.

[16] J. H. Park, H. C. Kim, and H. S. Lim, "Many-to-Many Disjoint Path Covers in Hypercube-Like Interconnection Networks with Faulty Elements," *IEEE Trans. Parallel Distrib. Syst.*, vol. 17, pp. 227-240, 2006.

[17] M. O. Rabin, "Efficient dispersal of information for security, load balancing, and fault tolerance," *J. ACM*, vol. 36, pp. 335-348, 1989.

[18] Y. Saad and M.H. Schultz, "Topological properties of hypercubes," *IEEE Trans. Computers*, vol. 37, pp. 867-872, 1988.

[19] G. Simmons, "Almost all n-dimensional rectangular lattices are hamilton laceable," *Congressus Numerantium*, vol. 21, pp. 649-661, 1978.

[20] R. Y. Wu, G. H. Chen, Y. L. Kuo, and G. J. Chang, "Node-disjoint paths in hierarchical Hypercube networks," *Inf. Sci.*, vol. 177, pp. 4,200-4,207, 2007.

[21] S. K. Yun and K. H. Park, "Hierarchical Hypercube Networks (HHN) for Massively Parallel Computers," *J. Parallel Distrib. Comput.*, vol. 37, pp. 194-199, 1996.