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The Balanced Hamiltonian Cycle in Hypercube

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Abstract

3-D scanning is an important issue in nowadays, and the application of the bit conversion in Gray-Code is one of the ways in decreasing consumption and increasing precision. The problem of finding a average bit conversion of Gray-Code is equal to the problem of finding the Balanced Hamiltonian Cycle in hypercube.

The Balanced Hamiltonian Cycle in hypercube means a Hamiltonian Cycle C in hypercube that for the set of all i-dimensional edge $E(C)_i$ in $E(C)$, $\left||E(C)_i| - |E(C)_j|\right| \leq 1$, for $i \neq j$. We can separate the problem into two cases, $n \neq 2^k$ and $n = 2^k$. In this paper, we have proved that there are no Balanced Hamiltonian Cycles in hypercube Q_n as $n \neq 2^k$. For the case of $n = 2^k$, we find the way to construct a Balanced Hamiltonian Cycle in Q_n, for k = 1, 2 and 3.

1 Introduction

In the application of 3-D scanning, the research of optimal encode will use gray-scale encode to signify the information of n-bits, which has mentioned in the reference [2], [3], [8] and [10]. The result after identification will decrease the consumption of resource and increase the precision. However, dealing with such information that contains many transformations between 1 and 0 will cause some problem, like that will spend much more cost in identification. What we proposed is a way to decrease the cost in dealing with such problems. Thus, we hope to decrease the transformations between 0 and 1 in the same dimension.

In a graph $G = (V, E)$, V is the set of all vertices of G. E is the set of all edges of G. $P = x_1e_1x_2e_2\ldots x_n$, for $x_i \in V(G)$, $e_i \in E(G)$, $1 \leq i \leq n$, is called a *path* of G if all vertices x_i in P are distinct, and x_1 is called origin vertex, x_n is called the terminus vertex. A *cycle* is defined as a path except that $x_1 = x_n$. A *Hamiltonian cycle* of G is defined as a cycle of G that contains all vertices of G.

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$Q_n = (V, E)$ is a *hypercube* if $V(Q_n) = \{x_1x_2\ldots x_n \mid x_i \in \{0, 1\}, i = 1, 2, \ldots, n\}$, $E(Q_n) = \{xy \mid x, y \in V(Q_n), \sum_{i=1}^{n}(x_i \oplus y_i) = 1\}$. The *dimension* of Q_n is n. It is well known that $Q_n = (K_2)^n$. The set of i-th dimensional edges of a Hamiltonian Cycle C of Q_n means that the set $E(C)_i = \{xy \mid x, y \in V(Q_n), x_i \oplus y_i = 1\}$. If a Hamiltonian Cycle C in Q_n satisfied that $\left||E(C)_i| - |E(C)_j|\right| \leq 1$, for $1 \leq i < j \leq n$, then we call C a *Balanced Hamiltonian Cycle*.

The way of bit-changing of Gray-Code by the definitions above. This question is quietly the same as the problem of the Balanced Hamiltonian Cycle in Q_n. Thus, we hope that we can solve the problem of bit-changing of Gray-Code by researching the problem of the Balanced Hamiltonian Cycle.

2 The Balanced Hamiltonian Cycle of $n \neq 2^k$

In this section, we consider the case for $n \neq 2^k$, for any positive integer k. We prove that there is no any Balanced Hamiltonian Cycles in Q_n for $n \neq 2^k$. First, we give two Lemmas, in order to support the proof of Theorem 1. These two Lemmas are shown as below:

Lemma 1. *For each set $E(C)_i$ that contains all i-th dimensional edges of Balanced Hamiltonian Cycle C, $|E(C)_i|$ is even.*

Proof. For any positive inter n, Q_n can be divided into two components Q_{n-1}^0 and Q_{n-1}^1 which are both isomorphic to Q_{n-1} that without i-th dimensional edges for some $i \in \{1, 2, \ldots, n\}$. If there exists a Balanced Hamiltonian Cycle C in Q_n such that $|E(C)_i|$ is odd for some integer $i \in \{1, 2, \ldots, n\}$. Without lost of generality, we say $i = 0$ and the origin vertex of C in Q_{n-1}^0. After tracing all edges of C, we have the terminate vertex of C is in Q_{n-1}^1 because $|E(C)_i|$ is odd. Then the origin vertex and the terminate vertex of the Balanced Hamiltonian

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Cycle C are different. That is a contradiction. So $|E(C)_i|$ is even for any $1 \le i \le n$. □

Lemma 2. *For each set $E(C)_i$ that contains all i-th dimensional edges of Balanced Hamiltonian Cycle C,* $\lfloor 2^n/n \rfloor \le |E(C)_i| \le \lceil 2^n/n \rceil$.

Proof. Since the definition of Balanced Hamiltonian Cycle C is that for the set $E(C)_i$ for all i-th dimensional edges of C, $\big||E(C)_i| - |E(C)_j|\big| \le 1$, for $1 \le i < j \le n$, and $|V(Q_n)| = 2^n$, we have $|E(C)| = \sum_{i=1}^{n}|E(C)_i| = 2^n$. Hence, without lost of generality, there exists $0 \le m < n$ such that $|E(C)_1| = |E(C)_2| = \cdots = |E(C)_m| = k+1$, and $|E(C)_{m+1}| = |E(C)_{m+2}| = \cdots = |E(C)_n| = k$, for some positive integer k. So $\sum_{i=1}^{n}|E(C)_i| = (n-m)k + m(k+1) = nk + m$. So $nk \le \sum_{i=1}^{n}|E(C)_i| = 2^n < nk + n = n(k+1)$. Thus, $(2^n/n) - 1 < k < 2^n/n$. So $\lfloor 2^n/n \rfloor \le |E(C)_i| \le \lceil 2^n/n \rceil$. □

Theorem 1. *There is no Balanced Hamiltonian Cycle in Q_n when $n \ne 2^k$ for any positive integer k.*

Proof. Suppose that there exists a Balanced Hamiltonian Cycle C in Q_n. Then for the set $E(C)_i$ that contain all i-th dimensional edges of C, we know $|E(C)_i| = \lfloor 2^n/n \rfloor$ or $\lceil 2^n/n \rceil$ for all $1 \le i \le n$ by Lemma 2. Since $n \ne 2^k$, then $\lfloor 2^n/n \rfloor$ and $\lceil 2^n/n \rceil$ must be two continuous integers.

So we have that one of $\lfloor 2^n/n \rfloor$ and $\lceil 2^n/n \rceil$ is odd. That is, $|E(C)_i|$ is odd for some $i \in \{1, 2, \ldots, n\}$. That is a contradiction by Lemma 1. So it is proved that there is no any Balanced Hamiltonian Cycles in Q_n when $n \ne 2^k$ for any positive integer k. □

3 The Balanced Hamiltonian Cycle in Q_2, Q_4 and Q_8

In this section, we first propose the Balanced Hamiltonian Cycle in Q_2 and Q_4 as shown in Figures 1 and 2. Then we propose a scheme to find a Balanced Hamiltonian Cycle in Q_8. In Figure 1, it is easy to see $Q_{2^1} = Q_2$ is a Balanced Hamiltonian Cycle of itself. For $Q_{2^2} = Q_4$, since $2^4 / 4 = 4$, we have $|E(C)_i| = 4$ for any $1 \le i \le n$ and any Balanced Hamiltonian Cycle C of Q_4. Figure 2 give a Balanced Hamiltonian Cycle in Q_4.

In the following, we give a scheme for finding a Balanced Hamiltonian Cycle of $Q_{2^3} = Q_8$. Note that the calculations of the footnode in the following steps are taken modula by 16.

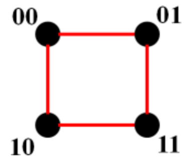

Figure 1. A Balanced Hamiltonian Cycle in Q_2.

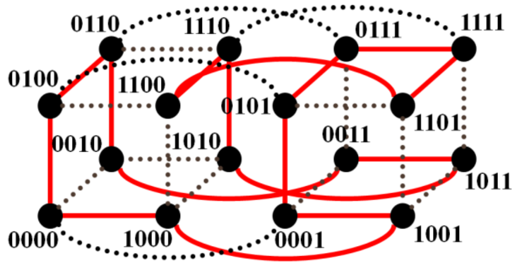

Figure 2. A Balanced Hamiltonian Cycle C in Q_4.

Algorithm: Finding_BHC_of_Q_8
Input: $Q_8 = (K_2)^8 = ((K_2)^4)^2 = (Q_4)^2$
Output: a Balanced Hamiltonian Cycle C^* of Q_8
Begin
Step 1. Copy sixteen Balanced Hamiltonian Cycles C in Q_4. Then take them as sixteen supernodes in the structure of the Balanced Hamiltonian Cycle C in Q_4. We denote these sixteen supernodes as $S_0, S_1, \ldots, S_{15}$, and denote the corresponding vertices in each supernode S_i as x_j^i, $0 \le i, j \le 15$, That is, $x_j^i \in V(S_i)$, $0 \le i, j \le 15$ as shown in Figure 3.

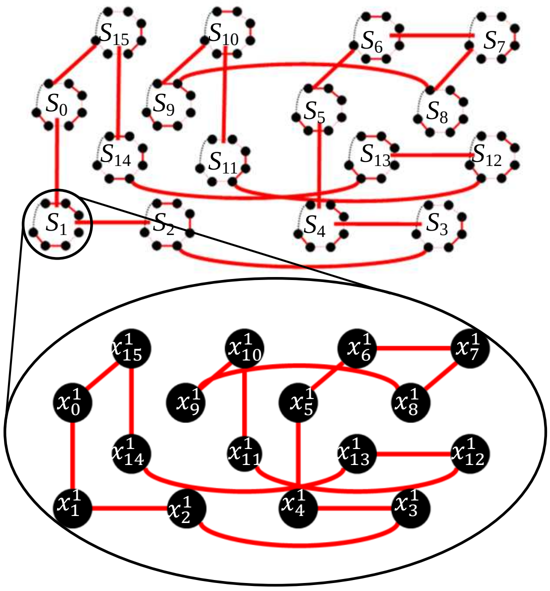

Figure 3. The structure of the Balanced Hamiltonian Cycle in Q_8 with supernodes which is the Balanced Hamiltonian Cycle in Q_4.

Step 2. Define four spanning subgraphs M_0^i, M_1^i, M_2^i and M_3^i of the Balanced Hamiltonian Cycle S_i in Q_4, for $0 \le i \le 15$, denoted by $V(M_j^i) = V(S_i)$, for $0 \le j \le 3$, and

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where

$E(M_0^i) = \{x_j^i x_{j+1}^i \mid j = 2t, 0 \le t \le 7\}$,

$E(M_1^i) = \{x_{j-1}^i x_j^i, x_j^i x_{j+1}^i \mid j = 4t - 2, 1 \le t \le 4\}$,

$E(M_2^i) = \{x_{j-1}^i x_j^i \mid j = 2t, 1 \le t \le 8\}$,

$E(M_3^i) = \{x_{j-1}^i x_j^i, x_j^i x_{j+1}^i \mid j = 4t, 0 \le t \le 3\}$.

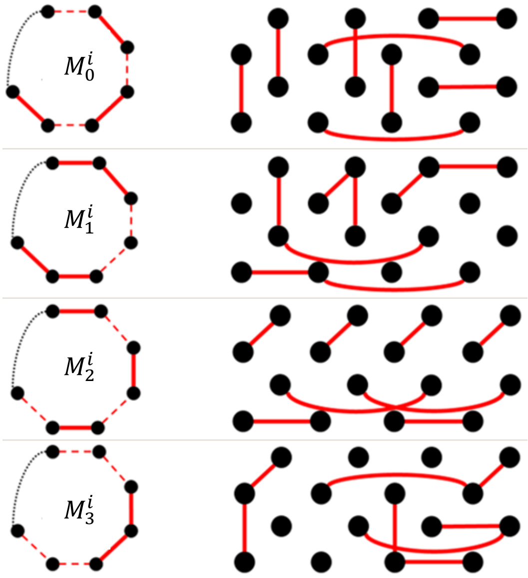

Figure 4. Four spanning subgraphs of the Balanced Hamiltonian Cycle S_i.

Step 3. Let $S_i' = M^i_{(i \bmod 4)}$, for $0 \le i \le 15$, and $E(C^*) \leftarrow \bigcup_{i=0}^{15} S_i'$ shown as Figure 5.

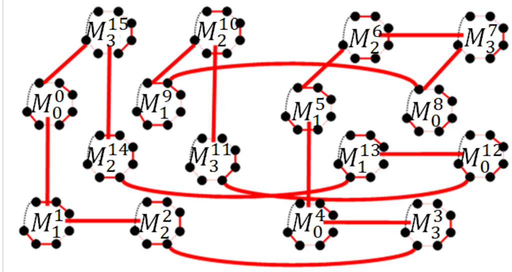

Figure 5. The structure of the Balanced Hamiltonian Cycle in Q_8 with supernodes which is replaced by M_0^i, M_1^i, M_2^i and M_3^i.

Step 4. For $0 \le i \le 15$, if $x_{j-1}^i x_j^i$, $x_j^i x_{j+1}^i \in E(S_i')$ for some $0 \le j \le 15$. Detect the two neighborhood supernodes $E(S_k')$ and $E(S_r')$ of $E(S_i')$. If $x_{j-1}^k x_j^k \in E(S_k')$, then add the edge $x_{j-1}^k x_{j-1}^i$ into $E(C^*)$. If $x_j^r x_{j+1}^r \in E(S_r')$, then add the edge $x_{j+1}^r x_{j+1}^i$ into $E(C^*)$. For all isolated points $x_m^i \in V(S_i')$, add two edges $x_m^i x_m^k$ and $x_m^i x_m^r$ into $E(C^*)$. Then C^* will be eight disjoint cycles that through all vertices of Q_8 now. Figure 6 is one of these cycle.

Step 5. Replace three spanning subgraphs S_0', S_2', and S_4' in C^* into the three spanning subgraphs N_1, N_2 and N_3, respectively, shown as Figure 7.

End of Algorithm

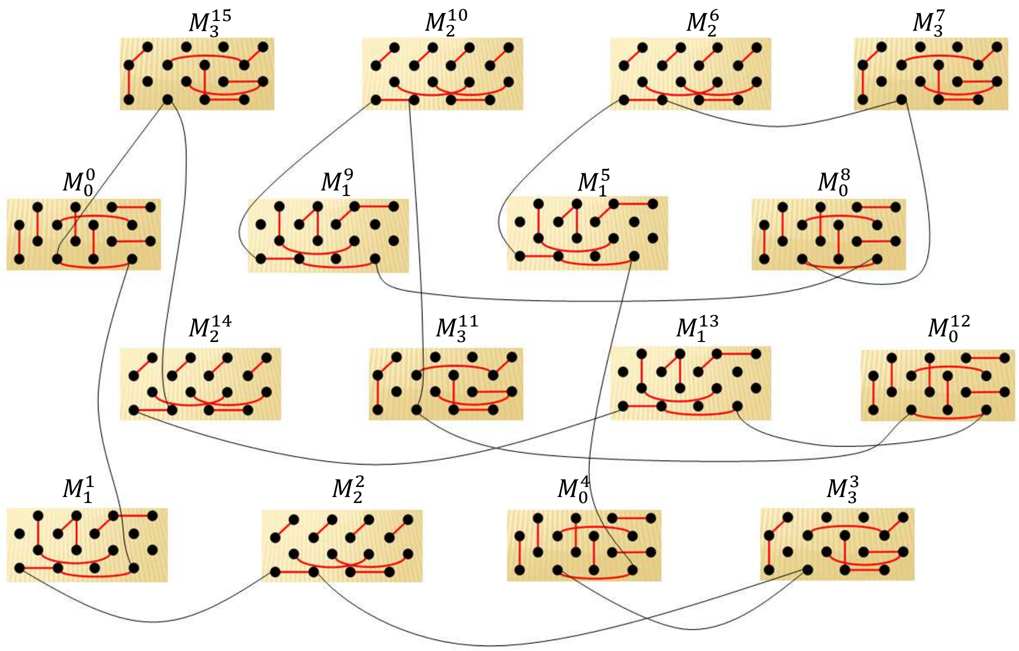

Figure 6. One of the eight cycles in C^* after executing step 4.

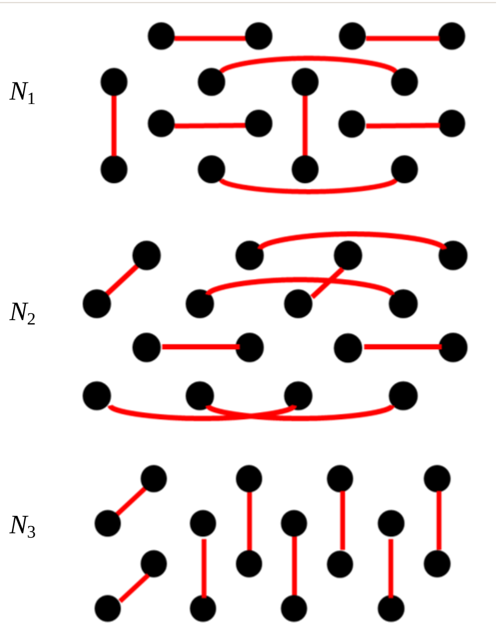

Figure 7. Three spanning subgraphs of Q_4.

Theorem 2. There is a Balanced Hamiltonian Cycle in Q_8.

Proof. The Output C^* of Algorithm Finding_BHC_of_Q_8 is stated in Appendix. One can check it is a Balanced Hamiltonian Cycle of Q_8. □

4 Conclusion

In this paper, we give a method to find a Balanced Hamiltonian Cycle in hypercube Q_2, Q_4 and Q_8. We also prove that there is no balanced Hamiltonian Cycle in hypercube Q_n for $n \neq 2^k$ for any positive integer k. By observing the regularity, we expect to find a solution for constructing a Balanced Hamiltonian Cycle in Q_n, when $n = 2^k$ for any positive integer $k \geq 4$ in the future.

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Appendix

The Output C^* of Algorithm Finding_BHC_of_Q_8

1	00000000	53	11101110	105	01010110	157	01001011	209	11110101
2	00001000	54	10101110	106	01010111	158	00001011	210	11110001
3	00011000	55	10101100	107	01110111	159	00001001	211	11010001
4	00010000	56	11101100	108	01111111	160	01001001	212	11010011
5	00010100	57	11001100	109	01111110	161	01101001	213	11011011
6	01010100	58	11001101	110	11111110	162	01101101	214	11001011
7	01010101	59	11101101	111	11111100	163	01001101	215	11001001
8	00010101	60	11101001	112	01111100	164	01001100	216	11011001
9	10010101	61	11101011	113	01011100	165	01001110	217	11111001
10	10010001	62	10101011	114	01011101	166	00001110	218	11111101
11	00010001	63	10100011	115	01111101	167	00001111	219	11011101
12	00010011	64	11100011	116	01111001	168	01001111	220	11011100
13	00011011	65	11000011	117	01111011	169	01101111	221	11011110
14	01011011	66	11000001	118	11111011	170	01100111	222	11001110
15	01011001	67	11100001	119	11110011	171	01000111	223	11001111
16	00011001	68	11100101	120	01110011	172	01000110	224	11011111
17	10011001	69	11100100	121	01010011	173	01000010	225	11111111
18	10011101	70	01100100	122	01010001	174	01100010	226	11110111
19	00011101	71	01100000	123	01110001	175	01100110	227	11010111
20	00011100	72	00100000	124	01110101	176	00100110	228	11010110
21	00011110	73	00110000	125	01110100	177	00110110	229	11010010
22	01011110	74	00111000	126	11110100	178	00110111	230	10010010
23	01011111	75	10111000	127	11110000	179	00100111	231	10010110
24	00011111	76	10110000	128	01110000	180	00101111	232	10000110
25	10011111	77	10110100	129	01010000	181	00101110	233	00000110
26	10010111	78	00110100	130	01011000	182	01101110	234	00000111
27	00010111	79	00110101	131	01111000	183	01101100	235	10000111
28	00010110	80	10110101	132	01111010	184	00101100	236	10001111
29	00010010	81	10100101	133	01110010	185	00111100	237	10001110
30	01010010	82	10100001	134	00110010	186	00111101	238	10011110
31	01011010	83	10110001	135	00111010	187	00101101	239	10011100
32	00011010	84	10110011	136	10111010	188	00101001	240	10001100
33	10011010	85	10111011	137	10101010	189	00101011	241	00001100
34	10011000	86	00111011	138	10101000	190	01101011	242	00001101
35	10001000	87	00111001	139	00101000	191	01100011	243	10001101
36	10001010	88	10111001	140	00101010	192	00100011	244	10001001
37	10000010	89	10101001	141	00100010	193	00110011	245	10001011
38	11000010	90	10101101	142	00000010	194	00110001	246	10011011
39	11001010	91	10111101	143	00001010	195	00100001	247	10010011
40	11011010	92	10111100	144	01001010	196	00100101	248	10000011
41	11111010	93	10111110	145	01101010	197	00100100	249	00000011
42	11111000	94	00111110	146	01101000	198	10100100	250	00000001
43	11101000	95	00111111	147	01001000	199	10100000	251	10000001
44	11101010	96	10111111	148	01000000	200	11100000	252	10000101
45	11100010	97	10101111	149	01000100	201	11000000	253	10000100
46	10100010	98	10100111	150	00000100	202	11001000	254	10010100
47	10100110	99	10110111	151	00000101	203	11011000	255	10010000
48	11100110	100	10110110	152	01000101	204	11010000	256	10000000
49	11000110	101	10110010	153	01100101	205	11010100		
50	11000111	102	11110010	154	01100001	206	11000100		
51	11100111	103	11110110	155	01000001	207	11000101		
52	11101111	104	01110110	156	01000011	208	11010101		