A Diagnosis Algorithm on the Star Graph under the PMC Model

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Abstract

In this paper, we propose a specific structure in the star graph for local diagnosis under the PMC model. We design an adaptive local diagnosis algorithm for a star graph $S_n$. We prove that $S_n$ is $(n-1)$-diagnosable with our proposed algorithm under the PMC model. Moreover, with our algorithm, a diagnosis is completed in three test rounds.

1 Introduction

A multiprocessor system consists of processors and communication links between processors. In practice, most multiprocessor systems are based on an underlying bus structure, or fabric, and are perfectly feasible for a central test controller (an independent processor acting as a controller) to check each processor in the system. In such a scheme, the central controller itself can be tested externally. Some research is related to the issue of network-on-chip (NoC); for example, Pande et al. [9] developed an evaluation methodology to compare the performance and characteristics of a variety of NoC topologies; Bartic et al. [2] presented an NoC design which is suitable for building networks with irregular topologies.

The problem of identifying faulty processors in a multiprocessor system has been widely addressed in [3, 4, 5, 6, 8, 10, 11]. Throughout this paper, the underlying topology of a multiprocessor system is modeled as a graph: each processor is represented by a vertex, and the communication bus, or fabric, is represented by a single edge between two vertices. A diagnosis testing signal is supposed to be delivered from one vertex to another one through the communication bus at one time. A system performs a so-called system-level diagnosis by making each processor act as a tester to test each of the directly connected ones. It is noticed that such a scheme contains no central test controller instead. Several well-known approaches to system diagnosis have been developed. One classic approach, called the PMC diagnosis model (or PMC model for short), was first proposed by Preparata et al. [10]. This model performs a diagnosis by sending a test signal from a processor to another linked one and then receiving a response.

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in the reverse direction. According to the collection of all test results, the fault status of every processor can be identified.

In some circumstances, however, we are only concerned about some substructure of a multiprocessor system, which is implementable in very large-scale integration (VLSI). Such a substructure, for example, can be a ring, a path, a tree, a mesh, and so on. If all processors in these substructures can be guaranteed to be fault free, a procedure is still workable even though there are many faulty processors in the remaining part of the system. Thus, the local substructure plays a more critical role than the global status of the entire system. Motivated by such a concept, Hsu and Tan [6] presented an elegant measure of diagnosability, known as local diagnosability, to identify the diagnosability of a system by computing the local diagnosability with respect to each individual processor. For any processor in a system, a useful structure [6] was proposed to determine its local diagnosability under the PMC model.

The $n$-dimensional star network $S_n$ was proposed in [1] as an attractive alternative to the $n$-cube topology for interconnecting processors in parallel computers. In this article, we concern with the local diagnosability on the star graph. We give an adaptive local diagnosis algorithm for a star graph with three test rounds. The rest of this article is organized as follows. Section 2 provides preliminary background for system diagnosis and the star graph. Section 3 introduces how to diagnose a star graph with our algorithm. Finally, our conclusions are given in Section 4.

2 Preliminaries

For the graph definitions and notations we follow the standard terminology [7]. Let an undirected graph $G = (V, E)$ denote the underlying topology of a multiprocessor system. Under the PMC diagnosis model [10], we assume that adjacent processors are capable of performing tests on each other. For any two adjacent vertices $u, v \in V$, the ordered pair $(u, v)$ represents a test that processor $u$ can diagnose processor $v$. In this situation, $u$ is a tester, and $v$ is a testee. The outcome of a test $(u, v)$ is 1 (respectively, 0) if $u$ evaluates $v$ to be faulty (respectively, fault-free). Because the faults considered here are permanent, the outcome of a test is reliable if and only if the tester is fault-free. A test assignment for system $G$ is a collection of tests and can be modeled as a directed graph $T = (V, L)$, where $(u, v) \in L$ implies that $u$ and $v$ are adjacent in $G$. The collection of all test results from the test assignment $T$ is called a syndrome. Formally, a syndrome of $T$ is a mapping $\sigma : L \to \{0, 1\}$. The set $F$ of all faulty processors in $G$ is called a faulty set. It is noticed that $F$ can be any subset of $V$. The process of identifying all faulty vertices is said to be the system diagnosis. Furthermore, the maximum number of faulty vertices that can be correctly identified in a system $G$ is called the diagnosability of $G$, denoted by $\tau(G)$.

A system is $t$-diagnosable if, given any syndrome, all faulty units can be identified without replacement, provided that the number of faulty units present does not exceed $t$ [10].

For a multiprocessor system, the random-fault model assumes that the probabilities of processor failures are identical and independent. Let $v$ be any vertex in a graph $G$. It is intuitive to observe that $(N_G(v), \{v\} \cup N_G(v))$ forms an indistinguishable pair of faulty sets. That is, the conventional diagnosability that has been addressed by many researchers mainly describes the global status of a system under the random-fault model. Instead, Hsu and Tan [6] turned their attention to the local connective substructure in a system. More precisely, given any single vertex $v$ in a graph it is only required to determine whether $v$ is faulty or not. The following concept is proposed in [6]. Let $G$ be a graph and let $v$ be any one of its vertices. Then $G$ is $t'$-diagnosable at vertex $v$ if, given a syndrome $\sigma_F$ produced by a set of faulty vertex $F \subseteq V$ with $v \in F$ and $|F| \leq t$, every set $F'$ of faulty vertex that is consistent with $\sigma_F$ must also contain vertex $v$. The local diagnosability of $v$ in $G$, denoted by $\tau^*_G(v)$, is defined to be the maximum integer of $t$ such that $G$ is $t'$-diagnosable at vertex $v$.

The star graph is proposed in [1]. We denote an $n$-dimensional star graph as $S_n$. The vertex set $V$ of $S_n$ is $\{u_1u_2 \ldots u_n | u_1u_2 \ldots u_n \text{ is a permutation of } 1, 2, \ldots, n\}$. The adjacency is defined as follows: $u_1u_2 \ldots u_i \ldots u_n$ is adjacent to $v_1v_2 \ldots v_i \ldots v_n$ through an edge of dimension $i$ with $2 \leq i \leq n$ if $v_j = u_j$ for $j \not\in \{1, i\}$, $v_1 = u_1$, and $v_n = u_n$. For example, in a $S_4$ containing 4! vertices, two vertices 1234 and 4231 are neighbors and joined through an edge labeled 4. Figure 1 illustrates the $S_4$. It is known that $S_n$ is a bipartite graph with one partite set containing all odd permutations and the other partite set containing all even permutations. For convenience, we refer an even permutation as a white vertex, and refer an odd permutation as a black vertex. By the definition of
3 A Diagnosis Algorithm on the Star Graph

In this section, we give a three-round adaptive local diagnosis algorithm under the PMC model for the star graph. For any two positive integers \( r \) and \( s \), we use \([r]_s\) to denote \( r \mod s\). For simplification, we use \( u_x,1, u_x,2, \ldots, \) and \( u_x,n-2 \) to denote \((u)^x, ((u)^x)^2, \ldots, \) and \((\ldots((u)^x)^2)^2 \ldots, ((u)^x)^2 \ldots, ((u)^x)^2 \ldots, ((u)^x)^2) \) for \( 2 \leq x \leq n \), respectively. To diagnose a vertex in a star graph \( S_n \), we give the following two algorithms, VOTE and SDA. See Figure 2 for an illustration.

Algorithm VOTE(\( T, u \))

Input: A vertex set \( T \) and a vertex \( u \).
Output: The value is 0 or 1 if \( u \) is fault-free or faulty, respectively.

Begin
\( n_0 \leftarrow 0; \)
\( n_1 \leftarrow 0; \)
if \( |T| = 1 \)
then let \( x \) be the vertex in \( T \);
return \( \sigma(x,u) \);
else
\( n_0 \leftarrow \{ i \mid \sigma(x_i,u) = 0 \} \) for each \( x_i \in T \)

End

Algorithm SDA(\( S_n, u \))

Input: A star graph \( S_n \) and a vertex \( u \).
Output: The value is 0 or 1 if \( u \) is fault-free or faulty, respectively.

Begin
\( T \leftarrow \emptyset; \)
\( p \leftarrow |\{ i \mid \sigma(u_{2,i}, u_{2,i−1}) = 1, 2 \leq i \leq n−2\}|; \)
\( q \leftarrow |\{ i \mid \sigma(u_{3,i}, u_{3,i−1}) = 1, 2 \leq i \leq n−2\}|; \)
for \( j = 4 \) to \( n \) do
\( \gamma_j \leftarrow |\{ i \mid \sigma(u_{j,i}, u_{j,i−1}) = 1, 2 \leq i \leq n−2\}|; \)
if \( \gamma_j = \min_{2 \leq j \leq n} \{ \gamma_j \} \)
then \( u' \leftarrow u_{j,1}; \)
\( r \leftarrow \gamma_j; \)

End
Let $\sigma$ be the PMC diagnosis model, we have the following two theorems.

**Theorem 1.** Suppose that $\sigma$ is a faulty vertex set in $P$. Then $e = 1$ for all $e$. If $e = 1$, then return VOTE $T(u)$. Otherwise, return VOTE $T(u)$. If $e = 0$ and $q = 0$, then return VOTE $T(u)$. If $e = 0$ and $q = 1$, then return VOTE $T(u)$. If $e = 0$ and $q = 0$, then return VOTE $T(u)$. If $e = 0$ and $q = 1$, then return VOTE $T(u)$. If $e = 0$ and $q = 0$, then return VOTE $T(u)$.

**End**

With the definition and the property of the PMC diagnosis model, we have the following two lemmas.

**Lemma 1.** Let $P = \langle p_1, p_2, p_3, \ldots, p_r \rangle$ be a path. If $p_i$ is faulty, and $\sigma(p_i, p_{i-1}) = 0$ for every $2 \leq i \leq r$, then all the $r$ vertices in $P$ are faulty.

**Lemma 2.** Let $P = \langle p_1, p_2, p_3, \ldots, p_r \rangle$ be a path. If $\sigma(p_i, p_{i-1}) = 1$ for some $i$ in $2 \leq i \leq r$, then there exists at least one fault vertex in $P$.

The following theorem ensures that a vertex in a star graph can be diagnosed correctly with the algorithm SDA under the PMC model.

**Theorem 1.** Suppose that $n \geq 4$. If $F$ is a faulty vertex set in $S_n$ with $|F| \leq n - 1$, then the fault/fault-free status of vertex $u$ can be identified correctly with the algorithm SDA $\langle S_n, u \rangle$.

**Proof.** For $2 \leq i \leq n$, let $\sigma_i = (\sigma(u_2, u_1), \sigma(u_3, u_2), \ldots, \sigma(u_n, u_{n-3}))$ and $e_0 = (z_1, z_2, \ldots, z_{n-3})$ where $z_{i'} = 0$ for every $1 \leq i' \leq n-3$. We consider the following cases.

**Case 1:** Suppose that there exists some $k$ such that $\sigma_k = e_0$ for $3 \leq k \leq n - 1$.

**Subcase 1.1:** Suppose that $\sigma_2 = e_0$ and $\sigma_3 = e_0$. We claim that $u \notin F$ if at least two of $\sigma(u_2, u_1)$, $\sigma(u_3, u_2)$, and $\sigma(u_4, u_3)$ are zero. Without loss of generality, suppose that $\sigma(u_2, u_1) = 0$ and $\sigma(u_3, u_2) = 0$. We assume that $u \in F$ by contradiction. Thus $\{u_2, u_3, u_1\} \subseteq F$. By Lemma 1, $|F| \geq 2(n - 2) + 1 > n - 1$ if $n \geq 4$, which contradicts the assumption that $|F| \leq n - 1$. Now we assert that $u \notin F$, and at least two of $\sigma(u_2, u_1)$, $\sigma(u_3, u_2)$, and $\sigma(u_4, u_3)$ are zero. Without loss of generality, suppose that $\sigma(u_2, u_1) = 0$ and $\sigma(u_3, u_2) = 1$. Then $\{u_2, u_3, u_1\} \subseteq F$. By Lemma 1, $|F| \geq 2(n - 2) + 1 > n - 1$ if $n \geq 4$, which contradicts the assumption that $|F| \leq n - 1$.

**Subcase 1.2:** Suppose that $\sigma_2 = e_0$ and $\sigma_3 \neq e_0$. We claim that $u \notin F$ if at least one of $\sigma(u_2, u_1)$ and $\sigma(u_3, u_2)$ is zero. Without loss of generality, suppose that $\sigma(u_2, u_1) = 0$. We assume that $u \in F$ by contradiction. Thus $u_2 \in F$. By Lemma 1 and Lemma 2, $|F| \geq (n - 2) + 1 > n - 1$, which contradicts the assumption that $|F| \leq n - 1$. Now we assert that $u \notin F$, and at least two of $\sigma(u_2, u_1)$ and $\sigma(u_3, u_2)$ are zero. Without loss of generality, suppose that $\sigma(u_2, u_1) = 0$ and $\sigma(u_3, u_2) = 1$. Thus $\{u_2, u_3, u_1\} \subseteq F$. By Lemma 1 and Lemma 2, $|F| \geq 2(n - 2) + 1 > n - 1$ if $n \geq 4$, which contradicts the assumption that $|F| \leq n - 1$.

**Subcase 1.3:** Suppose that $\sigma_2 \neq e_0$ and $\sigma_3 = e_0$. This case is similar to Subcase 1.2 by interchanging the roles of $u_2, u_1$ and $u_3, u_2$.

**Subcase 1.4:** Suppose that $\sigma_2 \neq e_0$ and $\sigma_3 \neq e_0$. We claim that $u \notin F$ if $\sigma(u_2, u_1)$ and $\sigma(u_3, u_2)$ are zero. Without loss of generality, suppose that $\sigma(u_2, u_1) = 0$. We assume that $u \in F$ by contradiction. Thus $u_2, u_1 \in F$. By Lemma 1 and Lemma 2, $|F| \geq (n - 2) + 2 + 1 > n - 1$, which contradicts the assumption that $|F| \leq n - 1$. Now we assert that $u \notin F$, and at least two of $\sigma(u_2, u_1)$ and $\sigma(u_3, u_2)$ are zero. Without loss of generality, suppose that $\sigma(u_2, u_1) = 0$ and $\sigma(u_3, u_2) = 1$. Thus $u_2, u_1 \in F$. By Lemma 1 and Lemma 2, $|F| \geq (n - 2) + 2 + 1 > n - 1$, which contradicts the assumption that $|F| \leq n - 1$.

**Case 2:** Suppose that $\sigma_k \neq e_0$ for $3 \leq k \leq n - 1$.

**Subcase 2.1:** Suppose that $\sigma_2 = e_0$ and $\sigma_3 = e_0$. We claim that $u \notin F$ if at least one of $\sigma(u_2, u_1)$ and $\sigma(u_3, u_2)$ is zero. Without loss of generality, suppose that $\sigma(u_2, u_1) = 0$. We assume that $u \in F$ by contradiction. Thus $u_2, u_1 \in F$. By Lemma 1 and Lemma 2, $|F| \geq (n - 2) + 1 + 1 > n - 1$, which contradicts the assumption that $|F| \leq n - 1$. Now we assert that $u \notin F$, and at least two of $\sigma(u_2, u_1)$ and $\sigma(u_3, u_2)$ are zero. Without loss of generality, suppose that $\sigma(u_2, u_1) = 0$ and $\sigma(u_3, u_2) = 1$. Thus $\{u_2, u_3, u_1\} \subseteq F$. By Lemma 1 and
Lemma 2, $|F| \geq 2(n - 2) + 1 > n - 1$ if $n \geq 4$, which contradicts the assumption that $|F| \leq n - 1$.

**Subcase 2.2:** Suppose that $\sigma_2 = e_0$ and $\sigma_3 \neq e_0$. We claim that $u \notin F$ if $\sigma(u_{2,1}, u) = 0$. We assume that $u \in F$ by contradiction. Thus $u_{2,1} \in F$. By Lemma 1 and Lemma 2, $|F| \geq (n-2) + 2 + 1 > n - 1$, which contradicts the assumption that $|F| \leq n - 1$. Now we assume that $u \notin F$ and $\sigma(u_{2,1}, u) = 1$. Thus $u_{2,1} \in F$. By Lemma 1 and Lemma 2, $|F| \geq (n-2) + 2 > n - 1$, which contradicts the assumption that $|F| \leq n - 1$.

**Subcase 2.3:** Suppose that $\sigma_2 \neq e_0$ and $\sigma_3 = e_0$. This case is similar as Subcase 2.2 by interchanging the roles of $u_{2,1}$ and $u_{3,1}$.

**Subcase 2.4:** Suppose that $\sigma_2 \neq e_0$ and $\sigma_3 \neq e_0$. By Lemma 2, there exist $n - 1$ fault vertices in $\{u_{i,j} \mid 2 \leq i \leq n, 2 \leq j \leq n - 2\}$. With the assumption that $|F| \leq n - 1$, the vertex $u \notin F$. \hfill $\square$

Now we describe the scheme that determining the fault status of a vertex $u$ in a star graph $S_n$ with three test rounds under the PMC model.

- The first test round: Perform the tests $(u_{2,1}, u), (u_{2,2i+1}, u_{2,2i})$ for every $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1$, and $(u_{2,2k}, u_{2,2k-1})$ for every $3 \leq j \leq n$ and $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.
- The second test round: Perform the tests $(u_{3,1}, u), (u_{2,2i}, u_{2,2i-1})$ for every $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, and $(u_{2,2k+1}, u_{2,2k})$ for every $3 \leq j \leq n$ and $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1$.
- The third test round: Let $j$ be an integer in $4 \leq j \leq n$ such that the number $|\{i \mid \sigma(u_{j,i}, u_{j,i-1}) = 1, 2 \leq i \leq n - 2\}|$ after the first two test rounds is minimal. Perform the test $(u_{j,1}, u)$.

4 Conclusion

In this paper, we concern with the local diagnosability. We propose an adaptive local diagnosis algorithm SDA for a star graph with three test rounds. Future work will try to find some specific structure for the existing practical interconnection networks. Then design the efficient diagnosis algorithm of the system with the useful structure in accordance with various conditions and diagnosis models.

References


