New Lower Bounds for the Three-dimensional Orthogonal Bin Packing Problem*

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Abstract

In this paper, we consider the three-dimensional orthogonal bin packing problem, which is a generalization of the well-known bin packing problem. We present new lower bounds for the problem and demonstrate that they improve the best previous results. The asymptotic worst-case performance ratio of the lower bounds is also proved. In addition, we study the non-oriented model, which allows items to be rotated.

1 Introduction

The bin packing problem (abbreviated as 1D-BP) is one of the classic NP-hard combinatorial optimization problems. Given a set of n items with positive sizes $v_1, v_2, \ldots, v_n \leq B$, the objective is to find a packing in bins of equal capacity B to minimize the number of bins required. The problem finds obvious practical relevance in many industrial applications, such as the container loading problem and the cutting stock problem.

The bin packing problem is strongly NP-hard, and it does not admit a $(\frac{3}{2} - \epsilon)$-factor approximation algorithm unless P=NP [13]. On the other hand, Johnson [14] showed that the simple *First Fit* approach can yield a $\frac{17}{10}$-approximation factor, and the *First Fit Decreasing* algorithm can approximate within an asymptotic $\frac{11}{9}$-factor [14]. Subsequently, Fernandez de la Vega and Lueker [12] proposed an asymptotic polynomial time approximation scheme (PTAS), and Karmarkar and Karp [15] presented an improved asymptotic fully PTAS. For further details of approximation algorithms, readers may refer to Coffman, Garey, and Johnson's survey [8].

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There are many variations of the bin packing problem, such as the strip packing, square packing, and rectangular box packing problems. In this paper, we consider the three-dimensional orthogonal bin packing problem (abbreviated as 3D-BP). Given an instance I of n 3D rectangular items $I_1, I_2, \ldots, I_n$, each item I_i is characterized by its width w_i, height h_i, depth d_i, and volume $v_i = w_i h_i d_i$. The goal is to determine a non-overlapping axis-parallel packing in identical 3D rectangular bins with width W, height H, depth D, and size $B = WHD$ that minimizes the number of bins required. First, we investigate the *oriented model*, which assumes that the orientation of the given items is fixed; that is, the items cannot be rotated and they are packed with each side parallel to the corresponding bin side. The *non-oriented model*, which allows items to be rotated, is also studied.

A considerable amount of research has been devoted to the design and analysis of lower bounds for the bin packing problem [5, 19]. Martello and Toth [22, 23] and Labbé *et al.* [17] proposed lower bounds for 1D-BP, and then extended the concept to multi-dimensional models [20, 21]. Fekete and Schepers [11] devised lower bounds based on *dual feasible functions* (for details, please refer to the Appendix A) and several related results were presented in [3, 7]. Boschetti [1] combined Martello and Toth's work with the above dual feasible functions and proposed the best lower bound for 3D-BP; that is, the lower bound *dominates*[1] all previous 3D-BP results. In contrast, there have been comparatively few studies of the non-oriented model [1, 6, 10], especially the three-dimensional model. Dell'Amico *et al.* [10] presented the first lower bound for the non-oriented model of the

[1]For two lower bounds L_i and L_j of a minimization problem, L_j is said to *dominate* L_i, denoted by $L_i \leq L_j$, if for any instance I, $L_i(I) \leq L_j(I)$, where $L(I)$ is the value provided by a lower bound L for an instance I.

two-dimensional orthogonal bin packing problem (abbreviated as 2D-BP). Clautiaux *et al.* considered different cases of the non-oriented 2D-BP model and proposed a new lower bound [6]; while Boschetti [1] investigated the non-oriented 3D-BP model and presented two new lower bounds.

In the following sections, we first review the previously proposed lower bounds and integrate the best of them for 1D-BP and 3D-BP to obtain a new lower bound for 3D-BP. Then, we propose a novel lower bound for 3D-BP and show that it dominates all the previous results. We also prove the asymptotic worst-case performance ratios of those results. Finally, we present a new lower bound for the non-oriented 3D-BP model.

2 The lower bounds for 1D-BP

An obvious lower bound for 1D-BP, called the *continuous lower bound*, can be computed as follows:

$$L_0 = \left\lceil \frac{\sum_{i=1}^{n} v_i}{B} \right\rceil$$

It is known that the asymptotic worst-case performance ratio of the continuous lower bound L_0 is $\frac{1}{2}$ for 1D-BP [22]. The lower bound can be easily extended to 3D-BP by considering the volume v_i of each item I_i. Martello *et al.* [20] showed that, for 3D-BP, the worst-case performance ratio of L_0 is $\frac{1}{8}$. Subsequently, the bound was improved by Martello and Toth [23]. Under the new bound, denoted by L_1, the set of items is partitioned into two subsets, one of which contains items that are larger than $B/2$ and the other contains the remainder. Since each item in the first subset needs one bin, at least $\mid V(B/2, B] \mid$[2] bins are required. Only items of size v_i, $p \leq v_i \leq B$ are considered, where p is an integer with $1 \leq p \leq B/2$. Hence, a valid lower bound L_1 can be computed if we allow the rest of the items (i.e., the items in $V[p, B/2]$) to be split. The other rounding scheme of L_1, denoted by $L'_1(p)$, is described in the Appendix A.

Labbé *et al.* [17] further improved L_1, denoted as L_2, by partitioning the set of items into three subsets ($V(B/2, B]$, $V(B/3, B/2]$, and $V[p, B/3]$, where $1 \leq p \leq B/3$) and applying the *First Fit Decreasing* algorithm [8, 14, 16]. The procedure is implemented as follows. The items in $V(B/2, B]$ are assigned to separate bins as L_1. It may be possible to assign some of the items in $V(B/3, B/2]$ to the open bins, but at most one item in $V(B/3, B/2]$ can fit in any of the open bins. Thus, the open bins are sorted in non-decreasing order based on their residual space, and the items in $V(B/3, B/2]$ are assigned in non-decreasing order of their size. The procedure proves that the items in $V(B/2, B]$ and $V(B/3, B/2]$ can be matched optimally in a pairwise manner. Let K be the subset of items in $V(B/3, B/2]$ that cannot be matched through the above procedure. The items in K can be paired, so at least $\lceil K/2 \rceil$ bins are required. It follows that a valid lower bound L_2 can be derived by allowing the items in $V[p, B/3]$ to be split as follows.

$$L_2 = \mid V(B/2, B] \mid + \lceil K/2 \rceil + \max_{1 \leq p \leq B/3}\{0, L_2(p)\}, \text{ where}$$

$$L_2(p) = \left\lceil \frac{\sum_{v_i \in V[p, B-p]} v_i}{B} - \mid V(B/2, B-p] \mid - \lceil K/2 \rceil \right\rceil$$

The lower bound L_2 can be obtained in $O(n)$ time provided that the items are pre-sorted according to their size. Bourjolly and Rebetez [2] and Crainic *et al.* [9] proved that $L_1 \leq L_2$ (excluding the rounding scheme $L'_1(p)$), and that the asymptotic worst-case performance ratio of L_2 for 1D-BP is $\frac{3}{4}$. Note that the primal concept of Labbé *et al.* cannot be easily extended to a new lower bound L_{m-1} for 1D-BP by partitioning the set of items into m subsets[3], even by using a brute-force approach.

3 Lower bounds for 3D-BP revisited

For 3D-BP, Boschetti [1] proposed a lower bound, denoted by L_B. Actually, it comprises three types of lower bounds: $L_B(p, q, r)$, $L'_B(p, q, r)$, and $L''_B(p, q, r)$[4], which we discuss in detail below. Note that no dominance relations hold between the three bounds.

$$L_B = \max_{\substack{1 \leq p \leq W/2 \\ 1 \leq q \leq H/2 \\ 1 \leq r \leq D/2}}\{L_B(p, q, r), L'_B(p, q, r), L''_B(p, q, r)\}$$

Boschetti [1] proved that L_B is currently the best lower bound for 3D-BP by applying L_1 to

[2] For convenience, we define $V(a, b] = \{I_i \mid a < v_i \leq b\}$ and its cardinality as $\mid V(a, b] \mid$.

[3] Scholl *et al.* [24] showed that the lower bound L_2 can be extended by considering the items in $V(B/4, B/3]$, but the process is quite complicated and it does not have any obvious extension.

[4] Note that the time required to compute the values of p, q, and r is polynomial in n, the size of an instance [1, 20].

$L_B(p,q,r)$ and $L'_B(p,q,r)$, denoted by $L_{B,1}$. In this section, we first review $L_{B,1}$ by applying L_1 to $L_B(p,q,r)$ and $L'_B(p,q,r)$. Then, based on the proofs in [2] and [3], which show, respectively, that $L_1 \leq L_2$ and $L'_1(p) \leq f_2^p$, we integrate L_2 in [17] and f_2^p in [3] with L_B to obtain a better lower bound for 3D-BP, denoted by $L_{B,2}$, and demonstrate that $L_{B,1} \leq L_{B,2}$.

The lower bound $L_{B,2}(p,q,r)$. First, we consider the lower bound $L_B(p,q,r)$. Given an item $I_i = (w_i, h_i, d_i)$ for every i, for convenience, we define $I^W(a,b] = \{I_i \mid a < w_i \leq b\}$. Similarly, $I^H(a,b] = \{I_i \mid a < h_i \leq b\}$ and $I^D(a,b] = \{I_i \mid a < d_i \leq b\}$. Thus, for example, $I^W(W-p,W] = \{I_i \mid W-p < w_i \leq W\}$, $I^H(H-q,H] = \{I_i \mid H-q < h_i \leq H\}$, and $I^D(D-r,D] = \{I_i \mid D-r < d_i \leq D\}$. We also let $I[p,q,r] = \{I_i \mid w_i \geq p,\ h_i \geq q,\ d_i \geq r\}$. The objective of $L_B(p,q,r)$ is to compute a valid lower bound for 1D-BP by using a simple rounding technique when considering the volume of each item in $I[p,q,r]$. For example, if $I_i \in I^W(W-p,W]$ is placed in a bin, it will occupy a volume equal to Wh_id_i, since no items in $I[p,q,r]$ can be packed side by side parallel to the width. Hence, we let $B = WHD$ and compute $L_B(p,q,r)$ as a continuous lower bound by rounding the volume of each item v_i for every i to $v_i(p,q,r) = w_i(p)h_i(q)d_i(r)$ such that if $I_i \in I^W(W-p,W]$, i.e., $w_i > W-p$, then $w_i(p) = W$; otherwise, $w_i(p) = w_i$. If $I_i \in I^H(H-q,H]$, then $h_i(q) = H$; otherwise, $h_i(q) = h_i$. Similarly, if $I_i \in I^D(D-r,D]$, then $d_i(r) = D$; otherwise, $d_i(r) = d_i$. This rounding technique is the so-called classic dual feasible function f_0^p [3, 11]. The resulting lower bound $L_B(p,q,r)$ can be formulated as follows:

$$L_B(p,q,r) = \left\lceil \frac{\Sigma_{i=1}^n v_i(p,q,r)}{B} \right\rceil$$

Since $L_B(p,q,r)$ can be computed as a continuous lower bound for 1D-BP by considering the volume of each item, L_1 can be applied to $L_B(p,q,r)$ to derive a valid lower bound, denoted by $L_{B,1}(p,q,r)$. By contrast, we apply L_2 and the dual feasible function f_2^p to $L_B(p,q,r)$ separately. Then, we select the larger of the two refined lower bounds, denoted by $L_{B,2}(p,q,r)$, and show that it is a valid lower bound and that it is not smaller than $L_{B,1}(p,q,r)$.

Lemma 1 *$L_{B,2}(p,q,r)$ is a valid lower bound for 3D-BP, and it dominates $L_{B,1}(p,q,r)$.*

Proof. Based on the above rounding scheme, each item in $V(B/2,B]$ that is rounded, say w_i is rounded to W if $w_i > W-p$, has two other dimensions larger than $H/2$ and $D/2$; otherwise, $v_i(p,q,r) \leq B/2$. Hence, the items in $V(B/2,B]$ are assigned to separate bins.

Consider the items in $V(B/3,B/2]$. Assume we fit item I_i in $V(B/3,B/2]$ in an open bin, and place item I_j in $V(B/2,B]$ in the same bin. In addition, suppose the original dimensions of I_j were $w_j > W-p$, $h_j > H/2$, and $d_j > D/2$. Then, we only need to determine the height and depth of I_i since we only consider the items in $I[p,q,r]$. We have $h_i > H/3$ and $d_i > D/3$ because $v_i(p,q,r) > B/3$. If $h_i < H/2$, it implies that $d_i > 2D/3$; similarly, if $d_i < D/2$, it implies that $h_i > 2H/3$. Thus, at most one item in $V(B/3,B/2]$ can fit in any of the open bins. Moreover, the rounded items in $V(B/3,B/2]$ that cannot be matched could not be matched originally either. Based on the above discussion, at most two items in $V(B/3,B/2]$ can be paired, so applying L_2 to $L_B(p,q,r)$ is valid.

Furthermore, $L_{B,2}(p,q,r)$ is a valid lower bound for 3D-BP because f_2^p is a dual feasible function, where $1 \leq p \leq B/2$ and it can be applied to $L_B(p,q,r)$ directly. Because $L_1 \leq L_2$ and $L'_1(p) \leq f_2^p$, $L_{B,2}(p,q,r)$ dominates $L_{B,1}(p,q,r)$.

The lower bound $L'_{B,2}(p,q,r)$. Regarding the lower bound $L'_B(p,q,r)$, as in the above proof, only the items in $I[p,q,r]$ are considered. Let $I(W-p,H-q,D-r) = I^W(W-p,W] \cap I^H(H-q,H] \cap I^D(D-r,D]$. Obviously, $|I(W-p,H-q,D-r)|$ is a valid lower bound and no items in $I[p,q,r]$ can be placed in the open bins. Next, the items in $I[p,q,r] \setminus I(W-p,H-q,D-r)$, denoted by $I'[p,q,r]$ are considered. The lower bound $L'_B(p,q,r)$ considers items in terms of their width, height, and depth. Let the respective subsets be:

$$I(p,H-q,D-r) = I^H(H-q,H] \cap I^D(D-r,D] \cap I'[p,q,r];$$
$$I(W-p,q,D-r) = I^W(W-p,W] \cap I^D(D-r,D] \cap I'[p,q,r].$$
$$I(W-p,H-q,r) = I^W(W-p,W] \cap I^H(H-q,H] \cap I'[p,q,r];$$

Any two items from the different subsets above cannot be matched in the same bin. That is, the items in $I(p,H-q,D-r)$, $I(W-p,q,D-r)$, and $I(W-p,H-q,r)$ can only be packed in separate bins. Thus, for each dimension, a continuous lower bound of 1D-BP can be computed similarly, which means that a valid lower bound $L'_B(p,q,r)$ can be derived as follows:

$$L'_B(p,q,r) = |\, I(W-p, H-q, D-r)\,| + \left\lceil \frac{\Sigma_{I_i \in I(p,H-q,D-r)} w_i}{W} \right\rceil + \left\lceil \frac{\Sigma_{I_i \in I(W-p,q,D-r)} h_i}{H} \right\rceil + \left\lceil \frac{\Sigma_{I_i \in I(W-p,H-q,r)} d_i}{D} \right\rceil$$

Because a continuous lower bound of 1D-BP can be computed for each dimension, Boschetti [1] applied L_1 to the lower bound $L'_B(p,q,r)$, denoted by $L'_{B,1}(p,q,r)$, with respect to the width, height, and depth. We derive our lower bound, denoted as $L'_{B,2}(p,q,r)$, by applying L_2 and f_2^p to $L'_B(p,q,r)$ separately and selecting the larger of the two refined lower bounds. The following lemma shows that $L'_{B,2}(p,q,r)$ is still a valid lower bound.

Lemma 2 *$L'_{B,2}(p,q,r)$ is a valid lower bound for 3D-BP, and it dominates $L'_{B,1}(p,q,r)$.*

Proof. Without loss of generality, we consider that the depth of each item in $I(W-p,H-q,r) = I^W(W-p,W] \cap I^H(H-q,H] \cap I'[p,q,r]$. Because $r \le d_i < D-r$, the lower bound L_2 for 1D-BP can be used directly due to the depth of these items. This is similar to the width and height of the items in $I(p,H-q,D-r)$ and $I(W-p,q,D-r)$ respectively. Moreover, the dual feasible function f_2^p can be used directly for each dimension of the items. Hence, $L'_{B,2}(p,q,r)$ is a valid lower bound for 3D-BP. Because $L_1 \le L_2$ and $L'_1(p) \le f_2^p$, $L'_{B,2}(p,q,r)$ dominates $L'_{B,1}(p,q,r)$. □

The lower bound $L''_{B,2}(p,q,r)$. The lower bound $L''_B(p,q,r)$, which is conceptually similar to $L_B(p,q,r)$, can be obtained by using another rounding technique proposed in [21, 23]. The objective is to pack items into a bin, e.g., small rectangular boxes whose dimensions are p, q, and r, where $1 \le p \le W/2$, $1 \le q \le H/2$, and $1 \le r \le D/2$. The maximum number of such boxes that can be placed in a bin is $\lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor$. Besides, every item is represented by small rectangular boxes whose dimensions are p, q, and r. For every i, the volume of each item v_i, can be rounded to $v'_i(p,q,r) = w'_i(p)h'_i(q)d'_i(r)$ such that, if $I_i \in I^W(W/2,W]$, then $w'_i(p) = \lfloor W/p \rfloor - \lfloor (W-w_i)/p \rfloor$; otherwise, $w'_i(p) = \lfloor w_i/p \rfloor$. If $I_i \in I^H(H/2,H]$, then $h'_i(q) = \lfloor H/q \rfloor - \lfloor (H-h_i)/q \rfloor$; otherwise, $h'_i(q) = \lfloor h_i/q \rfloor$. Similarly, if $I_i \in I^D(D/2,D]$, then $d'_i(r) = \lfloor D/r \rfloor - \lfloor (D-d_i)/r \rfloor$; otherwise, $d'_i(r) = \lfloor d_i/r \rfloor$. For each dimension, it can be proved that the rounding technique is a dual feasible function [3, 11]. $L''_B(p,q,r)$ can be computed as a continuous lower bound as follows:

$$L''_B(p,q,r) = \max_{\substack{1\le p\le W/2\\ 1\le q\le H/2\\ 1\le r\le D/2}} \left\{ \left\lceil \frac{\Sigma_{i=1}^n v'_i(p,q,r)}{\lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor} \right\rceil \right\}$$

We let the size of a bin B be equal to $\lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor$ and apply L_2 to $L''_B(p,q,r)$, denoted by $L''_{B,2}(p,q,r)$, and show that it is also a valid lower bound.

Lemma 3 *$L''_{B,2}(p,q,r)$ is a valid lower bound for 3D-BP, and it dominates $L''_B(p,q,r)$.*

Proof. When L_2 is applied to $L''_B(p,q,r)$, the dimensions of the items in $V(B/2,B]$ are larger than $\frac{1}{2}\lfloor W/p \rfloor$, $\frac{1}{2}\lfloor H/q \rfloor$, and $\frac{1}{2}\lfloor D/r \rfloor$. The width w_i of each item I_i with $w'_i(p) > \frac{1}{2}\lfloor W/p \rfloor$ was larger than $W/2$ originally. Similarly, $h_i > H/2$ and $d_i > D/2$. Hence, the items in $V(B/2,B]$ are assigned to separate bins.

Consider each item I_i in $V(B/3,B/2]$. Because $v'_i(p,q,r) > B/3$, we have $w'_i(p) > \frac{1}{3}\lfloor W/p \rfloor$, $h'_i(q) > \frac{1}{3}\lfloor H/q \rfloor$, and $d'_i(r) > \frac{1}{3}\lfloor D/r \rfloor$, which implies that $w_i > W/3$, $h_i > H/3$, and $d_i > D/3$. If item I_i can fit in an open bin, without loss of generality, there is one dimension of I_i, say $d'_i(r)$, that satisfies the condition $\frac{1}{2}\lfloor D/r \rfloor > d'_i(r) > \frac{1}{3}\lfloor D/r \rfloor$, which implies that $D/2 \ge d_i > D/3$. Furthermore, if $\frac{1}{2}\lfloor D/r \rfloor > d'_i(r)$, because $v'_i(p,q,r) > B/3$, we have $w'_i(p) > \frac{2}{3}\lfloor W/p \rfloor$ and $h'_i(q) > \frac{2}{3}\lfloor H/q \rfloor$, which implies that $w_i > 2W/3$ and $h_i > 2H/3$. Thus, at most one item in $V(B/3,B/2]$ can fit in any of the open bins; and at most two items in $V(B/3,B/2]$ can be paired.

On the other hand, we claim that if we cannot fit I_i in some open bin that contains I_j in $V(B/2,B]$, then I_j was not matched with I_i originally. More precisely, if $d'_j(r) + d'_i(r) > \lfloor D/r \rfloor$, then $d_j + d_i > D$. We know that $d'_j(r) = \lfloor D/r \rfloor - \lfloor (D-d_j)/r \rfloor$. Suppose that $d_i \le D/2$. Then, we have:

$$\left\lfloor \frac{D}{r} \right\rfloor - \left\lfloor \frac{D-d_j}{r} \right\rfloor + \left\lfloor \frac{d_i}{r} \right\rfloor > \left\lfloor \frac{D}{r} \right\rfloor$$

$$\Rightarrow \left\lfloor \frac{d_i}{r} \right\rfloor > \left\lfloor \frac{D-d_j}{r} \right\rfloor$$

$$\Rightarrow \frac{d_i}{r} \ge \left\lfloor \frac{d_i}{r} \right\rfloor \ge \left\lfloor \frac{D-d_j}{r} \right\rfloor + 1 > \left(\frac{D-d_j}{r} - 1 \right) + 1$$

$$\Rightarrow d_j + d_i > D$$

We also know that $d_j + d_i > D$ if $d_i > D/2$; therefore, $L''_{B,2}(p,q,r)$ is a valid lower bound. Because $L_0 \le L_2$, $L''_{B,2}(p,q,r)$ dominates $L''_B(p,q,r)$. $\square$

Thus, we have the following new lower bound $L_{B,2}$ for 3D-BP:

$$L_{B,2} = \max_{\substack{1\le p\le W/2\\ 1\le q\le H/2\\ 1\le r\le D/2}}\{L_{B,2}(p,q,r), L'_{B,2}(p,q,r), L''_{B,2}(p,q,r)\}$$

The theorem follows immediately.

Theorem 4 $L_B \le L_{B,1} \le L_{B,2}$.

4 A new lower bound for 3D-BP

In this section, we extend the approach in [17] to 3D-BP and propose a novel lower bound, denoted by L^*_B. We show that the new lower bound dominates all the results reported in the literature and provide a strictly better example. The worst-case performance ratios are also proved. First, we define some notations.

$$I(W/2,H/2,D/2) = I^W(W/2,W] \cap I^H(H/2,H] \cap I^D(D/2,D];$$
$$I(W/3,H/2,D/2) = I^H(H/2,H] \cap I^D(D/2,D] \cap I^W(W/3,W/2];$$
$$I(W/2,H/3,D/2) = I^W(W/2,W] \cap I^D(D/2,D] \cap I^H(H/3,H/2];$$
$$I(W/2,H/2,D/3) = I^W(W/2,W] \cap I^H(H/2,H] \cap I^D(D/3,D/2];$$
$$I((3-\sqrt{3})W/3,H/2,D/2) = I^H(H/2,H] \cap I^D(D/2,D] \cap I^W((3-\sqrt{3})W/3,W/2];$$
$$I(W/2,(3-\sqrt{3})H/3,D/2) = I^W(W/2,W] \cap I^D(D/2,D] \cap I^H((3-\sqrt{3})H/3,H/2];$$
$$I(W/2,H/2,(3-\sqrt{3})D/3) = I^W(W/2,W] \cap I^H(H/2,H] \cap I^D((3-\sqrt{3})D/3,D/2];$$

We compute the new lower bound as follows.

Step 1. The items in $I(W/2,H/2,D/2) \cap V(B/3,B]$ are assigned to separate bins because each dimension of the items in $I(W/2,H/2,D/2)$ is more than half the size of the corresponding side of the bins (see Figure 1(a) in the Appendix C). It is possible to assign the items in $I(W/3,H/2,D/2) \cap V(B/3,B]$, $I(W/2,H/3,D/2) \cap V(B/3,B]$, and $I(W/2,H/2,D/3) \cap V(B/3,B]$ to the open bins; however, at most one item can fit in any of the open bins because only the items in $V(B/3,B]$ are considered. In addition, an item from the above three subsets can only fit in the open bins if one of its dimensions is less than half the size of the corresponding side of the bin; that is, an item can only be packed in the open bins that have the appropriate width, height, and depth.

Step 2. Since the items in the open bins have at least two dimensions that are more than $\sqrt{3}/3$ the size of the corresponding sides of the bins, the open bins can be divided into three subsets based on the smallest dimension of the items they contain. The bins in each subset are sorted in non-increasing order based on the corresponding dimension. Therefore, the items in $I((3-\sqrt{3})W/3,H/2,D/2) \cap V(B/3,B]$, $I(W/2,(3-\sqrt{3})H/3,D/2) \cap V(B/3,B]$, and $I(W/2,H/2,(3-\sqrt{3})D/3) \cap V(B/3,B]$ must be assigned in non-decreasing order separately based on their width, height, and depth. Similar to the proof of Labbé *et al.* [17], the procedure proves that the items are matched optimally in a pairwise manner.

Step 3. Any remaining open bins are sorted in non-decreasing order based on their residual space. The items that cannot be matched in Step 2 are mixed with the remaining items in $I(W/3,H/2,D/2) \cap V(B/3,B]$, $I(W/2,H/3,D/2) \cap V(B/3,B]$, and $I(W/2,H/2,D/3) \cap V(B/3,B]$ and assigned in non-decreasing order according to their volume (see Figure 1(b)). Let $K^{HD} \subseteq I(W/3,H/2,D/2) \cap V(B/3,B]$ be the subset of items that cannot be matched through the above process; and let $K^{WD} \subseteq I(W/2,H/3,D/2) \cap V(B/3,B]$ and $K^{WH} \subseteq I(W/2,H/2,D/3) \cap V(B/3,B]$ be the other subsets of items that cannot be matched. Note that any two items from the different subsets listed above cannot be matched in the same bin because, without loss of generality, if one dimension of each item I_i, say w_i, is not larger than $W/2$, it implies that $h_i > 2H/3$ and $d_i > 2D/3$(see Figure 1(b)). Hence, the items in K^{HD}, K^{WD}, and K^{WH} can only be paired separately, and at least $\lceil K^{HD}/2\rceil + \lceil K^{WD}/2\rceil + \lceil K^{WH}/2\rceil$ bins are required.

Proposition 5 *For any instance I, all the items in $V(B/3,B]$ have been placed in bins before* Step 4.

Proof. For any instance I, each item in $V(B/3,B]$ has at least two dimensions that are more than half the size of the corresponding sides of its bin. In addition, the smallest dimension of each item is more than 1/3 the size of its corresponding side of the bin. Because all the items in $(I(W/2,H/2,D/2) \cup I(W/3,H/2,D/2) \cup I(W/2,H/3,D/2) \cup I(W/2,H/2,D/3)) \cap V(B/3,B]$ have been considered and placed in bins, the proof is complete. $\square$

Step 4. We consider the remaining items in $I(p,H-q,D-r)$, $I(W-p,q,D-r)$, and $I(W-p,H-q,r)$. The key point is that any two items from the different subsets listed above cannot be matched in the same bin. Thus, the items can only be packed in terms of each dimension. By Proposition 5, the remaining items can be classified into three types of subsets: $I(p, H-q, D-r) \cap I^W(0, W/3]$, $I(W-p, q, D-r) \cap I^H(0, H/3]$, and $I(W-p, H-q, r) \cap I^D(0, D/3]$. Note that we allow the remaining items to be split when they are assigned to the used open bins. First, we select the used open bins that contain the items in $I(p, H-q, D-r) \cap I^W(W/3, W-p]$, $I(W-p, q, D-r) \cap I^H(H/3, H-q]$, and $I(W-p, H-q, r) \cap I^D(D/3, D-r]$. The remaining items are assigned to these bins separately based on their smallest dimension. If these open bins are completely full, the other open bins are sorted in non-decreasing order based on their residual space, say b'_1, b'_2, ..., b'_k. Without loss of generality, the remaining items in $I(p, H-q, D-r) \cap I^W(0, W/3]$ are first assigned to b'_1, ..., b'_k in this order. If every open bin is completely full, we proceed to Step 5. Otherwise, we keep packing the items until some bin, say b'_j, $j \leq k$, cannot be filled completely. That is, each bin b'_i, $1 \leq i < j$, is completely full. The remaining items in $I(p, H-q, D-r) \cap I^W(0, W/3]$ will not be assigned to b'_j. Next, we consider the items in $I(W-p, q, D-r) \cap I^H(0, H/3]$. The items are assigned to b'_j, b'_{j+1}, ..., b'_k in the same way; then, we consider the items in $I(W-p, H-q, r) \cap I^D(0, D/3]$. If there are still open bins that are not completely full, the remaining items in each subset can be assigned to any open bin because the bins are sorted in non-decreasing order based on their residual space. Hence, we just assign each subset of remaining items to an open bin. If the number of open bins that are not completely full is less than the number of subsets of remaining items, go to Step 5.

Step 5. Let $I'(p, H-q, D-r)$, $I'(W-p, q, D-r)$, and $I'(W-p, H-q, r)$ be the subsets of items that cannot be packed in the $|I(W/2, H/2, D/2) \cap V(B/3, B]| + \lceil K^{HD}/2 \rceil + \lceil K^{WD}/2 \rceil + \lceil K^{WH}/2 \rceil$ open bins respectively. We compute a continuous lower bound of 1D-BP for each dimension (i.e., $L^*_B(3,4,5) = (3) + (4) + (5)$ below). Finally, a valid lower bound can be obtained by allowing the rest of the items to be split.

We use the rounding scheme, i.e., the dual feasible function f^p_0 for each dimension of every item I_i, to derive a rounded volume $v_i(p, q, r) = w_i(p) h_i(q) d_i(r)$. Next, we show that (1) L^*_B is a valid lower bound; and (2) after applying the rounding scheme f^p_0, L^*_B dominates $\max_{1 \leq p \leq W/2, 1 \leq q \leq H/2, 1 \leq r \leq D/2} \{L_{B,2}(p, q, r), L'_{B,2}(p, q, r)\}$.

$$L^*_B = |\, I(W/2, H/2, D/2) \cap V(B/3, B] \,| + \tag{1}$$

$$\left\lceil \frac{K^{HD}}{2} \right\rceil + \left\lceil \frac{K^{WD}}{2} \right\rceil + \left\lceil \frac{K^{WH}}{2} \right\rceil + \tag{2}$$

$$\left\lceil \frac{\sum_{I_i \in I'(p, H-q, D-r)} w_i}{W} \right\rceil + \tag{3}$$

$$\left\lceil \frac{\sum_{I_i \in I'(W-p, q, D-r)} h_i}{H} \right\rceil + \tag{4}$$

$$\left\lceil \frac{\sum_{I_i \in I'(W-p, H-q, r)} d_i}{D} \right\rceil + \tag{5}$$

$$\max_{\substack{1 \leq p \leq W/2 \\ 1 \leq q \leq H/2 \\ 1 \leq r \leq D/2}} \{0, L^*_B(p, q, r)\}, \text{ where } L^*_B(p, q, r) =$$

$$\left\lceil \frac{\sum_{I_i \in I'[p,q,r]} v_i}{B} - \alpha + |\, I(W-p, H-q, D-r) \,| \right\rceil,$$

$$\alpha = L^*_B(1, 2) + L^*_B(3, 4, 5),$$

and $L^*_B(1, 2) = (1) + (2)$, $L^*_B(3, 4, 5) = (3) + (4) + (5)$.

Lemma 6 *L^*_B is a valid lower bound.*

Proof. The dimensions of each item in $I(W/2, H/2, D/2)$ are more than half the size of the corresponding sides of its bin, even if the item is rounded. Hence, the items in $I(W/2, H/2, D/2)$ are assigned to separate bins.

Consider the items in $I(W/3, H/2, D/2) \cap V(B/3, B]$, $I(W/2, H/3, D/2) \cap V(B/3, B]$, and $I(W/2, H/2, D/3) \cap V(B/3, B]$. Without loss of generality, suppose we place item $I_j \in I(W/2, H/2, D/2) \cap V(B/3, B]$ in an open bin, and fit item $I_i \in I(W/3, H/2, D/2) \cap V(B/3, B]$ in the same bin. I_i may fit in terms of the width because $h_i(q) > H/2$ and $d_i(r) > D/2$ imply that $h_i > H/2$ and $d_i > D/2$. Besides, $w_i(p) = w_i$ because $W/2 \geq w_i > W/3$. $W/2 \geq w_i$ also implies that $h_i(q) > 2H/3$ and $d_i(r) > 2D/3$ because $v_i(p, q, r) > B/3$. Thus, if h_i is rounded, $h_i > H - q$; otherwise, $h_i > 2H/3$. Similarly, $d_i > \min\{D - r, 2D/3\}$. Because only the items in $I[p, q, r]$ are considered, at most one item in the above three subsets (every item I_k in the subsets has $w_k > W/3$, $h_k > H/3$, and $d_k > D/3$) can fit in any of the open bins.

On the other hand, since I_i may fit (in terms of the width) into the bin that contains I_j, we need to consider if w_j is rounded (because $w_i(p) = w_i$). We know that the rounded w_j that cannot be matched was not matched originally. In addition, based on the above discussion, for item $I_i \in K^{HD}$, $W/2 \geq w_i > W/3$ implies that $h_i > \min\{H - q, 2H/3\}$ and $d_i > \min\{D - r, 2D/3\}$. Thus, two items from any two of K^{HD}, K^{WD}, and K^{WH}

cannot be matched in the same bin; and at most two items from each subset can be paired.

Finally, similar to the lower bound $L'_B(p,q,r)$, we consider the remaining items in $I(W-p,H-q,r)$, $I(p,H-q,D-r)$, and $I(W-p,q,D-r)$. First, the items are assigned to the above open bins by allowing the items to be split. Note that Step 4 guarantees that any two items from the different subsets above will be assigned to different open bins after applying the rounding scheme f_0^p. If all the open bins are completely full, we compute a continuous lower bound of 1D-BP for each dimension of the remaining items; otherwise, $L_B^*(3,4,5)$ would be equal to zero. Thus, f_0^p can be applied to L_B^*, which becomes a valid lower bound for 3D-BP by allowing the rest of the items to be split. □

Lemma 7 *For each* $1 \le p \le W/2$, $1 \le q \le H/2$, $1 \le r \le D/2$, L_B^* *dominates* $L_{B,2}(p,q,r)$ *and* $L'_{B,2}(p,q,r)$.

Proof. First, we consider $L_{B,2}(p,q,r)$. Since f_0^p is applied to both $L_{B,2}(p,q,r)$ and our new lower bound L_B^*, we claim that the new partition scheme is better than that derived by Labbé *et al.*'s method [17], i.e., L_2. For the first part, we have $|I(W/2,H/2,D/2) \cap V(B/3,B]|$ open bins compared to $|V(B/2,B]|$ bins. Every item $I_k \in V(B/2,B]$ has $w_k(p) > W/2$, $h_k(q) > H/2$, and $d_k(r) > D/2$; thus, $I_k \in I(W/2,H/2,D/2) \cap V(B/3,B]$, and we have $V(B/2,B] \subseteq I(W/2,H/2,D/2) \cap V(B/3,B]$.

For the second part, each item $I_k \in V(B/3,B]$ has $w_k(p) > W/3$, $h_k(q) > H/3$, and $d_k(r) > D/3$. If one of the item's dimensions, say the width $w_k(p) \le W/2$, it implies that $h_k(q) > 2H/3$ and $d_k(r) > 2D/3$. By Proposition 5, we have $V(B/3,B] = (I(W/2,H/2,D/2) \cup I(W/3,H/2,D/2) \cup I(W/2,H/3,D/2) \cup I(W/2,H/2,D/3)) \cap V(B/3,B]$. Therefore, $|V(B/2,B]| + \lceil K/2 \rceil \le |I(W/2,H/2,D/2) \cap V(B/3,B]| + \lceil K^{WH}/2 \rceil + \lceil K^{HD}/2 \rceil + \lceil K^{WD}/2 \rceil$. It is obvious that the remainder of $L_{B,2}(p,q,r)$ is not larger than the remainder of L_B^*. Thus, L_B^* dominates $L_{B,2}(p,q,r)$.

Consider the lower bound $L'_{B,2}(p,q,r)$. For the first part, because f_0^p is applied to L_B^*, we have $I(W-p,H-q,D-r) \subseteq I(W/2,H/2,D/2) \cap V(B/3,B]$. Regarding the second part, without loss of generality, assume that $I(W-p,H-q,r)$ is considered in $L'_{B,2}(p,q,r)$. We explore the possibility of placing the items in $I(W/2,H/2,D/2) \cup I(W/2,H/2,D/3) \cup I(W-p,H-q,r)$ for the new lower bound L_B^*. Clearly, by considering each dimension separately in Steps 4 and 5, L_B^* dominates $L'_{B,2}(p,q,r)$. □

A strictly better example. Here, we present a strictly better example to emphasize the difference between the lower bounds $L_{B,2}(p,q,r)$, $L'_{B,2}(p,q,r)$, and L_B^*. Suppose the given instance has $|S_1|+|S_2|+|S_3|+|S_4|$ items, where S_1, S_2, S_3 and S_4 are defined as follows:

$$S_1 = I(W-p,H-q,r) \cap I^D(D/2,D] \cap V(2B/3,B]$$
$$S_2 = I(W/2,H/2,D/2) \cap I^H(H/2,H-q] \cap I^D(D/2,D-r] \cap V(2B/3,B]$$
$$S_3 = I(W/2,H/2,D/2) \cap I^H(H/2,H-q] \cap I^D(D/2,D-r] \cap I^W(\sqrt{3}W/3,W] \cap V(B/3,B/2]$$
$$S_4 = I(W/3,H/2,D/2) \cap I^H(H/2,H-q] \cap I^D(D/2,D-r] \cap I^W((3-\sqrt{3})W/3,W/2] \cap V(B/3,B/2]$$

Consider $L_{B,2}(p,q,r)$. The items in S_1 and S_2 are assigned to $|S_1|+|S_2|$ separate open bins. Besides, because the items in S_3 and S_4 are larger than $B/3$, it is impossible to assign them to those open bins. Hence, the items in S_3 and S_4 can be paired such that $L_{B,2}(p,q,r) = |S_1|+|S_2|+\lceil(|S_3|+|S_4|)/2\rceil$. In addition, because only the S_1 subset of items is considered for the lower bound $L'_{B,2}(p,q,r)$, we obtain $L'_{B,2}(p,q,r) = |S_1|$ by applying L_2. For our new lower bound L_B^*, the items in S_1, S_2 and S_3 are assigned to separate bins according to (1). The items in S_4 cannot be assigned to the bins that contain the items in S_1 and S_2 because each item in S_1 and S_2 is larger than $2B/3$. Besides, in Step 2, the items in S_4 cannot be matched with the items in S_3. Therefore, $L_B^* = |S_1|+|S_2|+|S_3|+\lceil|S_4|/2\rceil$ and $\max\{L_{B,2}(p,q,r), L'_{B,2}(p,q,r)\} < L_B^*$ when $|S_3| \ge 2$.

Next, we apply the dual feasible function f_2^p (instead of $L''_{B,2}(p,q,r)$) to each dimension of all the given items. Note that f_2^p is similar to the rounding technique of $L''_{B,2}(p,q,r)$ except when $w_i = W/2$, $h_i = H/2$, and $d_i = D/2$. We compute the summation of the rounded volume of each item, and obtain a continuous lower bound (referred to as $L''_{B,2}(p,q,r)$). It is also valid to apply L_2 to this continuous lower bound, denoted by L_{DF}^*, because f_2^p does not make any difference to the rounded volume ($x = B/2$ if the bin size is B, compared with $f_2^p(x) = \lfloor B/p \rfloor$ if the bin size is $2\lfloor B/p \rfloor$) when the uniquely different cases $w_i = W/2$, $h_i = H/2$, and $d_i = D/2$ between f_2^p and $L''_B(p,q,r)$ are considered [3].

Finally, we have:

$$L^*_{B,DF} = \max\{L^*_B, L^*_{DF}\}$$

Because $L''_{B,2}(p,q,r) \le L^*_{DF}$, the next theorem follows immediately.

Theorem 8 $L_{B,2} \le L^*_{B,DF}$.

4.1 Worst-case performance analysis

Given any instance I of a minimization problem, the *worst-case performance ratio* of a lower bound $L(I)$, which is the value provided by a lower bound L for the instance I, is defined as the maximum value $R(L)$ such that $R(L) \le L(I)/OPT(I)$, where $OPT(I)$ is the optimum solution for I.

We refer to the results reported in [2, 9, 20, 21, 23] and prove that the *asymptotic*[5] worst-case performance ratio of L^*_B is $\frac{3}{19}$ for any instance I with $L^*_B = L^*_B(1,2)$. The ratio is larger than that of L_0 for 3D-BP, i.e., $\frac{1}{8}$. To save space, we shall skip the following proofs, which can be found in the Appendix B.

Theorem 9 *For any instance I,*

$$R(L^*_B) \le \begin{cases} \frac{3}{19}, & \textit{if } L^*_B = L^*_B(1,2); \\ \frac{1}{8}, & \textit{otherwise.} \end{cases}$$

Similarly, the asymptotic worst-case performance ratio of L^*_{DF} can be obtained as follows.

Theorem 10 *For any instance I,*

$$R(L^*_{DF}) \le \begin{cases} \frac{1}{5}, & \textit{if } L_2(p) \le 0; \\ \frac{1}{8}, & \textit{otherwise.} \end{cases}$$

Here, $L_2(p)$ represents the difference between the continuous lower bound and the number of bins that are open before allowing the items in $V(0, B/3]$ to be split when applying L_2 to L^*_{DF}.

5 The non-oriented model

In this section, we extend our lower bounds to a more general case, the *non-oriented model* of 3D-BP, where rotation of each given item by 90° is allowed. Compared to the above *oriented model*, which assumes that the orientation of the given items is fixed, there is a dearth of research on this problem, especially on 3D-BP. We refer to the results reported in [1, 10] and propose new lower bounds for the non-oriented 3D-BP model.

The rationale behind our lower bound, which is similar to Dell'Amico *et al.*'s method [10], is to decompose the given rectangular items into square items, i.e., cubes. That is, every 3D rectangular item is converted into several 3D cubes by cutting it appropriately. Therefore, the new lower bound L^*_B can be applied directly, irrespective of whether the resulting items are rotated by 90°. We generalize the cutting approach for 2D-BP in [10] to the three-dimensional model. The decomposition procedure is implemented as follows.

Without loss of generality, we assume that $W \ge H \ge D$; and for each item I_i, $w_i \ge h_i \ge d_i$, $1 \le i \le n$. First, we decompose every item I_i into the maximum number of $d_i \times d_i \times d_i$ cubes, and consider its residual space in two cases, $(w_i \bmod d_i) \le H/2$ and $(w_i \bmod d_i) > H/2$, as shown in Figures 2 and 3 respectively.

Suppose $(w_i \bmod d_i) \le H/2$. If $(h_i \bmod d_i) > 0$ and we want to continue cutting the residual space S_{hd}, then $(h_i \bmod d_i) \times (h_i \bmod d_i) \times (h_i \bmod d_i)$ cubes will be produced because $(h_i \bmod d_i) < d_i$. However, because $(h_i \bmod d_i) < h_i/2 \le H/2 \le W/2$, the volume of a $(h_i \bmod d_i) \times (h_i \bmod d_i) \times (h_i \bmod d_i)$ cube is less than $B/3$ and such cubes will not be computed in $L^*_B(1,2)$ or $L^*_B(3,4,5)$. That is, the cubes can be split and computed as a continuous lower bound when we compute the lower bound L^*_B. Thus, it is not necessary to decompose the residual space S_{hd} for our lower bound L^*_B in cases (a) and (b) in Figure 2.

On the other hand, if $(w_i \bmod d_i) > 0$ and we want to continue cutting the residual space S_{wd}, then $(w_i \bmod d_i) \times (w_i \bmod d_i) \times (w_i \bmod d_i)$ cubes will be produced because $(w_i \bmod d_i) < d_i \le h_i$. Because we assume that $(w_i \bmod d_i) \le H/2 \le W/2$, the volume of a $(w_i \bmod d_i) \times (w_i \bmod d_i) \times (w_i \bmod d_i)$ cube is also less than $B/3$. So the residual space S_{wd} does not need to be decomposed for L^*_B either.

Consider the case $(w_i \bmod d_i) > H/2$ and the residual space S_{wd} shown in Figure 3. Because $d_i > (w_i \bmod d_i) > H/2 \ge h_i/2 \ge d_i/2$, at most one $(w_i \bmod d_i) \times (w_i \bmod d_i) \times (w_i \bmod d_i)$ cube will be produced from the residual space S_{wd} (Figure 3(a)). Similarly, the residual spaces S'_{wd} and S''_{wd} do not need to be decomposed for L^*_B because $h_i - (w_i \bmod d_i) < h_i/2 \le H/2 \le W/2$ and $d_i - (w_i \bmod d_i) < H/2 \le W/2$ respectively (Figure 3(b)).

Note that, for every given item I_i, the num-

[5]Given a positive integer s, the *asymptotic* worst-case performance ratio of a lower bound L is defined as $\limsup_{s\to\infty}\{L(I)/OPT(I), \forall I \text{ with } OPT(I) \ge s\}$ [9].

ber of cubes generated by the above decomposition procedure is at most $\lfloor \frac{w_i}{d_i} \rfloor \lfloor \frac{h_i}{d_i} \rfloor + 1$, which can be computed more simply than that produced by the cutting approach[6] for 2D-BP in [10], where the residual space is cut into squares repeatedly. In contrast, the residual space does not need to be decomposed for our lower bound L_B^* during the decomposition procedure. We remark that the above statement holds even when applying other *maximal* dual feasible functions based on the following property.

Proposition 11 [4] *If f is a maximal dual feasible function (MDFF), then f is superadditive: $\forall x \in [0, C]$, $f(x) \geq \max\{f(y) + f(z) \mid x = y + z,\ y, z \in [0, C]\}$*

Thus, for any instance, we use the above decomposition procedure to replace the given items by cubes. Then, we apply L_B^* to the resulting items to derive a lower bound, denoted by L_{BN}^*, for the non-oriented 3D-BP model. Next, we prove that the derived lower bound is valid.

Lemma 12 *L_{BN}^* is a valid lower bound for the non-oriented 3D-BP model.*

Finally, similar to Boschetti's approach [1], we specify all the possibilities for a given item when its rotation by 90° is allowed. That is, we consider each given item I_i with $v_i = w_i h_i d_i$ in six different orientations and let its rounded volume $v_i'(p, q, r)$ be:

$$\begin{aligned}\min\{&f_2^p(w_i, W) \times f_2^q(h_i, H) \times f_2^r(d_i, D),\\ &f_2^p(w_i, W) \times f_2^q(d_i, H) \times f_2^r(h_i, D),\\ &f_2^p(h_i, W) \times f_2^q(w_i, H) \times f_2^r(d_i, D),\\ &f_2^p(h_i, W) \times f_2^q(d_i, H) \times f_2^r(w_i, D),\\ &f_2^p(d_i, W) \times f_2^q(h_i, H) \times f_2^r(w_i, D),\\ &f_2^p(d_i, W) \times f_2^q(w_i, H) \times f_2^r(h_i, D)\},\end{aligned}$$

where $f_2^z(x, Z) : [0, Z] \rightarrow [0, 2\lfloor Z/z \rfloor]$ is the dual feasible function f_2^p described in Appendix B. Next, let the size of a bin $B = \lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor$ in a similar manner. Then, the lower bound can be computed as $L_B''(p, q, r)$, as described in Section 3. Note that the lower bound is valid for the non-oriented 3D-BP model because the minimum volume between the six cases is selected for each item [1]. We also apply L_2 to this lower bound, denoted by L_{DFN}^*.

Lemma 13 *L_{DFN}^* is a valid lower bound for the non-oriented 3D-BP model.*

[6] Dell'Amico *et al.* [10] showed that the number of square items produced by their method is pseudo-polynomial in the worst-case. However, the number of items produced is reasonably small in practical applications.

Boschetti [1] proposed the current best lower bound $L_B^M = \max\{L_1^M,\ L_2^M\}$ for the non-oriented 3D-BP model, where $L_1^M = \max\{L_1, L_{BN}(p, q, r)\}$ and $L_2^M = L_{BN}''(p, q, r)$. In [1], compared with $L_B(p, q, r)$, the volume $v_i(p, q, r)$ of each given item I_i is rounded from $d_i \times d_i \times d_i$ instead of $v_i = w_i h_i d_i$ when computing $L_{BN}(p, q, r)$ for the non-oriented model. Thus, our lower bound L_{BN}^* obviously dominates $L_{BN}(p, q, r)$. In addition, $L_{BN}''(p, q, r)$ is obtained for the non-oriented model by computing $L_B''(p, q, r)$ while considering the above six orientations. Hence, because $L_B''(p, q, r) \leq L_{DF}^*$, L_{DFN}^* dominates $L_{BN}''(p, q, r)$. The next theorem follows immediately.

Theorem 14 $L_B^M \leq L_N^* = \max\{L_2,\ L_{BN}^*,\ L_{DFN}^*\}$.

6 Concluding remarks

We have considered the 3D-BP problem and proposed two new lower bounds $L_{B,2}$ and $L_{B,DF}^*$. In addition, we have demonstrated that the lower bounds improve the best previous results, and that $L_{B,DF}^*$ dominates all the other lower bounds for 3D-BP proposed in the literature. We have also proved the worst-case performance ratios and provided a strictly better example. To the best of our knowledge, this is the first study to present better worst-case performance ratios than that of the continuous lower bound for the 3D-BP problem. Finally, we have presented the best lower bound L_N^* for the non-oriented 3D-BP model.

In our future research, we will explore the relationship between the lower bounds and the approximability of 3D-BP. A better approximation algorithm for 3D-BP based on the key concept of our lower bounds would be of interest.

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Appendix A

The lower bound L_1 of Martello and Toth [23]. The lower bound L_1 for 1D-BP is computed as follows:

$L_1 = \mid V(B/2, B] \mid + \max_{1 \le p \le B/2}\{0, L_1(p), L_1'(p)\}$, where

$$L_1(p) = \left\lceil \frac{\sum_{v_i \in V[p, B-p]} v_i}{B} - \mid V(B/2, B-p] \mid \right\rceil \text{ and}$$

$$L_1'(p) = \left\lceil \frac{\mid V[p, B/2] \mid - \sum_{v_i \in V(B/2, B-p]} \left\lfloor \frac{B - v_i}{p} \right\rfloor}{\left\lfloor \frac{B}{p} \right\rfloor} \right\rceil$$

The key concept of the lower bound L_1 was explained earlier, and $|V(B/2, B]| + \max_{1 \le p \le B/2}\{0, L_1(p)\}$ is specified more precisely in the above formula. The rounding technique $L_1'(p)$, where $1 \le p \le B/2$, also plays an important role. However, Carlier *et al.* [3] proved that the dual feasible function f_2^p, where $1 \le p \le B/2$, dominates this rounding scheme; therefore, later in the paper, we will apply f_2^p in some cases of our new lower bounds to improve the results reported in the literature. We introduce the dual feasible functions below. In addition, Martello *et al.* [20, 21] extended the lower bound L_1 for 1D-BP to the lower bounds of the multi-dimensional models (2D-BP and 3D-BP).

Dual feasible functions. A function $f : [0, 1] \to [0, 1]$ is called *dual feasible* if, for any finite set S of non-negative real numbers, the following condition holds.

$$\sum_{x \in S} x \le 1 \Rightarrow \sum_{x \in S} f(x) \le 1$$

The concept of dual feasible functions was first presented by Johnson [14] and subsequently extended by Lueker [18]. Dual feasible functions have been widely studied in the design and analysis of lower bounds for the bin packing problem and its variations. For more detailed information on a variety of dual feasible functions, readers may refer to Clautiaux *et al.*'s survey [5].

Fekete and Schepers [11] proposed using the concept of dual feasible functions to derive the properties of the lower bounds of the bin packing problem as follows.

Proposition 15 [11] *Given a dual feasible function f and an instance $I = \{v_1, v_2, \ldots, v_n\}$ of the bin packing problem, a lower bound for the instance $f(I) = \{f(v_1), f(v_2), \ldots, f(v_n)\}$ is also a lower bound for the instance I.*

In this paper, consider two dual feasible functions. The first is the classic dual feasible function for 1D-BP, $f_0^p : [0, B] \to [0, B]$, which is defined as follows [3, 11]:

$$f_0^p(x) = \begin{cases} B, & \text{if } x > B - p; \\ x, & \text{if } B - p \ge x \ge p; \\ 0, & \text{otherwise.} \end{cases}$$

where $1 \le p \le B/2$.

The second is the dual feasible function for 1D-BP, f_2^p, proposed by Carlier *et al.* [3]. For $1 \le p \le B/2$, $f_2^p : [0, B] \to [0, 2\lfloor B/p \rfloor]$ is defined as follows:

$$f_2^p(x) = \begin{cases} 2\left(\left\lfloor \frac{B}{p} \right\rfloor - \left\lfloor \frac{B-x}{p} \right\rfloor\right), & \text{if } x > B/2; \\ \left\lfloor \frac{B}{p} \right\rfloor, & \text{if } x = B/2; \\ 2\left\lfloor \frac{x}{p} \right\rfloor, & \text{otherwise.} \end{cases}$$

Clautiaux *et al.* [7] proved that the above functions f_0^p and f_2^p are *maximal dual feasible functions* (MDFFs), because there are no dual feasible functions larger than them [4, 7].

Moreover, by definition, composition and convex combinations of any dual feasible functions are still dual feasible. Thus, the concept of dual feasible functions can be extended to the lower bounds of the multi-dimensional bin packing problems [1, 3, 4, 7, 11].

Appendix B

Theorem 9 *For any instance I ,*

$$R(L_B^*) \le \begin{cases} \frac{3}{19}, & \text{if } L_B^* = L_B^*(1,2); \\ \frac{1}{8}, & \text{otherwise.} \end{cases}$$

Proof. Given an instance I, if $L_B^* > L_B^*(1,2)$, the first $L_B^*(1,2)$ bins that are already open are completely full. Thus, it is obvious that $L_0 \le L_B^* \le L_0 + 2$ because the rest of the items can be split. The ratio $R(L_B^*)$ asymptotically tends to $R(L_0) = \frac{1}{8}$.

Suppose $L_B^* = L_B^*(1,2)$. First, we have $OPT(I) \le OPT(V(B/3,B]) + OPT(V(0,B/3])$ for any instance I. Consider the packing process of $V(B/3,B]$ for L_B^*, i.e., the operations before Step 4. Let the numbers of open bins containing exactly one and two items that are larger than $B/3$ be x and y respectively; that is, $L_B^* = L_B^*(1,2) = x+y$. We claim that $OPT(V(B/3,B]) \le x+y+\lceil \frac{y}{2} \rceil +2$. In the worst case, at most y split items need to be removed from the $x+y$ open bins, and a feasible solution can be obtained by creating new bins for them. Since the y items are in $I(W/3,H/2,D/2) \cup I(W/2,H/3,D/2) \cup I(W/2,H/2,D/3)$, the items in each subset can be paired so that at most $\lceil \frac{y}{2} \rceil + 2$ supplementary bins will be needed.

Consider the items in $V(0,B/3]$ for L_B^*. For those items, there is at most $2B/3$ residual space and at most $B/3$ residual space in each of the x and y open bins respectively. According to Martello *et al.* [20], $OPT(V(0,B/3]) \le 8\left\lceil \frac{x\times 2B/3 + y \times B/3}{B} \right\rceil = 8\left\lceil \frac{2x+y}{3} \right\rceil$. Thus, the worst-case performance ratio can be computed as follows:

$$\begin{aligned} OPT(I) &\le OPT(V(B/3,B]) + OPT(V(0,B/3]) \\ &\le (x+y+\left\lceil \frac{y}{2} \right\rceil + 2) + 8\left\lceil \frac{2x+y}{3} \right\rceil \\ &< \frac{19}{3}x + \frac{25}{6}y + 11 \\ &< \frac{19}{3}(x+y) + 11 \\ &= \frac{19}{3}L_B^* + 11 \end{aligned}$$

Hence, when the instance size is sufficiently large and $L_B^* = L_B^*(1,2)$, we have $L_B^*/OPT(I) \ge \frac{3}{19} - \frac{33}{19OPT(I)}$, which asymptotically tends to $\frac{3}{19}$. □

Theorem 10 *For any instance I,*

$$R(L_{DF}^*) \le \begin{cases} \frac{1}{5}, & \text{if } L_2(p) \le 0; \\ \frac{1}{8}, & \text{otherwise.} \end{cases}$$

Here, $L_2(p)$ represents the difference between the continuous lower bound and the number of bins that are open before allowing the items in $V(0,B/3]$ to be split when applying L_2 to L_{DF}^*.

Proof. Given an instance I, if $L_2(p) > 0$, which means that the bins already open are completely full before allowing the items in $V(0,B/3]$ to be split, then L_{DF}^* asymptotically tends to L_0. Thus, the ratio $R(L_{DF}^*)$ asymptotically meets $R(L_0) = \frac{1}{8}$.

Suppose $L_2(p) \le 0$. Similar to the proof of Theorem 9, we first consider the packing process of $V(B/3,B]$. Let the numbers of open bins containing exactly one and two items that are larger than $B/3$ be x and y respectively, i.e., $L_{DF}^* = x+y$. We have $OPT(V(B/3,B]) \le x+2y$ because at most y split items need to be removed from the $x+y$ open bins in the worst case, and a feasible solution can be obtained by creating y supplementary bins for them.

Consider the items in $V(0,B/3]$ for L_{DF}^*. For those items, there is at most $B/2$ residual space and at most $B/3$ residual space in each of the x and y open bins respectively. We have $OPT(V(0,B/3]) \le 8\left\lceil \frac{x\times B/2 + y\times B/3}{B} \right\rceil = 8\left\lceil \frac{3x+2y}{6} \right\rceil$ [20], and the worst-case performance ratio can be computed as follows:

$$\begin{aligned} OPT(I) &\le OPT(V(B/3,B]) + OPT(V(0,B/3]) \\ &\le (x+2y) + 8\left\lceil \frac{3x+2y}{6} \right\rceil \\ &< 5x + \frac{14}{3}y + 8 \\ &< 5(x+y) + 8 \ = \ 5L_{DF}^* + 8 \end{aligned}$$

When the instance size is sufficiently large and $L_2(p) \le 0$, $L_{DF}^*/OPT(I) \ge \frac{1}{5} - \frac{8}{5OPT(I)}$, which asymptotically tends to $\frac{1}{5}$. □

Lemma 12 *L_{BN}^* is a valid lower bound for the non-oriented 3D-BP model.*

Proof. For each item I_i with $v_i = w_i h_i d_i$, the largest cube decomposed from I_i is a $d_i \times d_i \times d_i$ cube. If $d_i > W/2 \ge H/2 \ge D/2$ and $d_i^3 > B/3$, then at most one $d_i \times d_i \times d_i$ cube will be produced from I_i and assigned to a separate open bin. Item I_i (with $w_i \ge h_i \ge d_i$ and $v_i \ge d_i^3 > B/3$) was originally assigned to a separate open bin irrespective of whether it had been rotated. Furthermore, if some subset contains a $d_i \times d_i \times d_i$ cube (without loss of generality, say $I(W/3,H/2,D/2)$), then the corresponding item I_i had $w_i \geq h_i \geq d_i > \max\{W/3,\ H/2,\ D/2\}$ originally. That is, each dimension of item I_i was not less than all the constraints on the subset originally.

In addition, when computing the lower bound L^*_{BN}, if the resulting cubes cannot be matched, their corresponding items could not be matched originally, irrespective of whether they were rotated. Thus, in Step 2, at most one cube can fit in any open bin. Moreover, any two cubes from different K^{HD}, K^{WD}, and K^{WH} cannot be matched in the same bin; and at most two cubes from each subset can be paired in Step 3. The operations in Steps 4 and 5 still hold as well. The above procedure implies that all the steps could have worked originally if rotation of the corresponding items had been allowed. Finally, the rest of the items can also be split in a similar manner. On the other hand, f^p_0 can be applied because an item I_i could have been rounded originally (irrespective of whether it was rotated) if its resulting $d_i \times d_i \times d_i$ cube could be rounded. Hence, L^*_{BN} is a valid lower bound. □

Appendix C

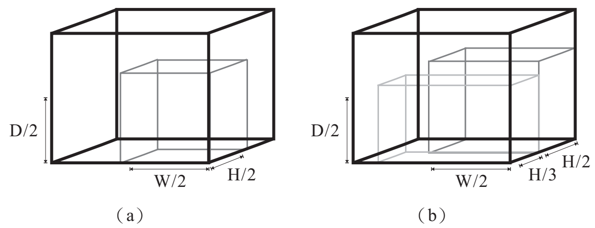

Figure 1: (a) An illustration of Step 1 (b) An example that illustrates the process of Step 3

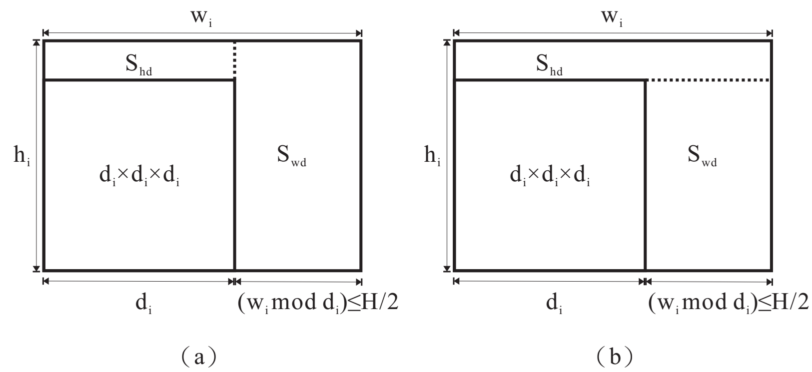

Figure 2: Consider the cases (a) and (b) from the perspective of the (width,height)-face. The residual spaces S_{hd} and S_{wd} do not need to be decomposed in these cases.

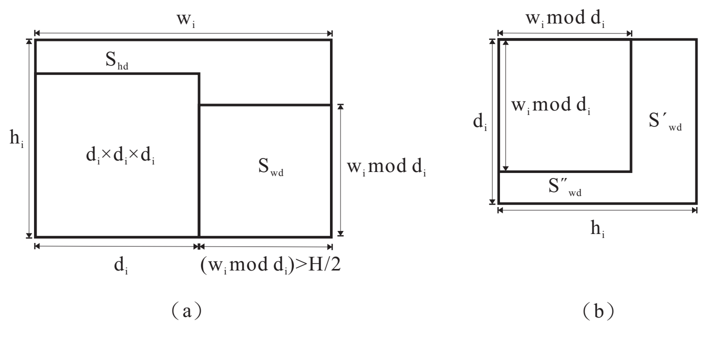

Figure 3: An illustration of the case $(w_i \bmod d_i) > H/2$