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# Circuits of Each Even Length in Hypercubes *

Yue–Li Wang[1],† Chien–Yi Li[1] and Hung–Chang Chan[2]

[1] Department of Information Management,
National Taiwan University of Science and Technology, Taipei, Taiwan
[2] Department of Computer Science and Information Engineering,
Yuanpei University, Hsinchu, Taiwan, ROC

## Abstract

*A circuit in a graph $G$ is an alternating sequence of vertices and edges of the form $v_1, e_1, v_2, e_2, v_3, \ldots, v_n, e_n, v_{n+1}$ in which edge $e_i = v_i v_{i+1}$, for $i = 1, 2, \ldots, n$ and $v_{n+1} = v_1$. Note that vertices in a circuit can occur repeatedly while no edge can be repeated in a circuit. In this paper, we are concerned with a circuit of every even length $k$ for $4 \leqslant k \leqslant \theta'(Q_n)$ in hypercube $Q_n$ where $\theta'(Q_n) = 2^{n-1}(n-1)$ if $n$ is odd and $\theta'(Q_n) = 2^{n-1}n - 4$ otherwise for $n \geqslant 3$.*

## 1 Introduction

In this paper, all considered digraphs $G = (V, E)$ are simple, i.e., there is at most one edge between any pair of vertices where $V(G)$ and $E(G)$ are the vertex and arc sets, respectively. For definitions of graph theoretic terms, we follow [25]. However, for ease of readability, we introduce some of them as follows. A *path* is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, v_2, e_2, v_3, \ldots, v_{n-1}, e_{n-1}, v_n$ in which edge $e_i = v_i v_{i+1}$, for $i = 1, 2, \ldots, n-1$. A *circuit* is also an alternating sequence $v_1, e_1, v_2, e_2, v_3, \ldots, v_{n-1}, e_{n-1}, v_n$ in which all edges are distinct while vertices might be repeated. A circuit is called a *cycle* if all vertices are distinct except $v_1 = v_n$. A circuit is called an *eulerian circuit* if it contains all the edges in $E(G)$. A graph with an eulerian circuit is called an *eulerian graph*. Let $E'$ be a subset of $E$. We use $G - E'$ to denote the graph with vertex set $V$ and edge set $E - E'$.

A graph of $n$ vertices is said to be *s-pancyclic* for some $3 \leqslant s \leqslant n$ if it contains a cycle of each length $t$ for $s \leqslant t \leqslant n$. If $s = 3$, then $s$-pancyclic is the so-called *pancyclic* [1, 3, 12, 22]. A graph of $n$ vertices is said to be *bipancyclic* if it contains a cycle of each even length $t$ for $4 \leqslant t \leqslant n$. The concept of pancyclicity has been extended to vertex-pancyclicity [13] and edge-pancyclicity [2]. A bipartite graph is *vertex-bipancyclic* [21] if every vertex lies on a cycle of every even length $t$ for $4 \leqslant t \leqslant n$. Similarly, a bipartite graph is *edge-bipancyclic* if every edge lies on a cycle of every even length $t$ for $4 \leqslant t \leqslant n$.

Path and cycle are two of the most fundamental networks for parallel and distributed computation, and suitable for designing simple algorithms with low communication costs. The pancyclicity of a network represents its power of embedding cycles of all possible lengths. It is an important measurement for determining whether the topology of a network is suitable for an application in which embedding rings of arbitrary length into the topology is required. Pancyclic graphs (or bipancyclic) and its related problems have been studied intensively, e.g., WK-recursive networks [4, 9], arrangement graphs [5], Möbius cubes [6, 14, 16], cross cubes [7], bijective connection graphs[8], hypercubes [10, 24], cube-connected cycles [11], augmented cubes [15], butterfly graphs [17], pancake graphs [18], twisted cubes [20, 26, 27], and so on.

In [23], Wang, Guo, Hung and Chan extended the concept of pancyclic to pancircuitous. A graph

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†All correspondence should be addressed to Professor Yue–Li Wang, Department of Information Management, National Taiwan University of Science and Technology, Taipei, Taiwan, ROC(Email: ylwang@cs.ntust.edu.tw).

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$G$ of $n$ vertices, for $n \geqslant 3$, is said to be $x$-*pancircuitous* if it contains a circuit of each length $k$ for $3 \leqslant k \leqslant x$ and no circuit of length $x+1$ exists in $G$. If $G$ is $x$-pancircuitous, then $x$ is called the *pancircuitous number* of $G$, denoted $\theta(G)$. In this paper, we define that a graph is $x$-*bipancircuitous* if it contains a circuit of each even length $k$ for $4 \leqslant k \leqslant x$ and no circuit of length $x+2$ exists in $G$. We use $\theta'(G)$ to denote the *pancircuitous number* $x$ for $x$-bipancircuitous graph $G$. In this paper, we shall find out $\theta'(Q_n)$, $n \geqslant 2$, for hypercubes $Q_n$ which will be introduced in Section 2.

## 2 Bipancircuitous of Hypercubes

Let $u = u_{n-1}u_{n-2}\cdots u_0$ and $v = v_{n-1}v_{n-2}\cdots v_1v_0$ be two $n$-bit binary strings. The *Hamming distance* $h(u,v)$ between $u$ and $v$ is the number of different bits in their corresponding strings. If only one bit is different between them, say $u_k$ and $v_k$ for some $0 \leqslant k < n$, then $u$ and $v$ are denoted by $v^k$ and $u^k$, respectively. The $n$-*dimensional hypercube*, denoted $Q_n$, consists of all $n$-bit binary strings as its vertices and two vertices $u$ and $v$ are adjacent if and only if $h(u,v) = 1$. Thus, $Q_n$ is a bipartite graph with partite sets $\{u|w(u) \text{ is odd}\}$ and $\{u|w(u) \text{ is even}\}$, where $w(u)$ stands for the number of 1's in $u_{n-1}u_{n-2}\cdots u_0$. It is well-known that $d_{Q_n}(u,v) = h(u,v)$. An edge $e = uv$ in $E(Q_n)$ is of dimension $k$ if $u = v^k$ (and $v = u^k$). For an edge $e = uv$ which is not of dimension $k$, denote by $e_b^k = u^kv^k$, $e_1^k = uu^k$, and $e_2^k = vv^k$.

For convenience, we use $Q_{n-1}^0$ to denote the subgraph of $Q_n$ induced by $\{x \in V(Q_n)|x_0 = 0\}$ and $Q_{n-1}^1$ to denote the subgraph of $Q_n$ induced by $\{x \in V(Q_n)|x_0 = 1\}$. Thus, $Q_{n-1}^0$ and $Q_{n-1}^1$ are isomorphic to $Q_{n-1}$. By the above definition, for an edge $e = uv \in E(Q_{n-1}^0)$, $e_b^0 = u^0v^0 \in E(Q_{n-1}^1)$ while $e_1^0 = uu^0$ and $e_2^0 = vv^0$ are two edges of dimension 0. Note that both $uu^0$ and $vv^0$ are not in $E(Q_{n-1}^0) \cup E(Q_{n-1}^1)$. Let $C_1$ and $C_2$ be two circuits in $Q_{n-1}^0$ and $Q_{n-1}^1$, respectively, in which $C_1$ contains edge $e$ and $C_2$ contains edge $e_b^0$. Denote by $C_1 \circ C_2$ the circuit containing all edges in $((E(C_1) \cup E(C_2)) \setminus \{e, e_b^k\}) \cup \{e_1^k, e_2^k\}$. It is obvious that $|C_1 \circ C_2| = |C_1| + |C_2|$ where $|C_x|$ denotes the length of circuit $C_x$.

**Theorem 1 (Theorem 2 in [19])** *$Q_n$ is $(n-2)$-edge-fault-tolerant edge-bipancyclic.*

**Proposition 2** $\theta'(Q_3) = 8$.

**Theorem 3** *For any hypercube $Q_n$, $n \geqslant 3$,*

$$\theta'(Q_n) = \begin{cases} 2^{n-1}(n-1) & \text{if } n \text{ is odd} \\ 2^{n-1}n - 4 & \text{if } n \text{ is even.} \end{cases}$$

**Proof.** We prove this theorem by induction on $n$. By Proposition 2, the basis holds for $n = 3$. Note that every vertex in $Q_n$ is of degree $n$, and the numbers of vertices and edges in $Q_n$ are $2^n$ and $2^{n-1}n$, respectively. In the induction step, there are two cases to consider for $n > 3$.

**Case 1**: $n$ is even.

Since $n$ is even, every vertex in $Q_n$ is of even degree. This means that $Q_n$ is an eulerian graph and the length of an eulerian circuit in $Q_n$ is $2^{n-1}n$. Clearly, removing one, two or three edges from $Q_n$ does not yield an eulerian subgraph. However, removing any $C_4$ from $Q_n$, the resulting graph is an eulerian subgraph. Therefore, $\theta'(Q_n) \leqslant 2^{n-1}n-4$. By the induction hypothesis, there exit circuits of every even length from 4 to $2^{n-2}(n-2)$ in both $Q_{n-1}^0$ and $Q_{n-1}^1$. All we have to prove is that there exist circuits of each even length from $2^{n-2}(n-2)+2$ to $2^{n-1}n-4$ in $Q_n$. In the following, we describe how to construct those circuits.

**(1)** Constructing a circuit of length $2^{n-2}(n-2)+2$.

Assume that $C_\ell$ is a circuit of length $2^{n-2}(n-2)$ in $Q_{n-1}^0$ and $e = uv$ is an edge in $C_\ell$. It is obvious that edge $e_b^0$ is in $Q_{n-1}^1$, and $e_1^0$ and $e_2^0$ are of dimension 0. Note that both $e_1^0$ and $e_2^0$ are not in $E(Q_{n-1}^0) \cup E(Q_{n-1}^1)$. Thus, after adding $e_b^0$, $e_1^0$, and $e_2^0$ to and removing $e$ from $C_\ell$, a circuit of length $2^{n-2}(n-2)+2$ can be constructed.

**(2)** Constructing circuits of each even length from $2^{n-2}(n-2)+4$ to $2^{n-1}(n-2)$.

Assume that $C_\ell$ is a circuit of length $2^{n-2}(n-2)$ in $Q_{n-1}^0$ and $e = uv$ is an edge in $C_\ell$. For circuits $C_r$ of each even length from 4 to $2^{n-2}(n-2)$, we may assume without loss of generality that $e_b^0$ is also an edge of $C_r$. Thus, $C_\ell \circ C_r$ is also a circuit in $Q_n$ whose length is even and is in the range from $2^{n-2}(n-2)+4$ to $2^{n-1}(n-2)$.

**(3)** Constructing circuits of each even length from $2^{n-1}(n-2)+2$ to $2^{n-1}n-4$.

By Theorem 1, $Q_n$ is bipancyclic. Thus, there are cycles $C_m$ of each even length from 4 to $2^n-2$ in $Q_n$ for $n \geqslant 4$. After removing all of the edges in $C_m$ from $Q_n$, namely $Q_n - E(C_m)$, the resulting graph is still an

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eulerian subgraph. Since $n \geqslant 4$ and exact two incidence edges of each vertex in $C_m$ are removed, $Q_n - E(C_m)$ is connected. Notice that the number of edges in $Q_n - E(C_m)$ is even and is in the range from $2^{n-1}(n-2)+2$ to $2^{n-1}n-4$. Thus, this lemma holds for $n$ even.

**Case 2**: $n$ is odd.

Since $n$ is odd, every vertex in $Q_n$ is of odd degree. Removing $2^{n-1}$ matching edges from $Q_n$ yields an eulerian graph. Note that those $2^{n-1}$ matching edges cannot be all in the same dimension. This means that the length of a longest circuit in $Q_n$ is $2^{n-1}n - 2^{n-1} = 2^{n-1}(n-1)$. By the induction hypothesis, there exit circuits of every even length from 4 to $2^{n-2}(n-1)-4$ in both $Q^0_{n-1}$ and $Q^1_{n-1}$. In the following, we introduce how to construct circuits of each even length from $2^{n-2}(n-1)-2$ to $2^{n-1}(n-1)-2$.

**(1)** Constructing a circuit of length $2^{n-2}(n-1)-2$.

Let $C_\ell$ be a circuit of length $2^{n-2}(n-1)-4$ in $Q^0_{n-1}$ and $e = uv$ an edge in $C_\ell$. After adding $e^0_b$, $e^0_1$, and $e^0_2$ to and removing $e$ from $C_\ell$, a circuit of length $2^{n-2}(n-1)-2$ can be constructed.

**(2)** Constructing a circuit of length $2^{n-2}(n-1)$.

Since $Q^0_{n-1}$ itself is an eulerian graph, there is a circuit of length $2^{n-2}(n-1)$, i.e., all edges in $Q^0_{n-1}$.

**(3)** Constructing a circuit of length $2^{n-2}(n-1)+2$.

Let $C_\ell$ be an eulerian circuit of length $2^{n-2}(n-1)$ in $Q^0_{n-1}$ and $e = uv$ is an edge in $C_\ell$. It is obvious that edge $e^0_b$ is in $Q^1_{n-1}$, and $e^0_1$ and $e^0_2$ are of dimension 0. Note that both $e^0_1$ and $e^0_2$ are not in $E(Q^0_{n-1}) \cup E(Q^1_{n-1})$. Thus, after adding $e^0_b$, $e^0_1$, and $e^0_2$ to and removing $e$ from $C_\ell$, a circuit of length $2^{n-2}(n-1)+2$ can be constructed.

**(4)** Constructing circuits of each even length from $2^{n-2}(n-1)+4$ to $2^{n-1}(n-1)-4$.

Let $C_\ell$ be an eulerian circuit of length $2^{n-2}(n-1)$ in $Q^0_{n-1}$ and $C_r$ a circuit of even length in the range from 4 to $2^{n-2}(n-1)-4$ in $Q^1_{n-1}$. We may assume without loss generality that edge $e = uv$ in $C_\ell$ has a corresponding edge $e^0_b$ in $C_r$. Thus, by constructing $C_\ell \circ C_r$, all circuits of each even length from $2^{n-2}(n-1)+4$ to $2^{n-1}(n-1)-4$ can be constructed in $Q_n$.

**(5)** Constructing a circuit of length $2^{n-1}(n-1)-2$.

Assume that $C_\ell$ is an eulerian circuit in $Q^0_{n-1}$ and $C_r$ is a symmetric eulerian circuit of $C_\ell$ in $Q^1_{n-1}$. That is, if $C_\ell$ is the circuit $u_1u_2\cdots u_{|C_\ell|}$, then $C_r$ is the circuit $u^0_1u^0_2\cdots u^0_{|C_\ell|}$. Let $uv$ and $vw$ be two successive edges in $C_\ell$. Adding edges $uu^0$ and $ww^0$ to and removing edges $uv$, $vw$, $u^0v^0$, $v^0w^0$ from $E(C_\ell) \cup E(C_r)$ yields a circuit of length $2^{n-1}(n-1)-2$. This completes the proof. □

## References


[1] B. Alspach, Cycles of each length in regular tournaments, *Canadian Mathematical Bulletin* 10 (1967) 283–286.

[2] B. Alspach and D. Hare, Edge-pancyclic block-intersection graphs, *Discrete Mathematics* 97 (1991) 17–24.

[3] J.A. Bondy, Pancyclic Graphs I, *Journal of Combinatorial Theory, Series B* 11 (1971) 80–84.

[4] G.H. Chen and D.R. Duh, Topological properties, communication, and computation on WK-recursive networks, *Networks* 24(6) (1994) 303–317.

[5] K. Day and A. Tripathi, Embedding of cycles in arrangement graphs, *IEEE Transactions on Computers* 12 (1993) 1002–1006.

[6] J.X. Fan, Hamilton-connectivity and cycle-embedding of Möbius cubes, *Information Processing Letters* 82 (2002) 113–117.

[7] J.X. Fan, X. Lin and X.H. Jia, Node-pancyclicity and edge-pancyclicity of crossed cubes, *Information Processing Letters* 93 (2005) 133–138.

[8] J.X. Fan and X.H. Jia, Edge-pancyclicity and path-embeddability of bijective connection graphs, *Information Sciences* 178 (2008) 340–351.

[9] J.F. Fang, Y.R. Wang and H.L. Huang, The $m$-pancycle-connectivity of a WK-Recursive network, *Information Sciences* 177 (2007) 5611–5619.

[10] J.F. Fang, The bipancycle-connectivity of the hypercube, *Information Sciences* 178 (2008) 4679–4687.

[11] A. Germa, M.C. Hydemann and D. Sotteau, Cycles in the cubeconnected cycle graph, *Discrete Applied Mathematics* 83 (1999) 135–155.

[12] F. Harary and L. Moser, The theory of round robin tournaments, *The American Mathematical Monthly* 73 (1966) 231–246.

[13] A. Hobbs, The square of a block is vertex pancyclic, *Journal of Combinatorics Theory Series B* 20 (1976) 1–4.

[14] S.Y. Hsieh and C.H. Chen, Pancyclicity on Möbius cubes with maximal edge faults, *Parallel Computing* 30 (2004) 407–421.

[15] H.C. Hsu, L.C. Chiang, Jimmy J.M. Tan and L.H. Hsu, Fault hamiltonicity of augmented cubes, *Parallel Computing* 31 (2005) 130–145.

[16] W.T. Huang, W.K. Chen and C.H. Chen, Pancyclicity of Möbius cubes, in: *Proceedings of the 9th International Conference on Parallel and Distributed Systems* (ICPADS'02) 2002, pp. 591–596.

[17] S.C. Hwang and G.H. Chen, Cycles in butterfly graphs, *Networks* 35 (2000) 161–171.

[18] A. Kanevsky and C. Feng, On the embedding of cycles in pancake graphs, *Parallel Computing* 21 (1995) 923–936.

[19] T.K. Li, C.H. Tsai, Jimmy J.M. Tan, and L.H. Hsu, Bipanconnected and edge-fault-tolerant bipancyclic of hypercubes, *Information Processing Letters* 87 (2003) 107–110.

[20] M.J. Ma and J.M. Xu, Panconnectivity of locally twisted cubes, *Applied Mathematics Letters* 19(7) (2006) 673–677.

[21] J. Mitchem and E. Schmeichel, Pancyclic and bipancyclic graphsXa survey, in: *the Proceedings First Colorado Symposium on Graphs and Applications*, Boulder, CO, Wiley-Interscience, New York, 1985, pp. 271–278.

[22] J.W. Moon, On subtournaments of a tournament, *Canadian Mathematical Bulletin* 9 (1966) 297–301.

[23] Y.L. Wang, J.L. Guo, C.H. Hung and H.C. Chan, Circuits of Each Length in Tournaments, manuscript, 2010.

[24] H.L. Wang, J.W. Wang and J.M. Xu, Edge-fault-tolerant bipanconnectivity of hypercubes, *Information Sciences* 179 (2009) 404–409.

[25] D.B. West, *Introduction to Graph Theory* (Prentice Hall, Upper Saddle River, NJ, 2001).

[26] X.F. Yang, G.M. Megson and D.J. Evans, Locally twisted cubes are 4-pancyclic, *Applied Mathematics Letters* 17(8) (2004) 919–925.

[27] M.C. Yang, T.K. Li, Jimmy J.M. Tang and L.H. Hsu, On embedding cycles into faulty twisted cubes, *Information Sciences* 176 (2006) 676–690.