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The Homotopy and Fundamental Groups of Topological Graphs Based on Khalimsky Arcs

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Abstract

In this paper, we aim to develop a suitable homotopy theory and fundamental groups of finite topological graphs by Khalimsky arcs. The notions developed are considered for the investigation of the algebraic invariants of topological graphs for their topological and graphical classifications. The present work is a precursor of approaching the general homotopy theory, fundamental groups, and fixed (and almost fixed) point theorems [6] of topological graphs.

1 Topological Graphs and Khalimsky Arcs

In order to relate topology with graphs, as proposed by Smyth in [3, 4], a topological space may be considered as the limit of an inverse sequence of finite graphs. The graphs in such a sequence are considered as increasingly better discrete representations of the space, and that the sequence as a whole approximates the space is justified mathematically by the fact that the space may be derived from its limit via a quotienting operation. Thus a formal framework, topological graphs, is a structure which embodies topology as well as a binary relation, thereby constituting a generalization of ordered topological spaces to accommodate standard topological spaces and general graphs.

A *topological graph* (X, T, R) is a structure such that T is a topology, R is a closed reflexive binary relation satisfying $(\forall u, v \in T, x \in u \ \& \ R(u) \subseteq v \Rightarrow y \in v) \Rightarrow xRy$. A *graph morphism* from the topological graph (X, T_X, R_X) to the topological graph (Y, T_Y, R_Y) is a continuous function w.r.t. the topologies T_X and T_Y, which is relation preserving. For a topological graph $G = (X, T, R)$, we say G is *compact Hausdorff* if (X, T) is compact Hausdorff. Moreover, G is said to be *connected* if for every partition $\{u, v\}$ of X into T-open sets, there exist $x \in u, y \in v$ such that xRy or yRx. For more information about topological graphs, please refer [3, 4, 5, 8].

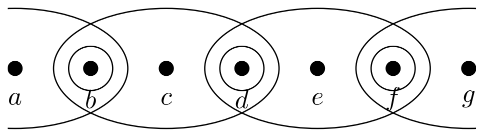

Figure 1: Khalimsky digital line

Note 1.1. Without any further notice, the topological graphs are finite; the maps between two topological graphs are graph morphisms.

We say a connected topological graph G satisfying the *COTS* (*connected ordered topological space* [2]) condition if and only if for any three distinct points $x, y, z \in G$, one of $\{x, y, z\}$ separates the other two.

Definition 1.2. [5] A *continuum* is a compact connected topological graph. A *linear* topological graph is a connected locally connected (read as usual) topological graph satisfying the *COTS* condition. An *arc* is a second countable continuum satisfying the *COTS* condition.

The so-called *Khalimsky integer line* is defined to be the set of all integers $\mathcal{Z}$ with its natural order together with its interval alternating topology T_{ia}, where T_{ia} is a topology defined by sets of types $(-\infty, 2i], [2j, \infty), i, j \in \mathcal{Z}$ (or alternately $(-\infty, 2i-1], [2j-1, \infty), i, j \in \mathcal{Z}$) as a subbasis. The *Khalimsky digital line* then can be defined as a finite connected subspace of the Khalimsky integer line. Fig.1 shows a portion of a standard Khalimsky digital line.

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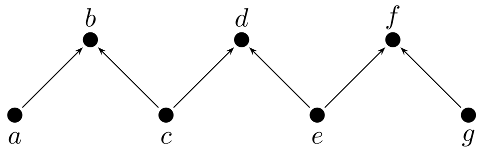

Figure 2: Khalimsky arc

From Theorem 2.7 of [2], for any Khalimsky digital line A with two end points a and b, there are only two natural linear orderings $\prec$ and $\succ$ $(=\prec^{-1})$ such that we can embed into A. For convenient, we may say that we travel A from the point a to the point b if $a \prec b$, and we denote it by ${}_aA_b$; otherwise we denote it by ${}_bA_a$ if and only if $a \succ b$.

Theorem 1.3. *[7] Let A be a Khalimsky digital line with at least three points. Then A has two end points which are both closed or open (but not both) w.r.t. the topology of* COTS *if and only if* $(\sharp A \mod 2) = 1$*, where* $\sharp A$ *is the numbers of points of* A.

Notice that a Khalimsky digital line is not necessarily an arc by regarding it as a topological space, i.e., with identity relation, since it is not compact if it contains infinite points. With the specialization pre-ordering embedded into a Khalimsky digital line, regarded as its relation, we can convey a Khalimsky digital line into a directed graph. As for a good example we can convey the Khalimsky digital line in Fig. 1 into a directed graph in Fig. 2.

Definition 1.4. [7] $G = (B, T, R)$ is a *Khalimsky arc* if B is the set of all points of a Khalimsky digital line A, T is the interval topology induced by the linear ordering $\prec$ (or $\succ$) of A, and R is the specialization pre-ordering $\le$ induced from the *COTS* topology of A.

2 Homotopy of Topological Graphs by Khalimsky Arcs

The material presented in this section can be found in [7].

For any point c of a Khalimsky digital line ${}_aA_b$, we denote the successor (resp. predecessor) of c by $c+$ (resp. $c-$) the immediate adjacent point of c such that $c- \prec c \prec c+$ when $c \notin \{a, b\}$ and $c- \equiv a$ if and only if $c \equiv a$ & $c+ \equiv b$ if and only if $c \equiv b$. Let A be a Khalimsky arc and $c \in A$.

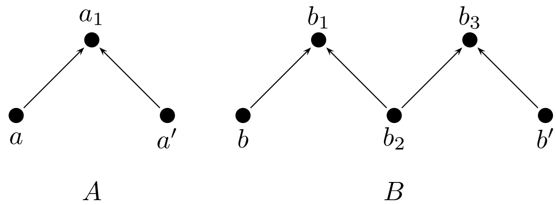

Figure 3: $A, B \in \mathfrak{p}\mathfrak{S}$

We say c is a *minimal* point of A if and only if $c \le c-$ & $c \le c+$, and c is a *maximal* point if and only if $c- \le c$ & $c+ \le c$. It is obvious that $c+$ and $c-$ (if exists) are minimal (resp. maximal) if and only if c is maximal (resp. minimal) when c is any point of a Khalimsky arc:

Lemma 2.1. *Let A be a Khalimsky arc with at least three points. Then A has two end points which are both minimal or maximal (but not both) if and only if* $(\sharp A \mod 2) = 1$.

Let $\mathfrak{S}$ be the collection of all Khalimsky arcs A which satisfy the condition $(\sharp A \mod 2) = 1$. By Lemma 2.1, A is a Khalimsky arc with two end points which are both minimal or maximal (or a singleton Khalimsky arc, which is a Khalimsky arc containing a single point). Therefore it is clear that $\mathfrak{S}$ can be classified into two classes whose intersection is a set containing all singleton Khalimsky arcs. One class denoted by $\mathfrak{p}\mathfrak{S}$ contains all elements of $\mathfrak{S}$ with two minimal end points (and all singleton Khalimsky arcs); the other class denoted by $\mathfrak{c}\mathfrak{S}$ contains all elements of $\mathfrak{S}$ with two maximal end points (and all singleton Khalimsky arcs).

Definition 2.2. Let $A, B \in \mathfrak{p}\mathfrak{S}$ such that $\sharp A = n$ with two end points a and a', and $\sharp B = m$ with two end points b and b'. We define $A \triangledown B \in \mathfrak{p}\mathfrak{S}$ to be a Khalimsky arc with $n + m - 1$ points such that the first $n - 1$ points of $A \triangledown B$ are considered as points of A and the last $m - 1$ points of $A \triangledown B$ are considered as points of B. And the topology of $A \triangledown B$ is the interval topology of the points of $A \triangledown B$, where the concatenation point of $A \triangledown B$ is one of $\{a \triangledown b, a \triangledown b', a' \triangledown b, a' \triangledown b'\}$ which is depended on the initial and final points of A and B. The operator is called *lexicographic union* or *oriented union*.

Clearly $A \triangledown B$ defined in Definition 2.2 is well-defined as the concatenation point of $A \triangledown B$ is always minimal. For example, the Fig 4 shows the lexicographic union of two Khalimsky arcs

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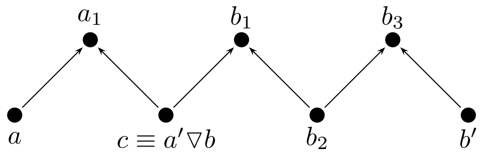

Figure 4: $A \nabla B \in \mathfrak{p}\mathfrak{S}$

$A, B \in \mathfrak{p}\mathfrak{S}$ in Fig 3. Moreover, we can extend Definition 2.2 into any arbitrary finite lexicographic union of Khalimsky arcs in $\mathfrak{p}\mathfrak{S}$. Also it is easy to check Definition 2.2 is still valid if we replace $\mathfrak{p}\mathfrak{S}$ by $\mathfrak{c}\mathfrak{S}$.

Definition 2.3. Let $A \in \mathfrak{p}\mathfrak{S}, B \in \mathfrak{c}\mathfrak{S}$ be two Khalimsky arcs, and G is a topological graph. We call the maps $f : A \to G$ and $g : B \to G$ a $\mathfrak{p}$-path and a $\mathfrak{c}$-path of G respectively.

Theorem 2.4. *Let $G_1 = (X, T_X, R_X)$ and $G_2 = (Y, T_Y, R_Y)$ be topological graphs. Then $(X \times Y, T_{X\times Y}, R_{X\times Y})$ is a topological graph, where $T_{X\times Y}$ is the product topology of T_X and T_Y, $R_{X\times Y} = \{((x, y), (x', y')) \mid (x, x') \in R_X \ \& \ (y, y') \in R_Y\}$.*

We denote $G_1 \times G_2 = (X \times Y, T_{X\times Y}, R_{X\times Y})$. From Theorem 2.4, it is easy to check that $T_{X\times Y}$ is Hausdorff if T_X and T_Y are Hausdorff, and $R_{X\times Y}$ is symmetric if R_X and R_Y are symmetric.

Definition 2.5. Let $(X, T_X, R_X), (Y, T_Y, R_Y)$ be topological graphs, f and g are maps of (X, T_X, R_X) into (Y, T_Y, R_Y). We say f is *$\mathfrak{p}$-homotopic* to g if there exists a Khalimsky arc $A \in \mathfrak{p}\mathfrak{S}$, and a map $H : (X, T_X, R_X) \times A \to (Y, T_Y, R_Y)$ such that $H(x, a) = f(x), H(x, b) = g(x)$ for all $x \in X$, where a, b are end points of A. We write $f \approx^{\mathfrak{p}} g$ if f is $\mathfrak{p}$-homotopic to g.

Similarly, we have $\mathfrak{c}$-homotopy if we replace $A \in \mathfrak{p}\mathfrak{S}$ by $A \in \mathfrak{c}\mathfrak{S}$ in Definition 2.5. We denote $f \approx^{\mathfrak{c}} g$ if f is $\mathfrak{c}$-homotopic to g.

Definition 2.6. With notations and terminology are same as Definition 2.5, we say f is homotopic to g if f is $\mathfrak{p}$-homotopic to g or f is $\mathfrak{c}$-homotopic to g. we denote $f \approx g$ if f is homotopic to g.

Definition 2.7. $G_1 = (X, T_X, R_X), G_2 = (Y, T_Y, R_Y)$ are topological graphs, and f, g are maps of G_1 into G_2. We say f and g are $\mathfrak{p}$-homotopic (resp. $\mathfrak{c}$-homotopic, homotopic) relative to G, G is a subgraph of G_1, if there exists a Khalimsky arc $A \in \mathfrak{p}\mathfrak{S}$ ($A \in \mathfrak{c}\mathfrak{S}$, $A \in \mathfrak{S}$), and a homotopy $H : G_1 \times A \to G_2$ such that $H(w, c) = f(w) = g(w)$ for all $w \in G$ & $c \in A$. The homotopy H is called a $\mathfrak{p}$-homotopy (resp. $\mathfrak{c}$-homotopy, homotopy) relative to G and we write $f \approx^{\mathfrak{p}}_G g$ (resp. $f \approx^{\mathfrak{c}}_G g$, $f \approx_G g$).

It is clear that we have $f \approx^{\mathfrak{p}} g \Leftrightarrow f \approx^{\mathfrak{p}}_G g$, $f \approx^{\mathfrak{c}} g \Leftrightarrow f \approx^{\mathfrak{c}}_G g$, and $f \approx g \Leftrightarrow f \approx_G g$ if and only if $G = \emptyset$.

Theorem 2.8. *With notations and terminology are same as Definitions 2.5, 2.6, and 2.7, the homotopy $\approx^{\mathfrak{p}}$ (resp. $\approx^{\mathfrak{c}}$, $\approx$) and $\approx^{\mathfrak{p}}_G$ (resp. $\approx^{\mathfrak{c}}_G$, $\approx_G$) are equivalence relations.*

3 Khalimsky Fundamental Groups of Topological Graphs

This section is an extension of the previous section *Homotopy of Topological Graphs by Khalimsky Arcs*. In this section, we develop the theory of fundamental groups of topological graphs mainly based on the Khalimsky arcs with two end points which are both minimal ($\in \mathfrak{p}\mathfrak{S}$). This may be due to the reason that two minimal end points are closed in the original *COTS* topology of the Khalimsky digital lines. However, it should be emphasized that the Khalimsky arcs with two maximal end points ($\in \mathfrak{c}\mathfrak{S}$) are also suitable for the same development.

Definition 3.1. Let $A, B \in \mathfrak{p}\mathfrak{S}$, and $f : {}_aA_b \to G$, $g : {}_cB_d \to G$ be two $\mathfrak{p}$-paths of G with $f(b) = g(c)$. We call the map $f * g : A \nabla B \to G$ such that

$$f * g(x) = \begin{cases} f(x), & \text{if } x \in A, \\ f(b) = g(c), & \text{if } x = b \nabla c, \\ g(x), & \text{if } x \in B, \end{cases}$$

the *product* of f and g.

Definition 3.2. Let $f : A \to G$ and $g : B \to G$ be two $\mathfrak{p}$-paths of G, where $A, B \in \mathfrak{p}\mathfrak{S}$. We say f is *path-homotopy* to g if there exist Khalimsky arcs $C, D \in \mathfrak{p}\mathfrak{S}$, and surjective maps $\hat{f} : C \to A, \hat{g} : C \to B$, and a homotopy $H : {}_{c_0}C_{c_1} \times {}_{d_0}D_{d_1} \to G$ such that $H(c, d_0) = f \circ \hat{f}(c), H(c, d_1) = g \circ \hat{g}(c)$, and $H(c_0, d) = f \circ \hat{f}(c_0) = g \circ \hat{g}(c_0)$, $H(c_1, d) = f \circ \hat{f}(c_1) = g \circ \hat{g}(c_1)$, for all $c \in C, d \in D$. We denote $f_p \approx^{\mathfrak{p}} g$ if the $\mathfrak{p}$-path f is path-homotopy to the $\mathfrak{p}$-path g.

In other words, two $\mathfrak{p}$-paths $f : A \to G$ and $g : B \to G$ are said to be path-homotopy if there

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exists an extended Khalimsky arc ${}_{c_0}C_{c_1} \in \mathfrak{p}\mathfrak{S}$ with two surjective maps $\hat{f} : C \to A$, $\hat{g} : C \to B$ such that $f \circ \hat{f} \approx^{\mathfrak{p}}_{\{c_0,c_1\}} g \circ \hat{g}$.

Lemma 3.3. *${}_p\approx^{\mathfrak{p}}$ is reflexive.*

Proof. Let $f : {}_aA_b \to G$ be a $\mathfrak{p}$-path, then clearly the Khalimsky arc ${}_aA_b$ and the identity map $id_A : A \to A$ satisfy $f \circ id_A = f \approx^{\mathfrak{p}}_{\{a,b\}} f = f \circ id_A$. Therefore ${}_p\approx^{\mathfrak{p}}$ is reflexive. □

Lemma 3.4. *${}_p\approx^{\mathfrak{p}}$ is symmetric.*

Proof. Let $f_p \approx^{\mathfrak{p}} g$, where $f : A \to G$ and $g : B \to G$ are two $\mathfrak{p}$-paths of G, we claim $g_p \approx^{\mathfrak{p}} f$.

By Definition 3.2, $f_p \approx^{\mathfrak{p}} g$ means that there exists a Khalimsky arc ${}_{c_0}C_{c_1} \in \mathfrak{p}\mathfrak{S}$ and two surjective maps $\hat{f} : {}_{c_0}C_{c_1} \to A, \hat{g} : {}_{c_0}C_{c_1} \to B$ such that $f \circ \hat{f} \approx^{\mathfrak{p}}_{\{c_0,c_1\}} g \circ \hat{g}$. From Theorem 2.8, $\approx^{\mathfrak{p}}_{\{c_0,c_1\}}$ is an equivalence relation (hence symmetric). Therefore we have $g \circ \hat{g} \approx^{\mathfrak{p}}_{\{c_0,c_1\}} f \circ \hat{f}$, and which implies ${}_p\approx^{\mathfrak{p}}$ is symmetric. □

Lemma 3.5. *${}_p\approx^{\mathfrak{p}}$ is transitive.*

Proof. Suppose $f_p \approx^{\mathfrak{p}} g$ and $g_p \approx^{\mathfrak{p}} h$, where $f : A \to G, g : B \to G$, and $h : C \to G$ are $\mathfrak{p}$-paths of G.

The relation $f_p \approx^{\mathfrak{p}} g$ means that there exist Khalimsky arcs ${}_{d_0}D_{d_1}, {}_{e_0}E_{e_1} \in \mathfrak{p}\mathfrak{S}$, and two surjective maps $\hat{f} : {}_{d_0}D_{d_1} \to A, \hat{g} : {}_{d_0}D_{d_1} \to B$, and a homotopy $H : {}_{d_0}D_{d_1} \times {}_{e_0}E_{e_1} \to G$, such that $H(d, e_0) = f \circ \hat{f}(d), H(d, e_1) = g \circ \hat{g}(d)$, and $H(d_0, e) = f \circ \hat{f}(d_0) = g \circ \hat{g}(d_0)$, $H(d_1, e) = f \circ \hat{f}(d_1) = g \circ \hat{g}(d_1)$, for all $d \in D, e \in E$.

The relation $g_p \approx^{\mathfrak{p}} h$ means that there exist Khalimsky arcs ${}_{m_0}M_{m_1}, {}_{n_0}N_{n_1} \in \mathfrak{p}\mathfrak{S}$, and two surjective maps $\tilde{g} : {}_{m_0}M_{m_1} \to B, \hat{h} : {}_{m_0}M_{m_1} \to C$, and a homotopy $K : {}_{m_0}M_{m_1} \times {}_{n_0}N_{n_1} \to G$, such that $K(m, n_0) = g \circ \tilde{g}(m), K(m, n_1) = h \circ \hat{h}(m)$, and $K(m_0, n) = g \circ \tilde{g}(m_0) = h \circ \hat{h}(m_0), K(m_1, n) = g \circ \tilde{g}(m_1) = h \circ \hat{h}(m_1)$, for all $m \in M, n \in N$.

As $\hat{g}$ and $\tilde{g}$ are surjective maps, hence they define a same path which itself is the Khalimsky arc B, therefore $\hat{g}$ and $\tilde{g}$ are path-homotopy. By the definition of $\hat{g}_p \approx^{\mathfrak{p}} \tilde{g}$, there exist Khalimsky arcs ${}_{s_0}S_{s_1}, {}_{t_0}T_{t_1} \in \mathfrak{p}\mathfrak{S}$, and two surjective maps $g' : {}_{s_0}S_{s_1} \to {}_{d_0}D_{d_1}, g'' : {}_{s_0}S_{s_1} \to {}_{m_0}M_{m_1}$, and a homotopy $L : {}_{s_0}S_{s_1} \times {}_{t_0}T_{t_1} \to B$, such that $L(s, t_0) = \hat{g} \circ g'(s), L(s, t_1) = \tilde{g} \circ g''(s)$, and $L(s_0, t) = \hat{g} \circ g'(s_0) = \tilde{g} \circ g''(s_0)$, $L(s_1, t) = \hat{g} \circ g'(s_1) = \tilde{g} \circ g''(s_1)$, for all $s \in S, t \in T$.

Define $F : {}_{s_0}S_{s_1} \times I \to G$ by

$$F(s, i) = \begin{cases} H(g'(s), i), & \text{if } i \in E, \\ g(L(s, i)), & \text{if } i \in T, \\ K(g''(s), i), & \text{if } i \in N, \end{cases}$$

where $I = {}_{e_0}E_{e_1} \nabla {}_{t_0}T_{t_1} \nabla {}_{n_0}N_{n_1} \in \mathfrak{p}\mathfrak{S}$.

F is well-defined as if $x = e_1 \nabla t_0$, then we have $F(s, x) = H(g'(s), e_1) = g \circ \hat{g}(g'(s)) = g(\hat{g} \circ g'(s)) = g(L(s, t_0)) = F(s, x)$. And if $x = t_1 \nabla n_0$, then we have $F(s, x) = g(L(s, t_1)) = g(\tilde{g} \circ g''(s)) = g \circ \tilde{g}(g''(s)) = K(g''(s), n_0) = F(s, x)$.

Clearly $\hat{f} \circ g' : {}_{s_0}S_{s_1} \to A$ and $\hat{h} \circ g'' : {}_{s_0}S_{s_1} \to C$ are surjective maps as $\hat{f}, g', \hat{h}$, and g'' are surjective maps.

We claim F is a homotopy relative to $\{s_0, s_1\}$ of $f \circ (\hat{f} \circ g')$ and $h \circ (\hat{h} \circ g'')$. It is trivial that F is relation-preserving. Moreover, with a similar proof of Theorem 2.8, F can be shown to be continuous by the pasting lemma.

For showing $F(s, t_0) = f \circ (\hat{f} \circ g'(s))$ and $F(s, n_1) = h \circ (\hat{h} \circ g''(s))$, it is trivial as we have $F(s, t_0) = H(g'(s), t_0) = f \circ \hat{f}(g'(s)) = f \circ (\hat{f} \circ g'(s))$ and $F(s, n_1) = K(g''(s), n_1) = h \circ \hat{h}(g''(s)) = h \circ (\hat{h} \circ g''(s))$.

For showing $F(s_0, i) = (f \circ (\hat{f} \circ g'))(s_0) = (h \circ (\hat{h} \circ g''))(s_0)$ and $F(s_1, i) = (f \circ (\hat{f} \circ g'))(s_1) = (h \circ (\hat{h} \circ g''))(s_1)$, we have the following equations:

(1) $F(s_0, i)$

$$= \begin{cases} H(g'(s_0), i) \\ = H(d_0, i) = \begin{cases} f \circ \hat{f}(d_0) \\ \quad \| \\ g \circ \hat{g}(d_0) \end{cases} & i \in E, \\ g(L(s_0, i)) \\ = \begin{cases} g(\hat{g} \circ g'(s_0)) = g(\hat{g}(d_0)) \\ \quad \| \\ g(\tilde{g} \circ g''(s_0)) = g(\tilde{g}(m_0)) \end{cases} & i \in T, \\ K(g''(s), i) \\ = K(m_0, i) = \begin{cases} g \circ \tilde{g}(m_0) \\ \quad \| \\ h \circ \hat{h}(m_0) \end{cases} & i \in N, \end{cases}$$

(2) $(f \circ (\hat{f} \circ g'))(s_0) = f(\hat{f} \circ g'(s_0)) = f(\hat{f}(d_0)) = f \circ \hat{f}(d_0)$

(3) $(h \circ (\hat{h} \circ g''))(s_0) = h(\hat{h} \circ g''(s_0)) = h(\hat{h}(m_0)) = h \circ \hat{h}(m_0)$

From the equations (1), (2) and (3), it is clear that we have $F(s_0, i) = (f \circ (\hat{f} \circ g'))(s_0) = (h \circ$

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$(\hat{h} \circ g''))(s_0)$. The proof of showing $F(s_1, i) = (f \circ (\hat{f} \circ g'))(s_1) = (h \circ (\hat{h} \circ g''))(s_1)$ is similar.

Therefore we have $f \circ (\hat{f} \circ g') \approx^{\mathfrak{p}}_{\{s_0,s_1\}} h \circ (\hat{h} \circ g'')$, and which implies $f_p \approx^{\mathfrak{p}} h$ by Definition 3.2. ☐

Theorem 3.6. ${}_p\approx^{\mathfrak{p}}$ *is an equivalence relation.*

Proof. Directly follows from Lemmas 3.3, 3.4, and 3.5. ☐

Let G be a topological graph and f is a $\mathfrak{p}$-path of G. We denote the equivalence class of f by $[f] = \{g | f_p \approx^{\mathfrak{p}} g, g \text{ is a } \mathfrak{p}\text{-path of } G\}$. By the naturalistic of the equivalence relation ${}_p\approx^{\mathfrak{p}}$, clearly we have either $[f] = [g]$ or $[f] \cap [g] = \emptyset$ for any pair of $\mathfrak{p}$-paths f and g.

If $f * g$ is a well-defined product of two $\mathfrak{p}$-paths f and g, we define the product of two path-homotopy classes $[f]$ and $[g]$ by $[f] * [g] = [f * g]$.

Theorem 3.7. *The product* $*$ *is well-defined on path-homotopy classes.*

Proof. Let $f_{1p} \approx^{\mathfrak{p}} f_2$ and $g_{1p} \approx^{\mathfrak{p}} g_2$ such that $f_i * g_j$ is well-defined for $i, j = 1, 2$, where $f_i : {}_{a(i)_0}A^i_{a(i)_1} \to G, g_j : {}_{b(j)_0}B^j_{b(j)_1} \to G$ are $\mathfrak{p}$-paths of G. We claim $f_i * g_{jp} \approx^{\mathfrak{p}} f_k * g_l$ for $i, j, k, l = 1, 2$. Clearly it is sufficient to show that $f_1 * g_{1p} \approx^{\mathfrak{p}} f_2 * g_2$ only, since ${}_p\approx^{\mathfrak{p}}$ is an equivalence relation.

The relation $f_{1p} \approx^{\mathfrak{p}} f_2$ means that there exist Khalimsky arcs ${}_{c_0}C_{c_1}, {}_{d_0}D_{d_1} \in \mathfrak{pG}$ and two surjective maps $\hat{f}_1 : {}_{c_0}C_{c_1} \to A^1, \hat{f}_2 : {}_{c_0}C_{c_1} \to A^2$ and a homotopy $H : {}_{c_0}C_{c_1} \times {}_{d_0}D_{d_1} \to G$ such that $H(c, d_0) = f_1 \circ \hat{f}_1(c), H(c, d_1) = f_2 \circ \hat{f}_2(c)$ and $H(c_0, d) = f_1 \circ \hat{f}_1(c_0) = f_2 \circ \hat{f}_2(c_0)$, $H(c_1, d) = f_1 \circ \hat{f}_1(c_1) = f_2 \circ \hat{f}_2(c_1)$ for all $c \in C, d \in D$.

The relation $g_{1p} \approx^{\mathfrak{p}} g_2$ means that there exist Khalimsky arcs ${}_{m_0}M_{m_1}, {}_{n_0}N_{n_1} \in \mathfrak{pG}$ and two surjective maps $\hat{g}_1 : {}_{m_0}M_{m_1} \to B^1, \hat{g}_2 : {}_{m_0}M_{m_1} \to B^2$ and a homotopy $K : {}_{m_0}M_{m_1} \times {}_{n_0}N_{n_1} \to G$ such that $K(m, n_0) = g_1 \circ \hat{g}_1(m), K(m, n_1) = g_2 \circ \hat{g}_2(m)$ and $K(m_0, n) = g_1 \circ \hat{g}_1(m_0) = g_2 \circ \hat{g}_2(m_0)$, $K(m_1, n) = g_1 \circ \hat{g}_1(m_1) = g_2 \circ \hat{g}_2(m_1)$ for all $m \in M, n \in N$.

Firstly we show $f_i \circ \hat{f}_i(c_1) = g_j \circ \hat{g}_j(m_0)$ for $i, j = 1, 2$. It is clear $f_1 \circ \hat{f}_1(c_1) = f_2 \circ \hat{f}_2(c_1)$ and $g_1 \circ \hat{g}_1(m_0) = g_2 \circ \hat{g}_2(m_0)$ by the relations $f_{1p} \approx^{\mathfrak{p}} f_2$ and $g_{1p} \approx^{\mathfrak{p}} g_2$. Moreover, $f_i \circ \hat{f}_i(c_1) = f_i(a(i)_1)$ and $g_j \circ \hat{g}_j(m_0) = g_j(b(j)_0)$ since $\hat{f}_i$ and $\hat{g}_j$ are surjective maps (hence $\hat{f}_i(c_1) = a(i)_1$ and $\hat{g}_j(m_0) = b(j)_0$. Therefore we have $f_i \circ \hat{f}_i(c_1) = g_j \circ \hat{g}_j(m_0)$ as $f_i(a(i)_1) = g_j(b(j)_0)$ by the assumption that $f_i * g_j$ is well-defined.

Define $F : X \times Y \to G$ by

$$F(x, y) = \begin{cases} H(x, y), & \text{if } x \in {}_{c_0}C_{c_1}, y \in {}_{d_0}D_{d_1}, \\ K(x, n_0), & \text{if } x \in {}_{m_0}M_{m_1}, y \in {}_{d_0}D_{d_1}, \\ H(x, d_1), & \text{if } x \in {}_{c_0}C_{c_1}, y \in {}_{n_0}N_{n_1}, \\ K(x, y), & \text{if } x \in {}_{m_0}M_{m_1}, y \in {}_{n_0}N_{n_1}, \end{cases}$$

where $X = {}_{c_0}C_{c_1} \nabla {}_{m_0}M_{m_1}, Y = {}_{d_0}D_{d_1} \nabla {}_{n_0}N_{n_1} \in \mathfrak{pG}$.

For the following proof, let $i, j = \{1, 2\}$. We have F is well-defined as:

- $F(c_1 \nabla m_0, y)$ is unique for all $y \in Y$:

$$F(c_1, y) = \begin{cases} H(c_1, y) \\ = f_i \circ \hat{f}_i(c_1), & \text{if } y \in {}_{d_0}D_{d_1}, \\ H(c_1, d_1) \\ = f_i \circ \hat{f}_i(c_1), & \text{if } y \in {}_{n_0}N_{n_1}, \end{cases}$$

$$F(m_0, y) = \begin{cases} K(m_0, n_0) \\ = g_j \circ \hat{g}_j(m_0), & \text{if } y \in {}_{d_0}D_{d_1}, \\ K(m_0, y) \\ = g_j \circ \hat{g}_j(m_0), & \text{if } y \in {}_{n_0}N_{n_1}, \end{cases}$$

  so we have $F(c_1, y) = f_i \circ \hat{f}_i(c_1) = g_j \circ \hat{g}_j(m_0) = F(m_0, y)$, hence $F(c_1 \nabla m_0, y) = F(c_1, y) = F(m_0, y)$; which shows $F(c_1 \nabla m_0, y)$ is unique for all $y \in Y$.

- $F(x, d_1 \nabla n_0)$ is unique for all $x \in X$:

$$F(x, d_1) = \left\{ \begin{array}{ll} H(x, d_1) & \\ = f_2 \circ \hat{f}_2(x), & \text{if } x \in {}_{c_0}C_{c_1} \\ H(x, n_0) & \\ = g_1 \circ \hat{g}_1(x), & \text{if } x \in {}_{m_0}M_{m_1} \end{array} \right\} = F(x, n_0)$$

  so we have $F(x, d_1 \nabla n_0) = F(x, d_1) = F(x, n_0)$, hence $F(x, d_1 \nabla n_0)$ is unique for all $x \in X$.

Clearly $\hat{f}_i$ and $\hat{g}_j$ are $\mathfrak{p}$-paths of $({}_{a(i)_0}A^i_{a(i)_1}) \nabla ({}_{b(j)_0}B^j_{b(j)_1})$, therefore we have $\hat{f}_i * \hat{g}_j : X \to A^i \nabla B^j$ is well-defined as $\hat{f}_i * \hat{g}_j(c_1 \nabla m_0) = \hat{f}_i(c_1) = a(i)_1 \nabla b(j)_0 = \hat{g}_j(m_0)$. Also we have $\hat{f}_i * \hat{g}_j$ is surjective as $\hat{f}_i$ and $\hat{g}_j$ are surjective.

Following we prove F is a homotopy relative to $\{c_0, m_1\}$ of $(f_1 * g_1) \circ (\hat{f}_1 * \hat{g}_1)$ and $(f_2 * g_2) \circ (\hat{f}_2 * \hat{g}_2)$.

We claim $(f_i * g_j) \circ (\hat{f}_i * \hat{g}_j) = (f_i \circ \hat{f}_i) * (g_j \circ \hat{g}_j)$ first, where $(f_i \circ \hat{f}_i) * (g_j \circ \hat{g}_j) : X \to A^i \nabla B^j$ is defined by $(f_i \circ \hat{f}_i) * (g_j \circ \hat{g}_j)(x) = (f_i \circ \hat{f}_i)(x)$ if $x \in C$, $(f_i \circ \hat{f}_i) * (g_j \circ \hat{g}_j)(x) = (g_j \circ \hat{g}_j)(x)$ if $x \in M$, and

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$(f_i \circ \hat{f}_i) * (g_j \circ \hat{g}_j)(c_1 \nabla m_0) = a(i)_0 \nabla b(j)_1$. Clearly the definition of $(f_i \circ \hat{f}_i) * (g_j \circ \hat{g}_j)$ is well-defined, and in fact it is a surjective map. If $x \in C$, then we have $(f_i * g_j) \circ (\hat{f}_i * \hat{g}_j)(x) = (f_i * g_j)(\hat{f}_i(x)) = f_i(\hat{f}_i(x)) = f_i \circ \hat{f}_i(x) = (f_i \circ \hat{f}_i) * (g_j \circ \hat{g}_j)(x)$. If $x \in M$, then we have $(f_i * g_j) \circ (\hat{f}_i * \hat{g}_j)(x) = (f_i * g_j)(\hat{g}_j(x)) = g_j(\hat{g}_j(x)) = g_j \circ \hat{g}_j(x) = (f_i \circ \hat{f}_i) * (g_j \circ \hat{g}_j)(x)$. Moreover, we have $(f_i * g_j) \circ (\hat{f}_i * \hat{g}_j)(x) = a(i)_0 \nabla b(j)_1 = (f_i \circ \hat{f}_i) * (g_j \circ \hat{g}_j)(x)$ if $x = c_1 \nabla m_0$. Therefore we have $(f_i * g_j) \circ (\hat{f}_i * \hat{g}_j) = (f_i \circ \hat{f}_i) * (g_j \circ \hat{g}_j)$.

By the definition of F, it is trivial that F is relation-preserving. Moreover, similar to the proof (3) of Theorem 2.8, F can be proved to be continuous by the pasting lemma.

For showing $F(x, d_0) = (f_1 * g_1) \circ (\hat{f}_1 * \hat{g}_1)(x)$ and $F(x, n_1) = (f_2 * g_2) \circ (\hat{f}_2 * \hat{g}_2)(x)$, we have

$$\begin{aligned} F(x, d_0) &= \left\{ \begin{array}{ll} H(x, d_0) = f_1 \circ \hat{f}_1(x), & \text{if } x \in C \\ K(x, n_0) = g_1 \circ \hat{g}_1(x), & \text{if } x \in M \end{array} \right\} \\ &= (f_1 \circ \hat{f}_1) * (g_1 \circ \hat{g}_1)(x) \\ &= (f_1 * g_1) \circ (\hat{f}_1 * \hat{g}_1)(x); \end{aligned}$$

and

$$\begin{aligned} F(x, n_1) &= \left\{ \begin{array}{ll} H(x, d_1) = f_2 \circ \hat{f}_2(x), & \text{if } x \in C \\ K(x, n_1) = g_2 \circ \hat{g}_2(x), & \text{if } x \in M \end{array} \right\} \\ &= (f_2 \circ \hat{f}_2) * (g_2 \circ \hat{g}_2)(x) \\ &= (f_2 * g_2) \circ (\hat{f}_2 * \hat{g}_2)(x). \end{aligned}$$

For showing $F(c_0, y) = (f_i * g_i) \circ (\hat{f}_i * \hat{g}_i)(c_0)$ and $F(m_1, y) = (f_i * g_i) \circ (\hat{f}_i * \hat{g}_i)(m_1)$, we have

- $F(c_0, y) = (f_i * g_i) \circ (\hat{f}_i * \hat{g}_i)(c_0)$:

$$\begin{aligned} F(c_0, y) &= \left\{ \begin{array}{ll} H(c_0, y) & \\ = f_i \circ \hat{f}_i(c_0) & \\ = f_i(a(i)_0), & \text{if } y \in D \\ H(c_0, d_1) & \\ = f_i \circ \hat{f}_i(c_0) & \\ = f_i(a(i)_0), & \text{if } y \in N \end{array} \right\} \\ &= f_i(a(i)_0) = (f_i * g_i)(a(i)_0) \\ &= (f_i * g_i) \circ (\hat{f}_i * \hat{g}_i)(c_0). \end{aligned}$$

- $F(m_1, y) = (f_i * g_i) \circ (\hat{f}_i * \hat{g}_i)(m_1)$:

$$\begin{aligned} F(m_1, y) &= \left\{ \begin{array}{ll} K(m_1, n_0) & \\ = g_i \circ \hat{g}_i(m_1) & \\ = g_i(b(i)_1), & \text{if } y \in D \\ K(m_1, y) & \\ = g_i \circ \hat{g}_i(m_1) & \\ = g_i(b(i)_1), & \text{if } y \in N \end{array} \right\} \\ &= g_i(b(i)_1) = (f_i * g_i)(b(i)_1) \\ &= (f_i * g_i) \circ (\hat{f}_i * \hat{g}_i)(m_1). \end{aligned}$$

Therefore we have shown $(f_1 * g_1) \circ (\hat{f}_1 * \hat{g}_1) \approx^{\mathfrak{p}}_{\{c_0, m_1\}} (f_2 * g_2) \circ (\hat{f}_2 * \hat{g}_2)$[3], and hence we have $f_1 * g_{1p} \approx^{\mathfrak{p}} f_2 * g_2$, which implies $*$ is well-defined on path-homotopy classes.

□

Definition 3.8. Let G be a topological graph and $A \in \mathfrak{p}\mathfrak{S}$. For $x \in G$, we say the $\mathfrak{p}$-path $e_x : A \to G$, $e_x(a) = x$ for all $a \in A$ is *constant* (based at x). For any $\mathfrak{p}$-path $f : {}_cB_d \to G$, we call the constant $\mathfrak{p}$-path e_x (resp. e_y) is a *left* (resp. *right*) *identity* $\mathfrak{p}$-path of f if $f(c) = x$ (resp. $f(d) = y$). We denote the left (resp. right) identity $\mathfrak{p}$-path of $f : {}_cB_d \to G$ by $e_{f(c)}$ (resp. $e_{f(d)}$).

Theorem 3.9. *Let $A \in \mathfrak{p}\mathfrak{S}$, and $f : {}_{a_0}A_{a_1} \to G$ be a $\mathfrak{p}$-path of G. Then we have $[e_{f(a_0)}] * [f] = [f] = [f] * [e_{f(a_1)}]$.*

Proof. We shall prove that $[e_{f(a_0)}] * [f] = [f]$ only, since the proof of $[f] = [f] * [e_{f(a_1)}]$ is similar. Let $e_{f(a_0)}$ be a constant $\mathfrak{p}$-path with co-domain any Khalimsky arc ${}_{b_0}B_{b_1} \in \mathfrak{p}\mathfrak{S}$. We claim $e_{f(a_0)} * f : I \to G$ is path-homotopy to f, where $I = {}_{b_0}B_{b_1} \nabla {}_{a_0}A_{a_1}$.

Let $\hat{f} : I \to {}_{a_0}A_{a_1}$ be defined by $\hat{f}(x) = a_0$ if $x \in B$, $\hat{f}(x) = a_0$ if $x = b_1 \nabla a_0$, $\hat{f}(x) = x$ if $x \in A$, and $id_I : I \to I$ be an identity map from I into I. Clearly $\hat{f}$ and id_I are surjective maps.

It is easy to check that $(e_{f(a_0)} * f) \circ id_I$ and $f \circ \hat{f}$ are homotopic relative to $\{b_0, a_1\}$, since for any $x \in I$, we have

$$\begin{aligned} (e_{f(a_0)} * f) \circ id_I(x) &= (e_{f(a_0)} * f)(x) \\ &= \left\{ \begin{array}{ll} f(a_0), & \text{if } x \in B \\ f(a_0), & \text{if } x = b_1 \nabla a_0 \\ f(x), & \text{if } x \in A \end{array} \right\} \\ &= f \circ \hat{f}(x). \end{aligned}$$

hence $(e_{f(a_0)} * f) \circ id_I$ and $f \circ \hat{f}$ define the same function, which implies $(e_{f(a_0)} * f) \circ id_I$ and $f \circ \hat{f}$ are automatically homotopic relative to $\{b_0, a_1\}$ by the reflexivity of the equivalence relation $\approx^{\mathfrak{p}}_{*}$.

Therefore we have $e_{f(a_0)} * f_p \approx^{\mathfrak{p}} f$, and hence $[e_{f(a_0)}] * [f] = [f]$. □

Definition 3.10. Let $f : {}_{a_0}A_{a_1} \to G$ be a $\mathfrak{p}$-path of the topological graph G. We call the $\mathfrak{p}$-path $g : {}_{a_1}A_{a_0} \to G$, $Ff(a) = Fg(a), \forall a \in A$, the *inverse* $\mathfrak{p}$-path of f, where F is the forgetful functor which forgets the binary relation. We denote the inverse $\mathfrak{p}$-path of f by $\overleftarrow{f}$.

[3]In fact F is a homotopy relative to $\{c_0, c_1 \nabla m_0, m_1\}$ of $(f_1 * g_1) \circ (\hat{f}_1 * \hat{g}_1)$ and $(f_2 * g_2) \circ (\hat{f}_2 * \hat{g}_2)$.

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Lemma 3.11. *$f : {}_{a_0}A_{a_1} \to G$ and $g : {}_{b_0}B_{b_1} \to G$ are $\mathfrak{p}$-paths of the topological graph G, and $\overleftarrow{f}$, $\overleftarrow{g}$ are the inverse $\mathfrak{p}$-paths of f and g respectively. Then we have $f_p \approx^{\mathfrak{p}} g$ if and only if $\overleftarrow{f}_p \approx^{\mathfrak{p}} \overleftarrow{g}$.*

Proof. We shall prove $(f_p \approx^{\mathfrak{p}} g) \Rightarrow (\overleftarrow{f}_p \approx^{\mathfrak{p}} \overleftarrow{g})$ only, the inverse proof is similar.

The relation $f_p \approx^{\mathfrak{p}} g$ means that there exist Khalimsky arcs ${}_{c_0}C_{c_1}, {}_{d_0}D_{d_1} \in \mathfrak{p}\mathfrak{S}$ and two surjective maps $\hat{f} : {}_{c_0}C_{c_1} \to A, \hat{g} : {}_{c_0}C_{c_1} \to B$ and a homotopy $H : {}_{c_0}C_{c_1} \times {}_{d_0}D_{d_1} \to G$ such that $H(c, d_0) = f \circ \hat{f}(c), H(c, d_1) = g \circ \hat{g}(c)$ and $H(c_0, d) = f \circ \hat{f}(c_0) = g \circ \hat{g}(c_0)$, $H(c_1, d) = f \circ \hat{f}(c_1) = g \circ \hat{g}(c_1)$ for all $c \in C, d \in D$.

For showing $\overleftarrow{f}_p \approx^{\mathfrak{p}} \overleftarrow{g}$, define $\tilde{f} : {}_{c_1}C_{c_0} \to {}_{a_1}A_{a_0}$ and $\tilde{g} : {}_{c_1}C_{c_0} \to {}_{b_1}B_{b_0}$ by $\tilde{f}(c) = \hat{f}(c)$ and $\tilde{g}(c) = \hat{g}(c)$ for all $c \in C$. Clearly $\tilde{f}$ and $\tilde{g}$ are surjective maps.

Moreover, it is easy to check that $\overleftarrow{f} \circ \tilde{f}(c+) = f \circ \hat{f}(c-)$ and $\overleftarrow{g} \circ \tilde{g}(c+) = g \circ \hat{g}(c-)$ for all $c \in C$. Remember that we have defined $c+ = c$ and $c- = c$ if c is the final point and the initial point of C respectively.

Define $K : {}_{c_1}C_{c_0} \times {}_{d_0}D_{d_1} \to G$ by

$$K(c+, d) = H(c-, d),$$

for all $c \in C, d \in D$.

We check K is a homotopy relative to $\{c_1, c_0\}$ of $\overleftarrow{f} \circ \tilde{f}$ and $\overleftarrow{g} \circ \tilde{g}$. Clearly K is a map. And we have $K(c+, d_0) = H(c-, d_0) = f \circ \hat{f}(c-) = \overleftarrow{f} \circ \tilde{f}(c+)$ and $K(c+, d_1) = H(c-, d_1) = g \circ \hat{g}(c-) = \overleftarrow{g} \circ \tilde{g}(c+)$. Finally K is trivially relative to $\{c_1, c_0\}$ as H is relative to $\{c_1, c_0\}$. So we have shown $\overleftarrow{f} \circ \tilde{f} \approx^{\mathfrak{c}}_{\{c_1, c_0\}} \overleftarrow{g} \circ \tilde{g}$, and which implies $\overleftarrow{f}_p \approx^{\mathfrak{p}} \overleftarrow{g}$. Hence we have $(f_p \approx^{\mathfrak{p}} g) \Rightarrow (\overleftarrow{f}_p \approx^{\mathfrak{p}} \overleftarrow{g})$. ☐

Theorem 3.12. *$f : {}_{a_0}A_{a_1} \to G$ is a $\mathfrak{p}$-path of the topological graph G, and $\overleftarrow{f} : {}_{a_1}A_{a_0} \to G$ is the inverse $\mathfrak{p}$-path of f. Then we have $[f] * [\overleftarrow{f}] = [e_{f(a_0)}]$ and $[\overleftarrow{f}] * [f] = [e_{f(a_1)}]$, where $e_{f(a_0)} : {}_{b_0}B_{b_1} \to G$ and $e_{f(a_1)} : {}_{c_0}C_{c_1} \to G$ are the left and right identity $\mathfrak{p}$-paths of f respectively.*

Proof. We shall only prove that $[f] * [\overleftarrow{f}] = [e_{f(a_0)}]$, the proof of $[\overleftarrow{f}] * [f] = [e_{f(a_1)}]$ is similar.

By Lemma 3.11, the problem of showing $[f] * [\overleftarrow{f}] = [e_{f(a_0)}]$ is equivalent to show that $f * \overleftarrow{f}_p \approx^{\mathfrak{p}} e_{f(a_0)}$. Define $g : I \to {}_{b_0}B_{b_1}$ and $h : I \to {}_{a_0}A_{a_1} \nabla_{a_1} A_{a_0}$ by

$$g(i) = \begin{cases} i, & \text{if } i \in B, \\ b_1, & \text{others;} \end{cases} \quad h(i) = \begin{cases} a_0, & \text{if } i \in B, \\ a_0, & \text{if } i = b_1 \nabla a_0, \\ i, & \text{others.} \end{cases}$$

where $I = {}_{b_0}B_{b_1} \nabla_{a_0} A_{a_1} \nabla_{a_1} A_{a_0}$. It is clear that g and h are surjective maps.

In order to show $e_{f(a_0)p} \approx^{\mathfrak{p}} f * \overleftarrow{f}$, we define $H : I \times {}_{a_0}A_{a_1} \to G$ by

$$H(i,a) = \begin{cases} e_{f(a_0)} \circ g(i), & \text{if } i \in B, \\ \begin{cases} f \circ h(i), & \text{if } i \prec a, \\ f \circ h(a), & \text{if } i = a,, \\ f \circ h(a), & \text{if } i \succ a. \end{cases} & \text{if } i \in {}_{a_0}A_{a_1}, \\ \begin{cases} \overleftarrow{f} \circ h(i), & \text{if } i \prec a, \\ \overleftarrow{f} \circ h(a), & \text{if } i = a,, \\ \overleftarrow{f} \circ h(a), & \text{if } i \succ a. \end{cases} & \text{if } i \in {}_{a_1}A_{a_0}; \end{cases}$$

where the ordering $\prec$ defined in the definition of H is w.r.t. ${}_{a_0}A_{a_1}$.

It is easy to check that H is well-defined, since we have $H(b_1 \nabla a_0, a) = e_{f(a_0)} \circ g(b_1 \nabla a_0) = e_{f(a_0)}(b_1) = f(a_0) = f \circ h(a_0) = H(b_1 \nabla a_0, a)$ and $H(a_1 \nabla a_1, a) = f \circ h(a_1) = f(a_1) = \overleftarrow{f}(a_1) = \overleftarrow{f} \circ h(a_1) = H(a_1 \nabla a_1, a)$. Also it is trivial that H is continuous and relation-preserving. Hence H is a map.

For showing H is a homotopy of $e_{f(a_0)} \circ g$ and $(f * \overleftarrow{f}) \circ h$, we have

$$H(i, a_0) = \left\{ \begin{array}{ll} e_{f(a_0)} \circ g(i), & \text{if } i \in B \\ f \circ h(a_0) & \\ = f(a_0) & \\ = e_{f(a_0)} \circ g(i), & \text{if } i \in {}_{a_0}A_{a_1} \\ \overleftarrow{f} \circ h(a_0) & \\ = \overleftarrow{f}(a_0) = f(a_0) & \\ = e_{f(a_0)} \circ g(i), & \text{if } i \in {}_{a_1}A_{a_0} \end{array} \right\}$$
$$= e_{f(a_0)} \circ g(i).$$

$$H(i, a_1) = \left\{ \begin{array}{ll} e_{f(a_0)} \circ g(i) & \\ = f(a_0) = f \circ h(i) & \\ = \overleftarrow{f} \circ h(i), & \text{if } i \in B \\ f \circ h(i), & \text{if } i \in {}_{a_0}A_{a_1} \\ \overleftarrow{f} \circ h(i), & \text{if } i \in {}_{a_1}A_{a_0} \end{array} \right\}$$
$$= ((f \circ h) * (\overleftarrow{f} \circ h))(i) = (f * \overleftarrow{f}) \circ h(i).$$

For showing H is relative to $\{b_0, a_0\}$, we have

$$(1) \begin{cases} H(b_0, a) = e_{f(a_0)} \circ g(b_0) = e_{f(a_0)}(b_0) = f(a_0), \\ e_{f(a_0)} \circ g(b_0) = e_{f(a_0)}(b_0) = f(a_0), \\ (f * \overleftarrow{f}) \circ h(b_0) = (f * \overleftarrow{f})(a_0) = f(a_0); \end{cases}$$

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hence we have $H(b_0, a) = e_{f(a_0)} \circ g(b_0) = (f * \overleftarrow{f}) \circ h(b_0)$, and

$$(2)\begin{cases} H(a_0, a) = \overleftarrow{f} \circ h(a_0) = \overleftarrow{f}(a_0) = f(a_0), \\ e_{f(a_0)} \circ g(a_0) = e_{f(a_0)}(b_1) = f(a_0), \\ (f * \overleftarrow{f}) \circ h(a_0) = (f * \overleftarrow{f})(a_0) = f(a_0); \end{cases}$$

hence we have $H(a_0, a) = e_{f(a_0)} \circ g(a_0) = (f * \overleftarrow{f}) \circ h(a_0)$.

Therefore we have shown that H is a homotopy relative to $\{b_0, a_0\}$ of $e_{f(a_0)} \circ g$ and $(f * \overleftarrow{f}) \circ h$, which implies $e_{f(a_0)p} \approx^{\mathfrak{p}} f * \overleftarrow{f}$. Hence we have $[f] * [\overleftarrow{f}] = [e_{f(a_0)}]$. □

Theorem 3.13. *$f : {}_{a_0}A_{a_1} \to G$, $g : {}_{b_0}B_{b_1} \to G$, and $h : {}_{c_0}C_{c_1} \to G$ are $\mathfrak{p}$-paths of the topological graph G, then we have $([f]*[g])*[h] = [f]*([g]*[h])$.*

Proof. The proof of this theorem is much easier than the similar case of the classical homotopy theory in algebraic topology.

We claim $(f * g) * h_p \approx^{\mathfrak{p}} f * (g * h)$. In fact, we shall show that $(f * g) * h = f * (g * h)$, and then $(f * g) * h$ is automatically path-homotopic to $f * (g * h)$.

Clearly we have $f * g : {}_{a_0}A_{a_1} \triangledown {}_{b_0}B_{b_1} \to G$ and $g * h : {}_{b_0}B_{b_1} \triangledown {}_{c_0}C_{c_1} \to G$ are defined by

$$(1)\ f * g(x) = \begin{cases} f(x), & \text{if } x \in A, \\ f(a_1) = g(b_0), & \text{if } x = a_1 \triangledown b_0, \\ g(x), & \text{if } x \in B; \end{cases}$$

$$(2)\ g * h(x) = \begin{cases} g(x), & \text{if } x \in B, \\ g(b_1) = h(c_0), & \text{if } x = b_1 \triangledown c_0, \\ h(x), & \text{if } x \in C. \end{cases}$$

So from the equations (1) and (2), we have $(f * g) * h : ({}_{a_0}A_{a_1} \triangledown {}_{b_0}B_{b_1}) \triangledown {}_{c_0}C_{c_1} \to G$ and $f * (g * h) : {}_{a_0}A_{a_1} \triangledown ({}_{b_0}B_{b_1} \triangledown {}_{c_0}C_{c_1}) \to G$ are defined by

$$(3)\ (f * g) * h(x) = \left\{ \begin{array}{ll} f * g(x), & x \in A \triangledown B \\ f * g(b_1) & \\ = h(c_0), & x = b_1 \triangledown c_0 \\ h(x), & x \in C \end{array} \right\}$$
$$= \begin{cases} f(x), & x \in A, \\ f(a_1) = g(b_0), & x = a_1 \triangledown b_0, \\ g(x), & x \in B, \\ g(b_1) = h(c_0), & x = b_1 \triangledown c_0, \\ h(x), & x \in C; \end{cases}$$

$$(4)\ f * (g * h)(x) = \left\{ \begin{array}{ll} f(x), & x \in A \\ f(a_1) & \\ = g * h(b_0), & x = a_1 \triangledown b_0 \\ g * h(x), & x \in B \triangledown C \end{array} \right\}$$
$$= \begin{cases} f(x), & x \in A, \\ f(a_1) = g(b_0), & x = a_1 \triangledown b_0, \\ g(x), & x \in B, \\ g(b_1) = h(c_0), & x = b_1 \triangledown c_0, \\ h(x), & x \in C; \end{cases}$$

Therefore from the equations (3) and (4), the problem of showing $(f*g)*h = f*(g*h)$ can be reduced to a simple problem of showing the associativity of the operator- lexicographic union $\triangledown$, that is, $({}_{a_0}A_{a_1} \triangledown {}_{b_0}B_{b_1}) \triangledown {}_{c_0}C_{c_1} = {}_{a_0}A_{a_1} \triangledown ({}_{b_0}B_{b_1} \triangledown {}_{c_0}C_{c_1})$. By Definition 2.2 of the lexicographic union $\triangledown$, it is trivial that $\triangledown$ is an associative operator. Hence we have $(f * g) * h = f * (g * h)$. So we have $([f] * [g]) * [h] = [f] * ([g] * [h])$. □

Denote $\mathfrak{p}\mathfrak{S}_G$ to be the set of all $\mathfrak{p}$-paths in a topological graph G, and let $\pi(G) = \mathfrak{p}\mathfrak{S}_G/_p \approx^{\mathfrak{p}}$. Hence $\pi(G)$ is the set of path-homotopy classes of $\mathfrak{p}$-paths in G. Then $\pi(G)$ becomes a *groupoid* under the operation $*$, but loses to be a group due to the lack of satisfying the closure axiom and the uniqueness identity axiom. We may call $(\pi(G), *)$ the *Khalimsky fundamental groupoid* of the topological graph G.

Definition 3.14. A *pointed* topological graph is a pair (G, x) consisting of a topological graph $G = (X, T, R)$ and a base point $x \in X$.

By Definition 2.3, we have the following definition of $\mathfrak{p}$-loops.

Definition 3.15. Let ${}_aA_b \in \mathfrak{p}\mathfrak{S}$ be a Khalimsky arc, and (G, x) is a pointed topological graph. We call the $\mathfrak{p}$-path $f : {}_aA_b \to (G, x)$ a $\mathfrak{p}$-*loop* (based at x) of (G, x) if $f(a) = f(b)(= x)$.

Like the definition of simple loops in general graph theory, we say a $\mathfrak{p}$-loop $f : {}_aA_b \to G$ is *simple* if $f(c) \neq f(d)$ for all $c, d \in A \setminus \{a, b\}, c \neq d$.

By Definition 3.1, we have the product of $\mathfrak{p}$-loops based at x is well-defined. That is, for any pair of $\mathfrak{p}$-loops based at x, $f : {}_aA_b \to (G, x)$ and $g : {}_cB_d \to (G, x)$, we have $f * g : A \triangledown B \to (G, x)$ the product of $\mathfrak{p}$-loops f and g. Moreover, by the mathematical induction, the product of $\mathfrak{p}$-loops base at x (resp. $\mathfrak{p}$-paths) is well-defined for any positive integer numbers of $\mathfrak{p}$-loops based at x (resp. $\mathfrak{p}$-paths), i.e., we have $f_1 * f_2 * \ldots * f_n = (f_1 * f_2 * \ldots * f_{n-1}) * f_n$ for any $f_1, f_2, \ldots, f_{n-1}, f_n$ the $\mathfrak{p}$-loops based at x (or $f_1, f_2, \ldots, f_{n-1}, f_n$ the $\mathfrak{p}$-paths such that the image of the final point of

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f_i equal to the image of the initial point of f_{i+1}, for all $1 \le i \le n-1$).

Let (G, x) be a pointed topological graph and $\mathfrak{p}\mathfrak{S}_{(G,x)}$ be the set of all $\mathfrak{p}$-loops in G. Clearly from Theorems 3.7, 3.9, 3.12, and 3.13, $\pi(G,x) = \mathfrak{p}\mathfrak{S}_{(G,x)}/_p \approx^{\mathfrak{p}}$ becomes a group under the operation $*$. Therefore we have the following theorem:

Theorem 3.16. *$(\pi(G,x),*)$ is a group.*

We call $(\pi(G,x),*)$ the *Khalimsky fundamental group* of the pointed topological graph (G,x). Sometimes we shall use the notation $\pi(G,x)$ to indicate the Khalimsky fundamental group of (G,x) if there has no other operators.

4 Some Properties of the Khalimsky Fundamental Groups of Topological Graphs

Definition 4.1. A topological graph G is said to be *path-connected* if given any pair of points $x, y \in G$, there exists a Khalimsky arc ${}_{a_0}A_{a_1} \in \mathfrak{p}\mathfrak{S}$ and a $\mathfrak{p}$-path $f \in \mathfrak{p}\mathfrak{S}_G$ such that $f(a_0) = x$ and $f(a_1) = y$.

Lemma 4.2. *Every path-connected topological graph is connected.*

Proof. Let G be a path-connected topological graph. Suppose G is not connected, then there exists a partition U, V of G such that $U, V \neq \emptyset$. If $x \in U$ and $y \in V$, clearly there exists a Khalimsky arc ${}_{a_0}A_{a_1} \in \mathfrak{p}\mathfrak{S}$ and a $\mathfrak{p}$-path $f \in \mathfrak{p}\mathfrak{S}_G$ such that $f(a_0) = x$ and $f(a_1) = y$ as G is path-connected. So $f^{-1}(U)$ and $f^{-1}(V)$ becomes a partition of ${}_{a_0}A_{a_1}$. Contradiction, since ${}_{a_0}A_{a_1}$ is connected. □

Theorem 4.3. *Let G be a topological graph, and $x, y \in G$ are two points of G. If there exists a $\mathfrak{p}$-path α from x to y, then $(\pi(G,x),*)$ is group-isomorphic to $(\pi(G,y),*)$.*

Proof. We define a map $\psi : (\pi(G,x),*) \to (\pi(G,y),*)$ by $\psi([f]) = [\overleftarrow{\alpha}] * [f] * [\alpha]$.

Clearly ψ is well-defined as $*$ is well-defined on the path-homotopy classes. Moreover, it is trivial that $[\overleftarrow{\alpha}] * [f] * [\alpha] \in (\pi(G,y),*)$ as f is a $\mathfrak{p}$-loop.

We check ψ is a group-isomorphism between $(\pi(G,x),*)$ and $(\pi(G,y),*)$:

1. ψ is a group-homomorphism: We have $\psi([f] * [g]) = \psi([f*g]) = [\overleftarrow{\alpha}]*[f*g]*[\alpha] = [\overleftarrow{\alpha}]*([f]*[g])*[\alpha] = [\overleftarrow{\alpha}]*([f]*[e_x]*[g])*[\alpha] = [\overleftarrow{\alpha}]*([f]*([\alpha]*[\overleftarrow{\alpha}])*[g])*[\alpha] = ([\overleftarrow{\alpha}]*[f]*[\alpha])*([\overleftarrow{\alpha}]*[g]*[\alpha]) = \psi([f]) * \psi([g])$. Here we have used the properties of the left identity, inverse and associativity of the Khalimsky fundamental groups.
2. ψ has a inverse: Let $\phi : (\pi(G,y),*) \to (\pi(G,x),*)$ be defined by $\phi([h]) = [\alpha] * [h] * [\overleftarrow{\alpha}]$. A similar proof as above can easily show that ϕ is well-defined and is a homomorphism. We claim ψ and ϕ are the inverse to each other. For each $[h] \in (\pi(G,y),*)$, we have $\psi(\phi([h])) = \psi([\alpha] * [h] * [\overleftarrow{\alpha}]) = [\overleftarrow{\alpha}] * ([\alpha] * [h] * [\overleftarrow{\alpha}]) * [\alpha] = ([\overleftarrow{\alpha}] * [\alpha]) * [h] * ([\overleftarrow{\alpha}] * [\alpha]) = [e_y] * [h] * [e_y] = [h]$. Similarly, for each $[f] \in (\pi(G,x),*)$, we shall have $\phi(\psi([f])) = [f]$.

Therefore from (1) and (2) above, ψ is a group-isomorphism of $(\pi(G,x),*)$ and $(\pi(G,y),*)$. Therefore $(\pi(G,x),*)$ and $(\pi(G,y),*)$ are group-isomorphic. □

Lemma 4.4. *If G is a path-connected topological graph, and $x, y \in G$ are any two points of G, then $(\pi(G,x),*)$ is group-isomorphic to $(\pi(G,y),*)$.*

Proof. Trivial. □

Definition 4.5. A topological graph G is said to be *simple connected* if G is path-connected and $(\pi(G,x),*)$ is a trivial group for some $x \in G$, hence for every $x \in G$.

Theorem 4.6. *G is a simple connected topological graph. If $f, g \in \mathfrak{p}\mathfrak{S}_G$ with the same initial and final points, then $f_p \approx^{\mathfrak{p}} g$.*

Proof. We claim $[f] = [g]$. Suppose f and g have initial and final points x and y, then clearly $f * \overleftarrow{g}$ is a $\mathfrak{p}$-loop based at x. So we have $[f * \overleftarrow{g}] = [e_x]$. By the identity and associativity axioms of the Khalimsky fundamental group, we have $[g] = [e_x * g] = [(f * \overleftarrow{g}) * g] = [f * (\overleftarrow{g} * g)] = [f * e_y] = [f]$. So we have $[f] = [g]$, and hence $f_p \approx^{\mathfrak{p}} g$. □

Lemma 4.7. *$(G,x), (G',x')$ are pointed topological graphs, and $\Phi : (G,x) \to (G',x')$ is a map with $\Phi(x) = x'$. Then we have*

1. *$f \in \mathfrak{p}\mathfrak{S}_{(G,x)} \Rightarrow \Phi(f) \in \mathfrak{p}\mathfrak{S}_{(G',x')}$,*
2. *$f_p \approx^{\mathfrak{p}} g \Rightarrow \Phi(f)_p \approx^{\mathfrak{p}} \Phi(g)$.*

Proof. (1) is trivial, and we claim (2) only. Since we have $f_p \approx^{\mathfrak{p}} g$, suppose H is a homotopy relative to $\{c_0, c_1\}$ of $f \circ \hat{f}$ and $g \circ \hat{g}$, where $\hat{f}$ and $\hat{g}$ are surjective maps with common domain ${}_{c_0}C_{c_1}$

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and respective co-domains the domains of f and g. Then clearly $\Phi \circ H$ is a homotopy relative to $\{c_0, c_1\}$ of $(\Phi \circ f) \circ \hat{f}$ and $(\Phi \circ g) \circ \hat{g}$. So we have $\Phi(f)_p \approx^{\mathfrak{p}} \Phi(g)$. ☐

Theorem 4.8. *$(G, x), (G', x')$ are pointed topological graphs, and $\Phi : (G, x) \to (G', x')$ is a map with $\Phi(x) = x'$. Then the map $\Phi_* : (\pi(G, x), *) \to (\pi(G', x'), *)$, $\Phi_*([f]) = [\Phi(f)]$ is well-defined, and is a group-homomorphism.*

Proof. By Lemma 4.7, it is clear that Φ_* is well-defined. We check that Φ_* is a group-homomorphism. Let $f, g \in \mathfrak{p}\mathfrak{S}_{(G,x)}$ such that $f : {}_{a_0}A_{a_1} \to (G, x)$ and $g : {}_{b_0}B_{b_1} \to (G, x)$, then we have $\Phi_*([f] * [g]) = \Phi_*([f * g]) = [\Phi(f * g)]$. As we have

$$\begin{aligned}\Phi(f * g) &= \Phi(\left\{\begin{array}{ll} f(x), & x \in A \\ f(a_1) = g(b_0), & x = a_1 \nabla b_0 \\ g(x), & x \in B \end{array}\right\}) \\ &= \left\{\begin{array}{ll} \Phi \circ f(x), & x \in A \\ \Phi(f(a_1)) = \Phi(g(b_0)), & x = a_1 \nabla b_0 \\ \Phi \circ g(x), & x \in B \end{array}\right\} \\ &= \Phi(f) * \Phi(g).\end{aligned}$$

So we have $\Phi_*([f] * [g]) = [\Phi(f) * \Phi(g)] = [\Phi(f)] * [\Phi(g)] = \Phi_*([f]) * \Phi_*([g])$; which shows that Φ_* is a group-homomorphism. ☐

With notations are same as Theorem 4.8, we call Φ_* the group-homomorphism induced by Φ.

Theorem 4.9. *$(G, x), (G', x')$ and (G'', x'') are pointed topological graphs, and $\Phi : (G, x) \to (G', x')$ and $\Psi : (G', x') \to (G'', x'')$ are maps such that $\Phi(x) = x'$ and $\Psi(x') = x''$. Then we have*

1. $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$,
2. *id_* is the identity homomorphism;*

where $id : (G, x) \to (G, x)$ is the identity map.

Proof. Let $f \in \mathfrak{p}\mathfrak{S}_{(G,x)}$, then we have

1. $(\Psi \circ \Phi)_*([f]) = [\Psi \circ \Phi(f)] = \Psi_*([\Phi(f)]) = \Psi_*(\Phi_*[f]) = \Psi_* \circ \Phi_*([f])$; and
2. $id_*([f]) = [id(f)] = [f]$.

☐

Theorem 4.10. *Let $(G, x) = (X, T, R, x)$ and $(G', x') = (X', T', R', x')$ be pointed topological graphs, and $\Phi : (G, x) \to (G', x')$ is a map such that $\Phi(x) = x'$. If Φ is a homeomorphism w.r.t. T and T', and is an isomorphism w.r.t. R and R', then the induced homomorphism $\Phi_* : (\pi(G, x), *) \to (\pi(G', x'), *)$ is a group-isomorphism.*

Proof. Let $\Psi : (G', x') \to (G, x)$ be the inverse of Φ. Clearly we have $= \Phi \circ \Psi = id_{G'}$ and $\Psi \circ \Phi = id_G$. So we have $\Phi_* \circ \Psi_* = (\Phi \circ \Psi)_* = (id_{G'})_*$ and $\Psi_* \circ \Phi_* = (\Psi \circ \Phi)_* = (id_G)_*$, where $(id_{G'})_*$ and $(id_G)_*$ are the identity homomorphism of $(\pi(G', x'), *)$ and $(\pi(G, x), *)$ respectively. Therefore Φ_* is a group-isomorphism. ☐

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