The Hamiltonian path passing through prescribed edges in a star graph with faulty edges *

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Abstract

Let $F_e$ be the set of faulty edges of $S_n$ and $E_0$ be the edge set of some pairwise vertex-disjoint paths of $S_n$. In [5], the authors showed that $E_0$ lies on a Hamiltonian path $P(u, v)$ of $S_n - F_e$ where $d(u, v)$ is odd, $|F_e| \leq n - 3$, $|E_0| \leq 2n - 7 - 2|F_e|$. In this paper we improve the previous result as follows: $E_0$ lies on a Hamiltonian path $P(u, v)$ of $S_n - F_e$ where $d(u, v)$ is odd, $|F_e| \leq n - 3$, $|E_0| \leq 2n - 6 - 2|F_e|$.

1 Introduction

The hypercube $Q_n$ and star graph $S_n$ are the most popular and efficient interconnection networks. They possess many excellent properties such as recursive structure, symmetry, small diameter, low degree, popular structure embedding, easy routing, and optimal connectivity and fault-tolerance. There are many literatures for fault tolerance of Hamiltonian properties of hypercube and star graphs.

The interconnection network is usually represented by an undirected graph where vertices represent processors and edges represent links. Let $G = (V_0 \cup V_1, E)$ be a bipartite graph where $V_0$ and $V_1$ are two disjoint vertex sets such that each edge of $E$ consists of one vertex from each set. A Hamiltonian path is a spanning path of $G$. A Hamiltonian cycle is a cycle that visits every vertex exactly once. A Hamiltonian graph is a graph that contains a Hamiltonian cycle. Let $F_e$ be a set of faulty edges and $|F_e|$ be the number of $F_e$. A graph $G = (V, E)$ is $k$ edges Hamiltonian if $G - F_e$ is a Hamiltonian graph for every $F_e \subset E$ and $|F_e| \leq k$. A bipartite graph $G = (V_0 \cup V_1, E)$ is Hamiltonian laceable if there exists a Hamiltonian path between $x, y$ for any $x \in V_0, y \in V_1$. A Hamiltonian-laceable graph $G = (V_0 \cup V_1, E)$ is hyper-Hamiltonian laceable if for any vertex $v \in V_i$, $i = 0, 1$, there is a Hamiltonian path of $G - \{v\}$ between any two vertices of $V_{1-i}$. A bipartite graph $G = (V_0 \cup V_1, E)$ is $k$ edges Hamiltonian laceable if $G - F_e$ is Hamiltonian laceable for every $F_e \subset E$ and $|F_e| \leq k$. In [3], Li et al. proved that $S_n$ is $(n - 3)$ edges Hamiltonian laceable and $(n - 4)$ edges hyper-Hamiltonian laceable with $n \geq 4$.

On the other hand, Dvořák investigated the problem about prescribed edges. Given a set of fault-free prescribed edges in a hypercube, there is a Hamiltonian cycle passing through all edges of this set in the hypercube. In [1], they showed that $Q_n$ contains a Hamiltonian cycle passing through all edges of any set of prescribed edges $E_0$ if and only if the subgraph induced by $E_0$ consists of pairwise vertex-disjoint paths and $|E_0| \leq 2n - 3$. Wang et al. further presented the result that all edges of $E_0$ lie on a Hamiltonian cycle of $Q_n - F_e$ if $E_0$ is a linear forest (i.e., pairwise vertex-disjoint paths) and $|F_e| \leq n - 2, |E_0| \leq 2n - 3 - 2|F_e|$ in [4]. Let $d(x, y)$ be the distance of vertices $x$ and $y$. The authors of [2] introduced that all edges of $E_0$ lie on a Hamiltonian path $P(x, y)$ of $Q_n$, where $E_0$ is a linear forest with $|E_0| \leq 2n - 4$ and at most one vertex of $x, y$ is incident with one edge of $E_0$ and $d(x, y)$ is odd.

Hung et al. [5] proved that there all edges of $E_0$ lie on a Hamiltonian cycle of $S_n - F_e$, if $|F_e| \leq n - 3$, $|E_0| \leq 2n - 5 - 2|F_e|$ and lie on a Hamiltonian path $P(u, v)$ where $d(u, v)$ is odd, $|F_e| \leq n - 3$, $|E_0| \leq 2n - 7 - 2|F_e|$. In this paper, we improve the results of [5]. We prove that there is a Hamiltonian path $P(x, y)$ passing through all edges in $E_0$ of $S_n - F_e$ if $|F_e| \leq n - 3$ and $|E_0| \leq 2n - 6 - 2|F_e|$ for $n \geq 4$ where $E_0$ is a linear forest.

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2 Preliminaries

In this section, we will introduce some definitions, notations and properties about star graphs which are useful for our proofs.

Definition 1 The n-dimensional star graph is denoted by $S_n$. The vertex set $V = \{v_i\}$ is a permutation of $1, 2, \ldots, n$ and edge set $E = \{(v_1v_2 \ldots v_{i-1}v_i v_{i+1}v_{i+2} \ldots v_n) | v_1 \ldots v_n \in V$ and $2 \leq i \leq n\}$.

A star graph $S_n$ is an edge symmetric and node symmetric graph. By definition, $S_n$ contains $n!$ vertices and $\frac{n(n-1)}{2}$ edges. The graph $S_n$ can be decomposed to $n$ copies of $(n-1)$-dimensional star graphs. Give $2 \leq k \leq n$, we can use $S_k$ to denote the subgraph of $S_n$ induced by the vertices whose nth bit is $i$. We further denote the induced subgraph $S_{i_k}^1 \cup S_{i_k}^2 \ldots \cup S_{i_k}^n$ for $1 \leq i_1, i_2, \ldots, i_k \leq n$. Let $S_{n}$ be the edge set of some pairwise vertix-disjoint paths of $S_n$. All edges of $E_0$ lie on a Hamiltonian cycle of $S_n - F_e$, if $|F_e| \leq 1$, $|E_0| \leq 3 - 2|F_e|$ and lie on a Hamiltonian path $P(u, v)$ of $S_n$ where $d(u, v)$ is odd, $|E_0| = 1$ for $n \geq 4$.

Lemma 3 (See[5].) Let $F_e$ be the set of faulty edges of $S_n$ and $E_0$ be the edge set of some pairwise vertix-disjoint paths of $S_n$. All edges of $E_0$ lie on a Hamiltonian cycle of $S_n - F_e$, if $|F_e| \leq 1$, $|E_0| \leq 3 - 2|F_e|$ and lie on a Hamiltonian path $P(u, v)$ of $S_n$ where $d(u, v)$ is odd, $|E_0| = 1$ for $n \geq 4$.

Lemma 4 (See[5].) Let $I = \{2, 3, \ldots, n\}$. Let $s$ and $t$ be arbitrary vertices of $S_n$, $E$ be the edge set of some pairwise vertix-disjoint paths of $S_n$. There exist the maximum number of $I$ edge symmetric and node symmetric graph.

Lemma 5 (See[5].) Let $F_e$ be the set of faulty edges of $S_n$ and $E_0$ be the edge set of some pairwise vertix-disjoint paths of $S_n$. All edges of $E_0$ lie on a Hamiltonian cycle of $S_n - F_e$, if $|F_e| \leq 1$, $|E_0| \leq 3 - 2|F_e|$ and lie on a Hamiltonian path $P(u, v)$ of $S_n$ where $d(u, v)$ is odd, $|E_0| = 1$ for $n \geq 4$.

3 Hamiltonian paths passing through prescribed edges

In this section, we will show that all edges of $E_0$ lie on a Hamiltonian path $P(x, y)$ of $S_n - F_e$ if $|F_e| \leq n - 3$ and $|E_0| \leq 2n - 6 - 2|F_e|$ for $n \geq 4$ where $E_0$ is a linear forest.

Lemma 6 Let $I = 2, 3, \ldots, n$. Let $s_1, s_2$ and $t_1, t_2$ be vertices of $S_n^I = (V_0 \cup V_1, E)$ for $s_1, s_2 \in V_0, t_1, t_2 \in V_1, [s_1], [t_2] = 1, n \geq 5$. There exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of $S_n^I$.

Proof. Let $U = \{s_1, s_2, t_1, t_2\}$ and $U' = U \cap V(S_n^I)$ for $2 \leq i \leq n$. We will prove this lemma with the following two lemmas.

Case 1: $|U'| = 4$ for some $2 \leq i \leq n$.

Without loss of generality, we can assume that $i = 2$. Applying Lemma 2, we can construct a Hamiltonian path between $s_1$ and $t_1$ of $S_n^I$. Without loss of generality, we can denote this Hamiltonian path as $\langle s_1, P(s_1, x) \rangle x, P(x, a) \rangle a, b \rangle b, P(b, \phi(y)) \rangle \phi(y) \rangle \phi(y) \rangle \phi(y) \rangle \phi(y) \rangle y, P(y, t_2) \rangle t_1$ and $P(s_2, t_2)$ are two spanning paths of $S_n^I$, as illustrated in Figure 1.

Figure 1: The illustration of case 1 of Lemma 6.

Suppose that $\phi(x) \in S_n^I, \phi(y) \in S_n^I, 3 \leq i \neq j \leq n$, Applying Lemma 2, we can construct a Hamiltonian path $P(\phi(x), \phi(y))$ of...
Thus, \( \langle s_1 \xrightarrow{P(s_1,x)} x, \phi(x) \xrightarrow{P(\phi(x),y)} \phi(y) \xrightarrow{P(\phi(y),t_1)} t_1 \rangle \) and \( P(s_2,t_2) \) are two spanning paths of \( S_n^I \).

**Case 2:** \(|U^I| = 3 \) and \(|U^J| = 1 \) for some \( 2 \leq i \neq j \leq n \).

Without loss of generality, we can assume that \( i = 2, j = n \) and \( s_1, s_2 \in S_n^I \).

**Case 2.1** \( t_2 \in S_n^J, t_1 \in S_n^J \).

Applying Lemma 2, we can construct a Hamiltonian path \( \langle s_1 \xrightarrow{P(s_1,x)} x, s_2 \xrightarrow{P(s_2,t_2)} t_2 \rangle \) of \( S_n^J \) with \( \phi(x) \notin S_n^J \).

Suppose that \( \phi(x) \in S_n^J \). Applying Lemma 1, we can obtain that \( \phi(x,y) \) and \( \phi(y) \in S_n^I \) with \( \phi(y) \in \phi(x) \). Applying Lemma 2, we can construct a Hamiltonian path \( \langle \phi(x), y, z \xrightarrow{P(z,t_1)} t_1 \rangle \) of \( S_n^I \) with \( \phi(y) \in \phi(z) \). We can construct the Hamiltonian path \( P(\phi(y), \phi(z)) \) of \( S_n^I \) with \( \phi(y) \notin \phi(z) \). Thus, \( \langle s_1 \xrightarrow{P(s_1,x)} x, \phi(x), \phi(y), \phi(z) \xrightarrow{P(\phi(y),\phi(z))} \phi(z), z \xrightarrow{P(z,t_1)} t_1 \rangle \) and \( \langle s_2 \xrightarrow{P(s_2,t_2)} t_2 \rangle \) are two spanning paths of \( S_n^I \), as illustrated in Figure 2.

![Figure 2](image-url)

**Case 2.2** \( t_1 \in S_n^J, t_2 \in S_n^J \).

Applying Lemma 2, we can construct a Hamiltonian path \( \langle s_2 \xrightarrow{P(s_2,x)} x, s_1 \xrightarrow{P(s_1,t_1)} t_1 \rangle \) of \( S_n^J \) with \( x \notin S_n^J \).

Applying Lemma 2, we can construct the Hamiltonian path \( P(\phi(x), t_2) \) of \( S_n^J \). Thus, \( P(s_1,t_1) \) and \( \langle s_2 \xrightarrow{P(s_2,x)} x, \phi(x) \xrightarrow{P(\phi(x),y)} \phi(y) \xrightarrow{P(\phi(y),t_2)} t_2 \rangle \) are two spanning paths of \( S_n^I \).

**Case 3:** \(|U^I| = 3 \) and \(|U^J| = 2 \) for some \( 2 \leq i \neq j \leq n \).

Without loss of generality, we can assume that \( i = 2, j = n \) and \( t_2 \in S_n^J \).

**Case 3.1** \( s_1 \in S_n^J \).

Applying Lemma 2, we can construct a Hamiltonian path \( \langle s_2 \xrightarrow{P(s_2,u)} u, v \xrightarrow{P(v,t_1)} t_1 \rangle \) of \( S_n^J \) for \( \phi(u) \notin \phi(v) \). Applying Lemma 2, we can construct a Hamiltonian path \( \langle s_1, y, x \xrightarrow{P(x,t_2)} t_2 \rangle \) of \( S_n^J \) for \( \phi(x) \in \phi(y) \). We can also construct the Hamiltonian path \( P(\phi(u), \phi(v)) \) of \( S_n^J \) for \( \phi(u) \notin \phi(v) \). Thus, \( \langle s_1, y, \phi(y), \phi(z) \xrightarrow{P(\phi(y),\phi(z))} \phi(z), z \xrightarrow{P(z,t_1)} t_1 \rangle \) and \( \langle s_2 \xrightarrow{P(s_2,u)} u, \phi(u) \xrightarrow{P(\phi(u),\phi(v))} \phi(v), v \xrightarrow{P(v,t_2)} t_2 \rangle \) are two spanning paths of \( S_n^J \), as illustrated in Figure 3.

![Figure 3](image-url)

**Case 3.2** \( s_2 \in S_n^J \).

Applying Lemma 2, we can construct a Hamiltonian path \( \langle s_1 \xrightarrow{P(s_1,x)} x, y \xrightarrow{P(y,t_1)} t_1 \rangle \) of \( S_n^J \) with \( [x], [y] \notin \{1, 2\} \). Thus, \( \langle s_1 \xrightarrow{P(s_1,x)} x, \phi(x) \xrightarrow{P(\phi(x),y)} \phi(y) \xrightarrow{P(\phi(y),t_1)} t_1 \rangle \) and \( P(s_2,t_2) \) are two spanning paths of \( S_n^I \) as illustrated in Figure 4.

**Case 3.3** \( t_1 \in S_n^J \).
Let \( x \) be a neighbor of \( t_1 \) in \( S^2_n \) with \( \phi(x) \in S^0_n \). Applying Lemma 2, we can construct a Hamiltonian path
\[
\langle s_2, P(x, \phi(y)) \rightarrow \phi(y), s_1, P(s_1, \phi(x)) \rightarrow \phi(x) \rangle
\]
of \( S^2_n \) for \( y \notin S^1_{n^2} \). Without loss of generality, we can assume that \( y \in S^{n}_{n} \). Applying Lemma 2, we can construct a Hamiltonian path
\[
\langle x, t_1, z, P(z, t_2) \rightarrow t_2 \rangle
\]
of \( S^2_n \) for \( \phi(x) \in S^1_{n^2} \). We can also construct the Hamiltonian path
\[
P(y, \phi(z)) \text{ in } S^{3, \ldots, n}_{n}.
\]
Thus,
\[
\langle s_1, P(s_1, \phi(x)) \rightarrow \phi(x), x, t_1 \rangle
\]
and
\[
\langle s_2, P(z, \phi(y)) \rightarrow \phi(y), y, P(y, \phi(z)) \rightarrow \phi(z), z, P(z, t_2) \rightarrow t_2 \rangle
\]
are two spanning paths of \( S^1_{n} \), as illustrated in Figure 5.

**Case 4:** \(|U^i| = 2, |U^j| = 1\) and \(|U^k| = 1\) for some different integers \( 2 \leq i, j, k \leq n \).

Without loss of generality, we can assume that \( i = 2, j = 3 \) and \( k = n \).

**Case 4.1** \( s_1, s_2 \in S^0_n \) or \( t_1, t_2 \in S^2_n \).

Without loss of generality, we can assume that \( t_1, t_2 \in S^0_n \) and \( s_2 \in S^1_{n^2}, s_1 \in S^0_n \). Applying Lemma 1, we can obtain that \( (y, t_2) \) is an edge in \( S^2_n \) with \( \phi(y) \in S^0_n \). Applying Lemma 2, we can construct a Hamiltonian path
\[
\langle y, t_1, x, P(x, t_2) \rightarrow t_2 \rangle \text{ of } S^2_n \text{ for } \phi(x) \in S^{n-1}_{n}.
\]
Applying Lemma 2, we can construct a Hamiltonian path \( P(s_1, \phi(y)) \) of \( S^0_{n^2} \) and a Hamiltonian path \( P(s_2, \phi(x)) \) in \( S^{3, \ldots, n}_{n} \). Thus,
\[
\langle s_1, P(s_1, \phi(y)) \rightarrow \phi(y), y, t_1 \rangle \text{ and } \langle s_2, P(s_2, \phi(x)) \rightarrow \phi(x), x, P(x, t_2) \rightarrow t_2 \rangle
\]
are two spanning paths of \( S^1_{n} \), as illustrated in Figure 6.

**Case 4.2** \( s_1, t_1 \in S^2_n \) or \( s_2, t_2 \in S^2_n \).

Without loss of generality, we can assume that \( s_2, t_2 \in S^2_n \) and \( t_1 \in S^0_n, s_2 \in S^0_n \). Applying Lemma 2, we can construct a Hamiltonian path \( P(s_2, t_2) \) of \( S^2_n \) and a Hamiltonian path \( P(s_1, t_1) \) in \( S^{3, \ldots, n}_{n} \). These two paths are the two spanning disjoint paths of \( S^1_{n} \).

**Case 4.3** \( s_2, t_1 \in S^2_n \).

Suppose that \( |s_2|_1 = n \) and \( |t_1|_1 = 3 \).

Applying Lemma 1, we can obtain that \( (x, t_1) \) is an edge in \( S^2_n \) with \( \phi(x) \in S^{n-1}_{n} \).

Applying Lemma 2, we can construct a Hamiltonian path \( P(s_2, y, x, t_1) \) of \( S^2_n \) for \( \phi(y) \in S^0_n \). Applying Lemma 2, we can construct a Hamiltonian path \( P(s_1, \phi(x)) \) in \( S^{3, \ldots, n}_{n} \). Thus,
\[
\langle s_1, P(s_1, \phi(x)) \rightarrow \phi(x), x, t_1 \rangle \text{ and } \langle s_2, P(s_2, \phi(y)) \rightarrow \phi(y), y, \phi(y), P(\phi(y), t_2) \rightarrow t_2 \rangle
\]
are two spanning paths of \( S^1_{n} \), as illustrated in Figure 7.

Suppose that \( |s_2|_1 \neq n \) or \( |t_1|_1 \neq 3 \). We can assume that \( |t_1|_1 \neq 3 \). Let \( (x, t_1) \) be an edge in \( S^2_n \) with \( \phi(x) \in S^3_n \). Applying Lemma 2, we can construct a Hamiltonian path \( P(s_2, y, x, t_1) \) of \( S^2_n \) for \( \phi(y) \in S^0_n \). Applying Lemma 2, we can construct a Hamiltonian path \( P(s_1, \phi(x)) \) in \( S^{3, \ldots, n}_{n} \). Thus,
\[
\langle s_1, P(s_1, \phi(x)) \rightarrow \phi(x), x, t_1 \rangle \text{ and } \langle s_2, P(s_2, \phi(y)) \rightarrow \phi(y), y, \phi(y), P(\phi(y), t_2) \rightarrow t_2 \rangle
\]
Figure 7: The illustration of case 4.3 of Lemma 6.

Let Lemma 7

Case 4.4 apply Lemma 2, we can construct Hamiltonian path $P(s, t)$ of $S_n^I - \{f\}$.

Case 1: $f \in S_n^I$ for some $i = 2, n$.

Without loss of generality, we can assume that $i = n$. Let $y$ be a vertex of $(V_0 \cap S_n^I)$. Applying Lemma 2, we can construct a Hamiltonian path $P(y, t)$ of $S_n^I - \{f\}$. Applying Lemma 2, we can construct the Hamiltonian paths of $S_n^I$ for $2 \leq i \leq n - 1$. Thus, we can construct a Hamiltonian path $P(s, \phi(y))$ of $S_n^I - \{f\}$.

Case 2: $f \notin (S_2^I \cup S_n^I)$.

Without loss of generality, we can assume that $f \in S_n^I$. Let $(x, \phi(x))$ and $(y, \phi(y))$ be edges of $S_n^I$ for $x \in (V_1 \cap S_1^I)$, $(x, \phi(x)) \in (V_0 \cap S_1^I)$ and $y \in (V_0 \cap S_1^I)$. Applying Lemma 2, we can construct the Hamiltonian paths $P(s, x)$ and $P(\phi(x), \phi(y))$ of $S_n^I$ and $S_n^I - \{f\}$, respectively. Applying Lemma 2, we can construct a Hamiltonian path $P(y, t)$ of $S_n^I - \{f\}$.

Case 4.4 $s_1, t_2 \in S_2^I$

Since we have not used the properties $[s_1] = [t_2] = 1$ in the proof of Case 4.3, the proof of this case is the same as Case 4.3 with exchanging $s_1$ and $s_2$ and exchanging $t_1$ and $t_2$.

Case 5: The four vertices $s_1, s_2, t_1, t_2$ are in different subgraphs.

Without loss of generality, we can assume that $s_1 \in S_2^I$, $s_2 \in S_2^I$, $t_1 \in S_n^I$. Applying Lemma 2, we can construct Hamiltonian path $P(s_1, t_1)$ of $S_2^I$. Applying Lemma 2, we can construct Hamiltonian paths $P(s_2, t_2)$ of $S_2^I$. Thus, $P(s_1, t_1)$ and $P(s_2, t_2)$ are two spanning paths of $S_n^I$.

Lemma 7 Let $I = 2, 3, \ldots, n$ for $n \geq 5$. The graph $S_n^I = (V_0 \cup V_1, E)$ is hyper-Hamiltonian laceable.

Proof. Let $s, t, f$ be arbitrary vertices of $S_n^I$ where $d(s, t)$ is even and $d(s, f)$ is odd. By the symmetry of star graphs, we can arrange the vertices $s \in S_n^I$ and $t \in S_n^I$ for $1 \leq i < j \leq n$. Without loss of generality, we can assume that $s, t \in V_0$ and $s \in S_2^I$, $t \in S_n^I$ and $f \in V_1$. We will construct a Hamiltonian path $P(s, t)$ of $S_n^I - \{f\}$.

Theorem 1 Let $u, v$ be two vertices of $S_n^I = (V_0 \cup V_1, E)$ where $d(u, v)$ is odd. Let $F_e$ be the set of faulty edges of $S_n$ and $E_0$ be the edge set of some pairwise vertex-disjoint paths of $S_n$ where every edge of $E_0$ is not incident to $u$ or $v$. There exists a Hamiltonian path $P(u, v)$ of $S_n^I - F_e$ passing through every edge of $E_0$ for $|F_e| \leq n - 3, |E_0| \leq 2n - 6 - 2|F_e|$ for $n \geq 4$.

Proof. We will prove this theorem by induction on $n$. This theorem is true for $n = 4$ by the brute force method. This theorem holds for $|F_e| = n - 3$ by Lemma 2. In the follows, we will assume that $|F_e| \leq n - 4$. By symmetry of star graph, we can arrange every faulty edge $F_e$ in some $S_n^I$ and at most one prescribed edge of $E_0$ is not in any $S_n^I$ for $1 \leq i \leq n$. We will construct a Hamiltonian path $P(u, v)$ passing through all edges of $E_0$ for any $u \in V_1, v \in V_0$. Let $E_0 = E_0 \cap E(S_n^I)$ for $1 \leq i \leq n$. Without loss of generality, we can assume that $2|F_e| + |E_0| \geq 2|E_0| + |E_0|$ for every $1 \leq k \leq n$.

Case 1: $2|F_e| + |E_0| = 2n - 6$

Let $e_0 = (b, b')$ be an edge of $E_0$ where at
most one edge of $E_0$ is incident to $e_0$ for $b \in V_1, b' \in V_0$. And let $E_1 = E_0 - \{e_0\}$.

**Case 1.1** $u,v \in S_1^1$

Applying Lemma 5, we can construct a Hamiltonian cycle $\langle u \xrightarrow{P(u,v)} x, b \xrightarrow{P(b,v)} v, z \xrightarrow{P(z,w)} w, b' \xrightarrow{P(b',v)} y, u \rangle$ of $S_1^1 - F_1^1$ passing through every edge of $E_1$ with $(b,x), (b',y) \notin E_0$. Applying Lemma 6, we can construct two spanning disjoint paths $P(\phi(x), \phi(u))$ and $P(\phi(z), \phi(y))$ in $S_n^2\cdots^n$. Thus, $\langle u \xrightarrow{P(u,v)} x, \phi(x) \xrightarrow{P(\phi(x),\phi(u))} \phi(y), y \xrightarrow{P(y,b')} b', b \xrightarrow{P(b,v)} v \rangle$ is the Hamiltonian path passing through $E_0$ of $S_n - F_1$.

**Case 1.2** One of $u,v \in S_n^1$

Without loss of generality, we can assume that $u \in S_n^1$. Applying Lemma 5, we can construct the Hamiltonian cycle $\langle u \xrightarrow{P(u,b)} b, z \xrightarrow{P(z,b')} b', x \xrightarrow{P(x,y)} y, u \rangle$ of $S_n^1 - F_1^1$ passing through $E_1$ with $(b,z), (b',x) \notin E_0$. Suppose that $v \neq \phi(x)$. Applying Lemma 6, we can construct two spanning disjoint paths $P(\phi(y), v)$ and $P(\phi(z), \phi(x))$ in $S_n^2\cdots^n$. Thus, $\langle u \xrightarrow{P(u,b)} b, b' \xrightarrow{P(b',z)} z, \phi(z) \xrightarrow{P(\phi(z),\phi(x))} \phi(y), y \xrightarrow{P(y,b')} b', b \xrightarrow{P(b,v)} v \rangle$ is the Hamiltonian path passing through $E_0$ of $S_n - F_1$.

Suppose that $v = \phi(x)$. Applying Lemma 7, we can construct the Hamiltonian path $P(\phi(z), \phi(y))$ in $S_n^2\cdots^n - \{v\}$. Thus, $\langle u \xrightarrow{P(u,b)} b, b' \xrightarrow{P(b',z)} z, \phi(z) \xrightarrow{P(\phi(z),\phi(y))} \phi(y), y \xrightarrow{P(y,b')} b', b \xrightarrow{P(b,v)} v \rangle$ is the Hamiltonian path passing through $E_0$ of $S_n - F_1$.

**Case 1.3** $u,v \notin S_1^1$

Applying Lemma 5, we can construct a Hamiltonian cycle $C = \langle b, x \xrightarrow{P(x,b')} b', y \xrightarrow{P(y,b)} b \rangle$ of $S_n^1 - F_1^1$ passing through every edge of $E_1$ with $(b,x), (b',y) \notin E_0$. Suppose that $u \neq \phi(x)$ and $v \neq \phi(y)$. Applying Lemma 6, we can construct two spanning disjoint paths $P(u, \phi(y))$ and $P(\phi(x), v)$ in $S_n^2\cdots^n$. Thus, $\langle u \xrightarrow{P(u,\phi(y))} \phi(y), y \xrightarrow{P(y,b')} b', b \xrightarrow{P(b',v)} v \rangle$ is the Hamiltonian path passing through $E_0$ of $S_n - F_1$.

Suppose that either $u = \phi(x)$ or $v = \phi(y)$. Without loss of generality, we can assume that $v = \phi(y)$. Applying Lemma 7, we can construct the Hamiltonian path $P(u, \phi(x))$ in $S_n^2\cdots^n - \{v\}$. Thus, $\langle u \xrightarrow{P(u,\phi(x))} \phi(x), x \xrightarrow{P(x,y)} y, \phi(y) = v \rangle$ is the Hamiltonian path passing through $E_0$ of $S_n - F_1$.

Suppose that both $u = \phi(x)$ and $v = \phi(y)$.

**Case 1.3.1** $u,v \in S_n^2$ for some $2 \leq i \leq n$

Without loss of generality, we can assume that $u,v \in S_n^2$. We can choose an edge $(z_1, z_2)$ of the Hamiltonian cycle $C$ with $z_1 \in V_1$ and $\phi(z_1), \phi(z_2) \notin S_n^1$. Without loss of generality, we can denote the Hamiltonian cycle $C$ as $\langle b, x \xrightarrow{P(x,b')} b', y \xrightarrow{P(y,z_2)} z_2, z_1 \xrightarrow{P(z_1,b)} b \rangle$.

Suppose that $\phi(1) \in S_n^{n-1}$. Let $(\phi(z_1), \phi(z_2))$ be an edge of $S_2^1$ with $z \in S_n^2$. Applying Lemma 2, we can construct a Hamiltonian path $(z_3 \xrightarrow{P(z_3,a)} a, v)$ of $S_n^2 - \{u\}$.

Suppose that $\phi(1) \in S_n^{n-1}$. Let $(\phi(z_1), \phi(z_2))$ be an edge of $S_n^{n-1}$ with $z \in S_n^2$. Let $F^n(\phi(z_1)) = \{((\phi(z_1), w)) | [\phi(w)]_1 \neq 1 \text{ and } [\phi(w)]_1 \neq n - 2\}$. Applying Lemma 2, we can construct a Hamiltonian path $\langle \phi(a) \xrightarrow{P(\phi(a),\phi(z_1))} \phi(z_3), \phi(z_4), \phi(z_2) \rangle$ of $S_n^{n-1} - F^{n-1}(\phi(z_1))$. Thus, $z_3 \in S_n^{n-2}$. Let $F^2(\phi(z_1)) = \{((\phi(z_3), w)) | [\phi(w)]_1 \neq 1 \text{ and } [\phi(w)]_1 \neq 3\}$. Applying Lemma 2, we can construct the Hamiltonian path $\langle \phi(z_1), \phi(z_2), \phi(z_6) \xrightarrow{P(\phi(z_1),z_2)} z_4 \rangle$ of $S_n^2 - F^2(\phi(z_1))$. Thus, $z_6 \in S_n^3$. Applying Lemma 2, we can construct the Hamiltonian path $P(z_2, z_6)$ of $S_n^{2\cdots n-2}$. Therefore, $\langle u \xrightarrow{P(u,b')} b', b \xrightarrow{P(b,z_1)} z_1, \phi(z_1), \phi(z_3), z_3 \xrightarrow{P(z_3,a)} a, \phi(a) \xrightarrow{P(\phi(a),\phi(z_3))} \phi(z_5), z_5 \xrightarrow{P(z_5,\phi(z_6))} \phi(z_6), \phi(z_5) \xrightarrow{P(\phi(z_5),z_2)} z_4 \rangle$
Case 1.3.2 \( u \in S^i_n \) and \( v \in S^j_n \) for \( 2 \leq i \neq j \leq n \)
Without loss of generality, we can assume that \( u \in S^2_n \) and \( v \in S^3_n \). We can choose an vertex \( z_2 \in V_1 \) of the Hamiltonian cycle \( C \) where there is not any edge of \( E_0 \) incident to \( z_2 \) and \( \phi(z_2) \in S^2_n \). Without loss of generality, we can denote the Hamiltonian cycle \( C \) as \( (b, x) P^{(x,y)} y, v) \).

Applying Lemma 2, every \( S^i_n \) is Hamiltonian laceable and hyper-Hamiltonian laceable. Thus, we can construct the Hamiltonian path \( P(\phi(z_1), \phi(z_2)) \) of \( S^2_n \).

Case 2: 2\(|F_1^i| + |E_0^i| = 2n - 7\)
Let \( e_0 = (b, b') \) be the edge of \( E_0 \) for \( e_0 \notin S^1_n \). Suppose one of \( b \) and \( b' \) is in \( S^1_n \). This case is similar as Case 1. In the follows, we assume that \( b, b' \notin S^1_n \).

Case 2.1 \( u, v \in S^1_n \)
Applying Lemma 5, we can construct a Hamiltonian cycle \( (u P^{(u,y)} y, v) \).

Case 2.2 \( u \notin S^1_n \) or \( v \notin S^1_n \)
Without loss of generality, we can assume that \( u \in S^1_n \) or \( v \in S^3_n \). Applying Lemma 5, we can construct a Hamiltonian cycle \( (u P^{(u,x)} x, v) \).
Case 2.3 $u, v \in S^i_n$ for $i \neq 1$.

Without loss of generality, we can assume that $i = n$. By induction hypothesis, we can construct a Hamiltonian path $\langle u \stackrel{P(u,x)}{\longrightarrow} x, \phi(x) \stackrel{P(\phi(x),w)}{\longrightarrow} w \rangle$ passing through a prescribed edge of $S^n_n$ where $[x]_1 = 1$ and there is not prescribed edge incident $\phi(x)$. By induction hypothesis, we can construct a Hamiltonian cycle $\langle \phi(x), P(\phi(x),z), \phi(z) \rangle$ passing through $E^1_0$ of $S^n_n - F^1_2$ with $[z]_1 \neq [y]_1$. There exists a Hamiltonian path between any two vertices with odd distance passing through a prescribed edge in $S^j_n$ for $2 \leq j \leq n - 1$. Thus, we can construct a Hamiltonian path $P(\phi(z), \phi(y))$ passing through a prescribed edge of $S^{2\cdots n-1}_n$. Thus, $\langle u \stackrel{P(u,x)}{\longrightarrow} x, \phi(x) \stackrel{P(\phi(x),z)}{\longrightarrow} z, \phi(z) \rangle \stackrel{P(\phi(z),\phi(y))}{\longrightarrow} \phi(y), y \stackrel{P(y,v)}{\longrightarrow} v \rangle$ is the Hamiltonian path passing through $E^1_0$ of $S^n_n - F^1_2$, as illustrated in Figure 12.

Case 2.4 $u \in S^1_n, v \in S^j_n$ for $2 \leq i < j \leq n$.

Without loss of generality, we can assume that $i = 2$ and $j = n$. Let $e_0 = (b, b')$ be the prescribed edge who is not in $E^1(S^1_n)$ for $b \in V_1; b' \in V_0$.

Case 2.4.1 $b \in S^2_n$ and $b' \in S^2_n$.

Let $y$ be a vertex of $S^1_n \cap V_0$ where there is no prescribed edge incident to $y$ and $[y]_1 = n$. Applying Lemma 5, there exists a Hamiltonian cycle $\langle x \stackrel{P(x,y)}{\longrightarrow} y, x \rangle$ of $S^1_n - F^2_2$ passing through $E^1_0$ for $[x]_1 \neq 2$. Without loss of generality, we can assume $[x]_1 = 3$. Applying Lemma 2, we can construct a Hamiltonian path $\langle \phi(y), P(\phi(y),\phi(w_1)) \rangle \phi(w_1), b', \phi(w_2) \rangle$ of $S^n_n$ with $w_1, w_2 \in S^{n-1}_n \cup S^{n-2}_n$. Applying Lemma 2, we can construct a Hamiltonian path $\langle u \stackrel{P(u,a)}{\longrightarrow} a, b, z \rangle$ of $S^2_n$ with $\phi(a) \in S^{n-2}_n$ and $\phi(z) \in S^{n-1}_n$.

Suppose that $w_2 \in S^{n-1}_n$. Applying Lemma 2, we can also construct a Hamiltonian path $\langle u \stackrel{P(u,a)}{\longrightarrow} a, \phi(a) \rangle$ of $S^3_n \cdots S^{n-2}_n$ and a Hamiltonian path $\langle \phi(z), w_2 \rangle$ in $S^{n-1}_n$. Thus, $\langle u \stackrel{P(u,a)}{\longrightarrow} a, \phi(a) \rangle \phi(x), x \stackrel{P(x,y)}{\longrightarrow} \phi(y) \stackrel{P(\phi(y),\phi(w_1))}{\longrightarrow} \phi(w_1), b', b, z, \phi(z) \stackrel{P(\phi(z),w_2)}{\longrightarrow} w_2, \phi(w_2) \rangle$ is a Hamiltonian path passing through $E^1_0$ of $S_n - F_e$, as illustrated in Figure 13.

Figure 12: The Case 2.3 of Theorem 1.

Figure 13: The Case 2.4.1 of Theorem 1.
Lemma 2, we can also construct a Hamiltonian path \( P(\phi(a), \phi(x)) \) of \( S_{3}^{3-\ldots,n-2} \) and a Hamiltonian path \( P(\phi(z), w_1) \) in \( S_{n}^{n-1} \).

Thus, \( \langle u \xrightarrow{P(u,a)} a, \phi(a) \xrightarrow{P(\phi(a), \phi(x))} \phi(x), x \xrightarrow{P(x,y)} y, \phi(y) \xrightarrow{P(\phi(y), \phi(w_1))} \phi(w_1), w_1 \xrightarrow{P(w_1, \phi(z))} \phi(z), z, b, b' \xrightarrow{P(b', v)} v \rangle \) is a Hamiltonian path passing through \( E_{0} \) of \( S_{n} - F_{e} \), as illustrated in Figure 14.

**Figure 14:** The Case 2.4.1 of Theorem 1.

**Case 2.4.2** \( b' \in S_{n}^{2} \) and \( b \in S_{n}^{n} \).

Let \( x \) be a vertex of \( S_{n}^{1} \cap V_{1} \) where there is no prescribed edge incident to \( x \) and \( |x| = n \). Applying Lemma 5, there exists a Hamiltonian cycle \( \langle y \xrightarrow{P(y,x)} x, y \rangle \) of \( S_{n}^{1} - F_{e}^{1} \) passing through \( E_{1}^{0} \) for \( |y| \neq 2 \). Without loss of generality, we can assume \( |y| = 3 \). Applying Lemma 2, we can construct a Hamiltonian path \( \langle b \xrightarrow{P(b,z_1)} z_1, \phi(x), z_2 \xrightarrow{P(z_2,v)} v \rangle \) of \( S_{n}^{n} \) for \( \phi(z_1), \phi(z_2) \in S_{n}^{n-1} \cup S_{n}^{n-2} \).

Suppose that \( \phi(z_1) \in S_{n}^{n-1} \). Applying Lemma 2, we can construct a Hamiltonian path \( P(u,b') \) in \( S_{n}^{2} \) and a Hamiltonian path \( P(\phi(z_1), \phi(y)) \) in \( S_{n}^{3-\ldots,n-1} \). Thus, \( \langle u \xrightarrow{P(u,a)} b', b \xrightarrow{P(b,z_1)} z_1, \phi(z_1) \xrightarrow{P(\phi(z_1), \phi(y))} \phi(y), y \xrightarrow{P(y,x)} x, \phi(x), z_2 \xrightarrow{P(z_2,v)} v \rangle \) is a Hamiltonian path passing through \( E_{0} \) of \( S_{n} - F_{e} \), as illustrated in Figure 15.

**Figure 15:** The Case 2.4.2 of Theorem 1.

Let \( y \) be a vertex of \( S_{n}^{1} \cap V_{0} \) where there is no prescribed edge incident to \( y \) and \( |y| = 1 \). Applying Lemma 5, there exists a Hamiltonian cycle \( \langle x \xrightarrow{P(x,y)} y, x \rangle \) of \( S_{n}^{1} - F_{e}^{1} \) passing through \( E_{1}^{0} \) for \( |x| \neq 2 \).

Suppose that \( |x| \neq 2 \). Without loss of generality, we can assume \( |x| = n - 1 \). Let \( \phi(a), b' \) be an edge of \( S_{n}^{2} \) for \( a \in S_{n}^{2} \). Applying Lemma 2, we can construct a Hamiltonian path \( P(\phi(a), b'), z \xrightarrow{P(z,v)} v \) of \( S_{n}^{n} \) for \( \phi(z) \in S_{n}^{n-2} \).

Applying Lemma 2, we can construct a Hamiltonian path \( P(u,a) \) of \( S_{n}^{2} \), a Hamiltonian path \( P(b, \phi(x)) \) of \( S_{n}^{n-1} \) and a Hamiltonian path \( P(\phi(y), \phi(z)) \) of \( S_{n}^{n-1} \) and a Hamiltonian path \( P(\phi(y), \phi(z)) \) of \( S_{n}^{n-1} \). Thus, \( \langle u \xrightarrow{P(u,a)} a, \phi(a), b', b \xrightarrow{P(b,z_1)} z_1, \phi(z_1) \xrightarrow{P(\phi(z_1), \phi(y))} \phi(y), y \xrightarrow{P(y,x)} x, \phi(x), z_2 \xrightarrow{P(z_2,v)} v \rangle \) is a Hamiltonian path passing through \( E_{0} \) of \( S_{n} - F_{e} \), as illustrated in Figure 16.

Suppose that \( |x| = 2 \). Let \( \phi(a), b' \) be an edge of \( S_{n}^{2} \) for \( a \in S_{n}^{n-2} \). Applying Lemma 2, we can construct a Hamiltonian path \( \langle \phi(a), b', z \xrightarrow{P(z,v)} v \rangle \) of \( S_{n}^{n} \) for \( \phi(z) \in S_{n}^{n} \). Applying Lemma 2, we can construct a Hamiltonian path \( \langle u \xrightarrow{P(u,b')} b', b \xrightarrow{P(b,z_1)} z_1, \phi(z_1) \xrightarrow{P(\phi(z_1), \phi(y))} \phi(y), y \xrightarrow{P(y,x)} x, \phi(x), z_2 \xrightarrow{P(z_2,v)} v \rangle \) is a Hamiltonian path passing through \( E_{0} \) of \( S_{n} - F_{e} \), as illustrated in Figure 16.
Case 2.4.4

Paying Lemma 2, we can construct a Hamiltonian path passing through $(x, \phi)$ of $S_n^2$ for $c_1, c_2 \in S_n^{n-1} \cup S_n^{n-2}$. Suppose that $c_1 \in S_n^{n-1}$. Applying Lemma 2, we can construct a Hamiltonian path $P(b, c_1)$ of $S_n^{n-1}$ and a Hamiltonian path $P(\phi(y), a)$ of $S_n^{1..n-2}$. Thus, $\langle u, y, \phi(y) \rangle_{P(\phi(y), a)} \phi(c_2), \phi(x), x \mapsto P(x, y) y, \phi(y) \mapsto a, \phi(a), b, b \mapsto P(b, c_1) c_1, \phi(c_1) \phi(z), \mapsto z \mapsto P(z, v) v \rangle$ is a Hamiltonian path passing through $E_0$ of $S_n - F_e$, as illustrated in Figure 17.

Figure 16: The Case 2.4.3 of Theorem 1.

Figure 17: The Case 2.4.3 of Theorem 1.

Suppose that $c_2 \in S_n^{n-1}$. Applying Lemma 2, we can construct a Hamiltonian path $P(c_2, b)$ of $S_n^{n-1}$ and a Hamiltonian path $P(\phi(y))$ of $S_n^{3..n-2}$. Thus, $\langle u, P(u, \phi(c_2)) \phi(c_2), c_2 \mapsto P(c_2, b) b, b', \phi(a), a \mapsto P(a, \phi(y)) \phi(y), y \mapsto P(u, x) x, \phi(x), \phi(c_1) \mapsto \phi(z), z \mapsto P(z, v) v \rangle$ is a Hamiltonian path passing through $E_0$ of $S_n - F_e$.

Case 2.4.4 $b \notin S_n^2$ and $b' \notin S_n^2$ and $\{[b], [b']\} \neq \{2, n\}$. Without loss of generality, we can assume that $b \in S_n^{n-1}$ and $b' \in S_n^{n-2}$. Let $x$ be a vertex of $S_n^2 \cap V_1$ where there is no prescribed edge incident to $x$ and $|x| = 2$. Applying Lemma 5, there exists a Hamiltonian cycle $\langle x, P(x, y) y, x \rangle$ of $S_n^1 - F_e$ passing through $E_0$ for $|y| = n - 1$. Suppose $|y| = n$. Without loss of generality, we can assume that $|y| = 3$. Applying Lemma 2, we can construct a Hamiltonian path $P(\phi(y), b')$ of $S_n^{1..n-2}$ and a Hamiltonian path $P(b, v)$ of $S_n^{n-1..n}$. Thus, $\langle u, P(u, \phi(z)) \phi(x), x \mapsto P(x, y) y, \phi(y) \mapsto b, b \mapsto P(b, v) v \rangle$ is a Hamiltonian path passing through $E_0$ of $S_n - F_e$, as illustrated in Figure 18.

Figure 18: The Case 2.4.4 of Theorem 1.

Suppose $|y| = n$. Let $(\phi(y), z)$ be an edge of $S_n^2$ for $\phi(z) \in S_n^3$. Applying Lemma 2, we can construct a Hamiltonian path $\langle \phi(y), z, \phi(a) \rangle \mapsto P(\phi(a), v) v$ of $S_n^a$ for $a \in S_n^{n-1}$. Applying Lemma 2, we can construct a Hamiltonian path $P(\phi(z), b')$ of $S_n^{1..n-2}$ and a Hamiltonian path $P(b, a)$ of $S_n^{n-1}$. Thus, $\langle u, P(u, \phi(z)) \phi(x), x \mapsto P(x, y) y, \phi(y) \mapsto b, b \mapsto P(b, a) a, \phi(a) \mapsto P(\phi(a), v) v \rangle$ is a Hamiltonian path passing through $E_0$ of $S_n - F_e$.

Case 2.4.5 $(b, b') \in S_n^i$ for some $i \neq 1$ Without loss of generality, we can assume that $(b, b') \notin S_n^2$. Let $x$ be a vertex of $S_n^2 \cap V_1$ where there is no prescribed edge incident to $x$ and $|x| = 2$. Applying Lemma 5, there
exists a Hamiltonian cycle $x \xrightarrow{P(x,y)} y, x$ (of $S^1_n - F^1_n$ passing through $E^1_0$ for $|y|_1 \neq n$. Applying Lemma 2, we can construct a Hamiltonian path $P(u, \phi(x))$ of $S^2_n$ and a Hamiltonian path $P(\phi(y), v)$ of $S^n_3, \ldots , n$ passing through $(b, b')$. Thus, $(u \xrightarrow{P(u,\phi(x))} \phi(x), x \xrightarrow{P(x,y)} y, \phi(y) \xrightarrow{P(\phi(y),v)} v)$ is a Hamiltonian path passing through $E_0$ of $S_n - F_e$, as illustrated in Figure 19.

Case 2.4.6 $b \in S^1_n$ or $b' \in S^1_n$

Without loss of generality, we can assume that $b \in S^1_n$. Applying Lemma 5, there exists a Hamiltonian cycle $(b \xrightarrow{P(b,y)} y, b)$ of $S^1_n - F^1_n$ passing through $E^1_0$.

Suppose that $\phi(y) \neq u$. Applying Lemma 6, we can construct two spanning disjoint paths $P(u, b')$ and $P(\phi(y), v)$ of $S^n_3, \ldots , n$. Thus, $(u \xrightarrow{P(u,b')} b', b \xrightarrow{P(b,y)} y, \phi(y) \xrightarrow{P(\phi(y),v)} v)$ is a Hamiltonian path passing through $E_0$ of $S_n - F_e$, as illustrated in Figure 19.

Figure 19: The illustration of case 2.4.6 of Theorem 1.

Suppose that $\phi(y) = u$. Applying Lemma 7, we can construct a Hamiltonian path $P(b', v)$ of $S^n_3, \ldots , n - \{u\}$. Thus, $(u, y \xrightarrow{P(y,b)} b, b' \xrightarrow{P(b',v)} v)$ is a Hamiltonian path passing through $E_0$ of $S_n - F_e$, as illustrated in Figure 20.

Case 3: $2|F^k_e| + |E^k_0| \leq 2n - 8$ for every $1 \leq k \leq n$.

Case 3.1 $u, v \in S^i_n$ for some $1 \leq i \leq n$.

Without loss of generality, we can assume that $i = n$. By induction hypothesis, we can construct a Hamiltonian path $(u \xrightarrow{P(u,x)} x, y \xrightarrow{P(y,v)} v)$ passing through $E^0_0$ of $S^n_3, \ldots , n$ for $|y|_1 \neq n$. By induction hypothesis, there exists a Hamiltonian path passing through $E_0$ of $S_n - F_e$, as illustrated in Figure 20.

Case 3.2 $u \in S^1_n, v \in S^j_n$ for some $1 \leq i \neq j \leq n$.

Without loss of generality, we can assume that $i = 1, j = n$ and $u, b \in V_1$ and $v, b' \in V_0$.

Suppose $b \in S^1_n$ and $b' \in S^0_n$. Let $(b, z)$ and $(b', \phi(y))$ be edges of $S^1_n$ and $S^0_n$, respectively, for $\phi(z) \in S^2_n$ and $y \in S^2_n$. By induction hypothesis, there exist the Hamiltonian paths $(u \xrightarrow{P(u,x)} x, b, z)$ and $(\phi(y), b', \phi(a) \xrightarrow{P(\phi(a),v)} v)$ of $S^1_n$ and $S^0_n$, respectively, for $\phi(x) \in S^2_n$ and $a \in S^{n-1}_n$. By induction hypothesis, there exist the Hamiltonian paths $P(\phi(x), y)$ and $P(\phi(z), a)$ of $S^2_n$ and $S^{n-1}_n$, respectively. Thus, $(u \xrightarrow{P(u,x)} x, \phi(x) \xrightarrow{P(\phi(x),y)} y, \phi(y), b', b, z, \phi(z) \xrightarrow{P(\phi(z),a)} a, \phi(a) \xrightarrow{P(\phi(a),v)} v)$ is a Hamiltonian path passing through $E_0$ of $S_n - F_e$, as illustrated in Figure 21.

Suppose that $b \in S^1_n$ or $b' \in S^0_n$ but not both. Without loss of generality, we can assume that $b \in S^1_n$ and $b' \in S^0_n$. Let $(b, \phi(z))$ be an edge of $S^1_n$ for $z \in S^{n-1}_n$. By induction hypothesis, there exists a
Hamiltonian path $\langle u \xrightarrow{P(u,x)} x, b, \phi(z) \rangle$ of $S_n^1$ for $\phi(x) \in S_n^2$. By induction hypothesis, there exist the Hamiltonian paths $P(\phi(x), b')$ and $P(z, v)$ of $S_n^2 \cdots S_n^2$ and $S_n^{-1,n}$, respectively. Thus, $\langle u \xrightarrow{P(u,x)} x, b, \phi(z) \xrightarrow{P(z,v)} z \rangle$ is a Hamiltonian path passing through $E_0$ of $S_n - F_c$, as illustrated in Figure 22.

Figure 21: The Case 3.2 of Theorem 1.

Figure 22: The Case 3.2 of Theorem 1.

Suppose $b \in S_n^0$ and $b' \in S_n^1$. By induction hypothesis, there exists a Hamiltonian path $\langle u \xrightarrow{P(u,x)} x, z \xrightarrow{P(z,v)} b \rangle$ of $S_n^1$ for $\phi(x), \phi(z) \notin S_n^3$. Without loss of generality, we can assume that $\phi(x) \in S_n^2$ and $\phi(z) \in S_n^3$. By induction hypothesis, there exist the Hamiltonian paths $P(b, v)$ and $P(\phi(x), \phi(z))$ of $S_n^0$ and $S_n^2 \cdots S_n^2$, respectively. Thus, $\langle u \xrightarrow{P(u,x)} x, \phi(x) \xrightarrow{P(\phi(x), \phi(z))} \phi(z) \xrightarrow{P(z,v)} z \xrightarrow{P(z,v)} v \xrightarrow{P(b,v)} b \rangle$ is a Hamiltonian path passing through $E_0$ of $S_n - F_c$.

Suppose that $b \notin S_n^1, b' \notin S_n^0$ and $\{b, b'\} \notin (S_n^0 \cup S_n^1)$. By induction hypothesis, there exists a Hamiltonian path between any two vertices with odd distance passing through $E_0'$ in $S_n^1 - F_c$

for $1 \leq j \leq n$. Thus, we can construct a Hamiltonian path $P(u, v)$ passing through $E_0$ of $S_n - F_c$.

4 Conclusion

In this paper, we show for $n \geq 5$ let $F_c$ be the set of faulty edges of $S_n$ and $E_0$ be the edge set of some pairwise vertex-disjoint paths of $S_n$. All edges of $E_0$ lie on a Hamiltonian path $P(u, v)$ where $d(u, v)$ is odd, $|F_c| \leq n - 3$, $|E_0| \leq 2n - 6 - 2|F_c|$. 

References