Rainbow connection number in triangular pyramids

Fu–Hsing Wang and Guo–Shi Sung
Department of Information Management
Chinese Culture University
wang.fuhsing@gmail.com

Abstract

Rainbow connection number of a connected graph G is the minimum number of colours needed to colour the edges of G, so that every pair of nodes is connected by at least one path whose edges have distinct colours. In this paper, we find the rainbow connection number for a triangular pyramid.

1 Introduction

Edge colouring of a graph is a function from its edge set to the set of natural numbers. A path P in an edge coloured graph with no two edges sharing the same colour is called a *rainbow path*. If P is coloured by using r colours, then P is also a *rainbow r path*. An edge-coloured graph G is *rainbow connected* if any two nodes are connected by a rainbow path. This concept of rainbow connection in graphs was introduced by Chartrand et al. in [5]. The rainbow connection number of a connected graph G, denoted by $rc(G)$, is the smallest number of colours that are needed in order to make G rainbow connected.

The problem of rainbow connection has application in secure transfer of classified information between various agencies [6]. Chakraborty et al. showed that computing the rainbow connection number of a general graph is NP-hard [4]. In fact, even deciding whether $rc(G) = 2$ holds for a graph G is an NP-complete problem [4].

Most recent research has been devoted to solving the problem for certain special graphs, e.g., trees, complete graphs and complete multi-partite graphs [5]. There are some approaches to study the bounds of the rainbow connection numbers of 3-connected graphs [8], connected bridgeless graphs [2], strongly regular graphs [1], etc. A good survey of rainbow connections can be found in [9].

We consider the problem for a particular interconnection network, namely, triangular pyramids which is based on triangular mesh instead of the traditional two-dimensional mesh employed by traditional pyramids. The triangular pyramid features include the fault-tolerate properties such as fault diameter, w-wide diameter [10]. Pyramid network preserves many desirable properties of traditional pyramid networks. Furthermore, the network can be exploited simulate other network topologies more efficiently compared to its equivalent pyramid network [10]. In this paper, we propose a linear time algorithm for finding a rainbow path for any two nodes of the pyramid network TP_n. As far as we know, no rainbow path algorithm exists for triangular pyramids.

The remaining part of this paper is organized as follows. In Section 2, we give the definition of triangular pyramid networks and introduce some basic terminology and notation. In Section 3, a result shows that the n-layer triangular pyramid TP_n has a rainbow n edge colouring and presents an algorithm for determining a rainbow path for any two nodes of a TP_n. Finally, some concluding remarks and future research are given in the last section.

2 Preliminaries

All graphs considered in this paper are simple, finite and undirected. For terms of graphs not defined here please refer to the notation and terminology of Bondy and Murty [3]. The *radix–n triangular mesh network*, denoted by T_n, has the node set $V(T_n) = \{(x, y) \mid 0 \le x, y < n \text{ and } 0 \le x + y < n\}$ and there is a *mesh edge* between nodes (x_1, y_1) and (x_2, y_2) if $|x_1 - x_2| + |y_1 - y_2| < n - 1$ and $x_1 + y_1 \le x_2 + y_2$. A set of nodes with sum of coordinates equals r is called *row r*. A higher radix triangular mesh is formed by adding rows to a lower radix mesh. For example, Figure 1 depicts an example of a radix-5 triangular mesh T_5. Row 4 is the set of nodes $(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)$. The dash area, the subgraph induced by the rows 0, 1, 2 and 3, establishes a T_4.

The *n-layered triangular pyramid*, denoted by TP_n, has the node set $V(TP_n) = V_0 \cup V_2 \cup \cdots \cup V_{n-1}$, where

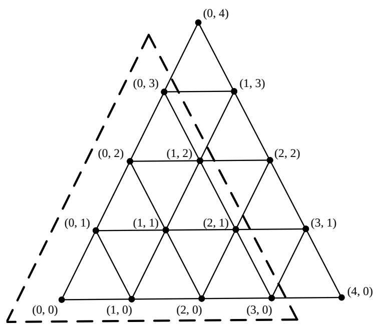

Figure 1: The T_5.

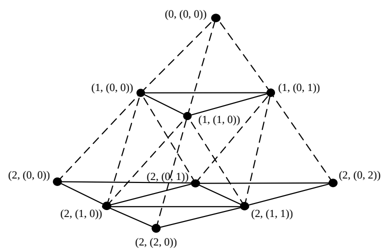

Figure 2: The TP_3.

$V_k = \{(k,(x,y)) \mid 0 \le x+y \le k\}$ is the set of nodes on layer k and $0 \le k \le n-1$. Each V_k is connected as a T_k and a mesh edge of T_k is also called a *triangular mesh edge* in TP_n. We use $(k; x_1, y_1, x_2, y_2)$ to denote a triangular mesh edge incident on nodes $(k,(x_1,y_1))$ and $(k,(x_2,y_2))$ if $|x_1-x_2|+|y_1-y_2| < n-1$ and $x_1+y_1 \le x_2+y_2$. Each node $(k,(x,y)) \in V_k$ is adjacent to exactly three nodes $(k+1,(x,y)), (k+1,(x+1,y)), (k+1,(x,y+1))$ of V_{k+1} where $0 \le k \le n-2$ and the edges connecting them of different layers are said to be *layer edges*. Figure 2 illustrates an example of a 3–layered triangular pyramid TP_3. The dash lines indicate layer edges, while the solid lines are mesh edges. The nodes $(1,(0,0))$ and $(1,(1,0))$ are connected by the triangular mesh edge $(1; 0,0,1,0)$. The node $(1,(0,0))$ has layer edges connecting to nodes $(2,(0,0))$, $(2,(1,0))$ and $(2,(0,1))$.

3 A Rainbow Colouring on Triangular Pyramid Networks

We define an edge colouring function to assign colours on the edges of a triangular mesh network.

Definition 1. *Let χ be an edge colouring on T_n and $e = (x_1,y_1,x_2,y_2)$ a triangular mesh edge of T_n, where $1 \le x_1, y_1, x_2, y_2 < n, 0 \le x_1+y_1 < n$ and $0 \le x_2+y_2 < n$.*

$$(i)\ \chi(e) = \begin{cases} y_1 & \text{if } x_1+y_1 = x_2+y_2, \\ x_2+y_2 & \text{if } x_1+y_1 < x_2+y_2. \end{cases}$$

Theorem 2. *Let T_n be a radix–n triangular mesh. T_n is rainbow connected under the edge colouring χ.*

Proof. We shall prove this theorem by induction on the number of radix n. A radix-2 triangular mesh T_2 has mesh edges $(0,0,0,1)$, $(0,0,1,0)$ and $(0,1,1,0)$. And the three mesh edges, by definition of χ, are all coloured 1. Since T_2 is indeed a 3-cycle, T_2 is clearly rainbow connected as the basis of induction. For the inductive hypothesis, assuming that the theorem is true for T_n, i.e., any two nodes are connected by a rainbow path using $n-1$ colours under the edge colouring χ. We now show that the theorem is also true for a T_{n+1}. Let $s,t \in V(T_{n+1})$. Note that a T_{n+1} has $n+1$ rows. Three cases are considered depending on positions of s and t.

Case 1. Assume that s and t are in rows $0,1,\ldots,$ or $n-1$.
The case follows from the inductive hypothesis.

Case 2. Assume that s is in row $0,1,\ldots,$ or $n-1$ and t is in row n.
Let w be a neighbor of t, where w is in row $n-1$. By induction hypothesis, there is an $s-w$ rainbow path and each edge of the path is assigned a distinct colour range form 0 to $n-2$. By the definition of χ, the mesh edge incident on the nodes w and t is assigned the colour $n-1$ (the sum of coordinates of w) since the sum of coordinates of w is less then the sum of coordinates of t. Then, the $s-t$ path, concatenated by $s-w$ path and the mesh edge between nodes w and t, is a rainbow path.

Case 3. Assume that s and t are in row n.
The subgraph induced by the row n is the path π: $(0,n), (1,n-1), \ldots, (n,0)$. By the definition of χ, the mesh edge $(0,n,1,n-1)$ is assigned the colour n (y-coordinate of the first node) since the sum of coordinates of both end nodes of the mesh edge are equal. By the same reason, the edges of π are coloured by consecutive colours from n to 1. Therefore, the $s-t$ path must be a rainbow path.

Q. E. D.

Since the diameter of a T_n is n, the next result follows immediately from Theorem 2.

Corollary 3. *If T_n is a radix–n triangular mesh, then $rc(T_n) = n$.*

We now show that a triangular pyramid TP_n has a rainbow n colouring. Note that the edge set of a triangular pyramid is classified into triangular mesh edges and layer edges. We shall colour mesh edges by using the colouring, mentioned above, which applied to colour the mesh edges of triangular meshes. So, we shall colour layer k of TP_n, where $1 \leq k \leq n$, by using colours range from 1 to k. On the other hand, the layer edges incident on the nodes of the same layer are assigned the same colour. The formal definition of the edge colouring χ in TP_n is as follows:

Definition 4. *Let χ_p be an edge colouring on TP_n, $e = (k; x_1, y_1, x_2, y_2)$ a triangular mesh edge on layer k and f a layer edge incidents on a node in V_{k-1} and a node in V_k, where $2 \leq k \leq n, 1 \leq x_1, y_1, x_2, y_2 < n, 0 \leq x_1 + y_1 < n$ and $0 \leq x_2 + y_2 < n$.*

(i) $\chi_p(e) = \begin{cases} y_1 & \text{if } x_1 + y_1 = x_2 + y_2, \\ x_2 + y_2 & \text{if } x_1 + y_1 < x_2 + y_2. \end{cases}$

(iii) $\chi_p(f) = k$.

Theorem 5. *If TP_n is an $n-$layered triangular pyramid, then $rc(TP_n) = n$.*

Proof. Let s and t be two nodes of TP_n. If one of the two nodes is the apex of TP_n, then we get a shortest $s - t$ path P, consisting of only layer edges, of length less than n. Since the layer edges in distinct layers are assigned different colours, P is a rainbow path under the edge colouring χ_p. Now, we suppose neither s nor t is the apex. Let $s = (k_1, x_1, y_1)$ and $t = (k_2, x_2, y_2)$. Without loss of generality, assuming $k_1 \leq k_2$. If $k_1 = k_2$, then s and t are on the same layer k_1 of TP_n. Since the layer k_1 connected as a T_{k_1}, s and t has a rainbow r path under the edge colouring χ_p where $r \leq k_1$. When $k_1 < k_2$. Let w be a node on layer k_1 such that $w - t$ path is of length $k_2 - k_1$. It can be seen that the edges along with the $w - t$ path, say $e_1, e_2, \ldots, e_{k_2-k_1}$, are layer edges and thus were assigned colour

The shortest $s-t$ path consisting of only layer edges, of length less than n. Since the layer edges in distinct layers are assigned different colours, P is a rainbow path under the edge colouring χ_p.

Q. E. D.

4 Concluding Remarks

For algorithmic purposes, an efficient algorithm is demonstrated for determining a rainbow path for any two nodes of a triangular pyramid. The algorithm considers balancing characteristic of a pyramid and avoids a possible bottleneck in the apex. When the network contains multiple pairs of source and destination nodes between which messages are transmitted along their path, the algorithm is applicable.

References

[1] A. Ahadi, A. Dehghan, On rainbow connection of strongly regular graphs, Arxiv preprint arXiv:1001.3413v1 [math.CO] (2010).

[2] M. Basavaraju, L.S. Chandran, D. Rajendraprasad, A. Ramaswamy, Rainbow connection number and radius, Arxiv preprint arXiv:1011.0620v1 [math.CO] (2010).

[3] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, Springer (2008).

[4] S. Chakraborty, E. Fischer, A. Matsliah, R. Yuster, Hardness and algorithms for rainbow connection, Journal of Combinatorial Optimization 21(3) (2011) 330–347.

[5] G. Chartrand, G.L Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, Math. Bohemica. 133(1) (2008), 85–98.

[6] A. Ericksen, A matter of security, Graduating Engineer & Computer Careers (2007), 24–28.

[7] H.J. Hsieh, D.R. Duh, ω-wide diameters of enhanced pyramid networks, Theoretical Computer Science 412(29) (2011) 3658–3675.

[8] X. Li, Y. Shi, Rainbow connection in 3-connected graphs, Arxiv preprint arXiv:1010.6131v1 [math.CO] (2010).

[9] X. Li, Y. Sun, Rainbow connection of graphs– A survay, manuscript (2011) arXiv:1101.5747v1 [math.CO].

[10] S. Razavi, H. Sarbazi-Azad, The triangular pyramid: Routing and topological properties, Inforation Sciences, 180 (2010) 2328–2339.

[11] Q.F. Stout, Pyramid computer solutions of the closest pair problem, Journal of Algorithms 6 (1985) 200–212.