Efficient Algorithms for the Flexible Longest Common Subsequence Problem

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Abstract

Given two sequences, the traditional longest common subsequence (LCS) problem is to obtain the common subsequence with the maximum number of matches, without considering the continuity of the matched characters. However, in many applications, the matching results with higher continuity are more meaningful than the sparse ones, even if the number of matched characters is a little lower. Accordingly, we define a new variant of the LCS problem, called the flexible longest common subsequence (FLCS) problem. In this paper, we design a scoring function to estimate the continuity of a matching result between two strings. We show that the optimal solution of FLCS can be determined in $O(n^2)$ time, where $n$ denotes the longer length of the two input sequences. Therefore, the results in this paper offer a new efficient tool for sequence analysis.

1 Introduction

The topic of evaluating how similar one string or one sequence to another is, such as the longest common subsequence [3, 4, 7–10, 15, 17] or the edit distance [2, 14], has been studied for several decades. Based on these algorithms, methods for solving verification [6], classification [1, 11], and clustering [13] problems are further developed.

Given two sequences $X$ and $Y$, the longest common subsequence (LCS) problem asks for the longest sequence that is a subsequence of both $X$ and $Y$. Meanwhile, the edit distance problem is to find the minimum number of operations (to insert, delete or substitute one character each time) required for modifying one string to another. However, to our knowledge, the continuity of matched characters has not yet been emphasized in previous methods. In Figure 1, there are two alignment results $Z_1$ and $Z_2$ of string $X = \text{applebreadcookieorangepeach}$ and string $Y = \text{orangemilkmeloncakebanana}$, where $Z_1$ has more matched characters (which is 9) than $Z_2$ (which is 8) does. However, $Z_1$ does not seem to be quite meaningful because of its scattered matches.

In bioinformatics, longer continuous matches in amino acid sequences mean that these sequence segments are likely to have the same function. For plagiarism detection, long and continuous matches infer that many sentences were copied from one article to the other. Therefore, it is interesting and necessary to devise a comparison scheme that considers the continuity of matches. For this purpose, we propose the flexible longest common subsequence (FLCS) problem.

The organization of this paper is as follows. In Section 2, we give a formal definition to the FLCS problem, and propose a straightforward algorithm with $O(n^2)$ time. In Section 3, we further propose improved $O(n^2 \times \min\{r \log r, n\})$-time, $O(n^2 r)$-time, and $O(n^2)$-time algorithms, where $r$ denotes the size of the dominant list in our algorithm, which is usually much less than $n$. Finally, the conclusions are given in Section 4.
2 The Flexible Longest Common Subsequence Problem

2.1 Problem Definition

For a given sequence of elements \( S = s_1s_2s_3\ldots s_{|S|} \), where \( |S| \) represents the length of \( S \), we use \( s_i \) to denote the \( i \)th element of \( S \). Also, let \( S_{i,j} = s_is_{i+1}s_{i+2}\ldots s_j \) denote a substring from the \( i \)th element to the \( j \)th element of \( S \). For two given sequences \( X \) and \( Y \), a pair of two characters from \( X \) and \( Y \) is defined as follows.

**Definition 1.** A pair \((i,j)\) is called a matched pair if \( x_i = y_j \). Otherwise, \((i,j)\) is a mismatched pair. A character \( x_i \) or \( y_j \) can form a mismatched pair with a gap, represented as \((i,\infty)\) or \((\infty,j)\), where \( x_\infty \) and \( y_\infty \) denote symbols that do not exist in \( X \) and \( Y \).

Thus, an alignment result (such as \( Z_1 \) in Figure 1) consists of a sequence \( P \) of pairs. However, for ease of reading, the alignment results are represented in characters, rather than pairs of indices.

**Definition 2.** Given an alignment result \( P = p_1p_2p_3\ldots p_{|P|} \), a matched segment of \( P \) is a substring \( p_ip_{i+1}p_{i+2}\ldots p_j \) of \( P \) that contains only matched pairs. Similarly, a mismatched segment of \( P \) is a substring of \( P \) that contains only mismatched pairs.

Suppose there are \( |Z| \) segments \( z_1z_2z_3\ldots z_{|Z|} \) in \( P \). Let \( Z_M \) be the set of matched segments. To emphasize continuous matches, each matched segment \( z_i \) in \( P \) is measured by a polynomial function \( f(x) = x^\gamma \) referring to the length of \( z_i \), where \( \gamma \) is a constant. That is, the score of the whole alignment is the summation of the scores of all matched segments, denoted as \( F(P) \), as follows.

\[
F(P) = \sum_{z_i \in Z_M} |z_i|^\gamma.
\]

Figure 2 shows an example for computing the scoring function \( F(P) \).

![Figure 1: Two alignment results with different continuity.](image)

\( X: \text{applebreadcookie} \ldots \text{peach} \)
\( Y: \text{orangemilkmelon} \ldots \text{cakebanana} \)
\( Z_1: \text{apple} \ldots \text{e} \ldots \text{an} \ldots \text{a} \)
\( |Z_1| = 9 \)

\( X: \text{app} \ldots \text{e-bread} \ldots \text{peach} \)
\( Y: \text{orangemilkmelon} \ldots \text{k-ebana} \ldots \text{a} \)
\( Z_2: \text{apple} \ldots \text{e} \ldots \text{ban} \ldots \text{a} \)
\( |Z_2| = 8 \)

\( Y: \text{ab-acb-a} \)
\( F(P) = 2^2 + 3^2 + 1^2 = 14 \)

Figure 2: An example for computing the scoring function \( F(P) \).
the length of the last matched segment ending at $x_i = y_j$. Clearly, the time complexity of the algorithm is $O(n^3)$, where it is assumed that $m \leq n$.

### 3 Improved Algorithms for the FLCS problem

In this section, we first propose an algorithm with the dominant strategy, which solves the FLCS problem in $O(n^2 \times \min\{r \log r, n\})$ time, where $r$ denotes the maximal size of dominant lists. After that, we improve the algorithm by predicting the dominance, which gives an $O(n^2 r)$-time algorithm. Finally, we show that the computation time can be further reduced to $O(n^2)$.

#### 3.1 The Algorithm with the Dominant Strategy

To reduce the time spent on checking the length, the concept of dominant points [5] is applied. If there is always one solution better than another, we say that this one dominates another, and the dominated solutions can be removed from the dominant list. Algorithm 1 is a general implementation for solving the FLCS with dominant lists, whose notations are explained as follows. We use $D_{i,j} = \{e_1, e_2, e_3, \ldots, e_{|D_{i,j}|}\}$ to denote the dominant list that preserves the candidate solutions for $L(X_i, Y_j)$ in lattice cell $(i, j)$. Each candidate solution is represented by a vector $e_k = (s_k, v_k)$, where $s_k$ and $v_k$ record its score and the length of the last matched segment ending at $x_i = y_j$, respectively.

By updating these values, one solution can be extended from one list to the next. When $x_i = y_j$, the function $E_{X_1}(e_k)$ extends each solution $e_k = (s_k, v_k) \in D_{i-1,j} \cup D_{i,j-1}$ to $e' = (s', v') \in D_{i,j}$, as shown in Equation 3.

$$E_{X_1}(e_k) = (s_k - v_k^{e_1} + (v_k + 1)^{e_2}, v_k + 1). \quad (3)$$

On the other hand, when $x_i \neq y_j$, the function $E_{Y_1}$ extends each solution $e_k = (s_k, v_k) \in D_{i-1,j} \cup D_{i,j-1}$ to $e' = (s', v') \in D_{i,j}$, as follows.

$$E_{Y_1}(e_k) = (s_k, 0). \quad (4)$$

For $\gamma \geq 1$, we say that $e_{k_1} = (s_{k_1}, v_{k_1})$ dominates $e_{k_2} = (s_{k_2}, v_{k_2})$ if $s_{k_1} \geq s_{k_2}$ and $v_{k_1} \geq v_{k_2}$. That is, no matter how these two solutions are extended, the score of the extended solution $e'_{k_1} = (s'_{k_1}, v'_{k_1})$ will not be worse than the extended solution $e'_{k_2} = (s'_{k_2}, v'_{k_2})$. Once a solution is dominated by any other solution, it can be removed from the dominant list. In Algorithm 1, the function $\text{DOM}(E)$ removes all dominated solutions in the given list $E$. By applying the algorithm for finding 2D maximal on bounded integer points [15], it takes $O(n)$ time for computing each $D_{i,j}$ because $0 \leq v_k \leq n$. Therefore, the time complexity of this algorithm is bounded by $O(n^3)$, or $O(n^2 r \log r)$, where the maximal size of $D_{i,j}$ is assumed to be bounded by $r$. By our observation, $r$ is usually much less than $n$.

#### 3.2 The Algorithm with Dominance Prediction

Based on Equation 3, we can predict the score $f_k(t)$ of $e_k$ for the next $t$ extensions with $E_{X_1}$, which can be written as Equation 5.

$$f_k(t) = s_k - v_k^{e_1} + (v_k + t)^{e_2}. \quad (5)$$

For two solutions $e_{k_1}$ and $e_{k_2}$, the intersection point $t$ of $f_{k_1}(t)$ and $f_{k_2}(t)$ can be computed by Equation 6. By computing such $t$, which we call the survival time, the timing for the future dominance of $e_{k_1}$ and $e_{k_2}$ can be predicted.

$$f_{k_1}(t) = f_{k_2}(t), \quad s_{k_1} - v_{k_1}^{e_1} + (v_k + t)^{e_2} = s_{k_2} - v_{k_2}^{e_1} + (v_k + t)^{e_2}. \quad (6)$$

For conciseness, in this paper we consider the special case with $\gamma = 2$, whose intersection point can be calculated by the formula shown in Equation 7.

$$t = \frac{s_{k_1} - s_{k_2}}{2(v_k - v_k)}. \quad (7)$$

For other positive integer $\gamma = 3, 4, 5$, one can refer to Galois theory [16]. For other values of $\gamma$, we have to approximate the integer-precise answer by numerical analysis, like the Newton-Raphson...
method [12], which can be achieved in a few iterations for nonnegative integer t.

Note that only the nonnegative t is meaningful. If the obtained t is negative, it means that solutions \( e_{k_1} \) and \( e_{k_2} \) cannot dominate each other in the future.

From hereafter, we include an additional survival time \( t_k \) in the vector of each solution, i.e., \( e_k = (s_k, v_k, t_k) \), where \( t_k \) is a nonnegative integer. The survival time \( t_k \) means that \( e_k \) is alive from time 0 (now) to time \( t_k \), and it will be expired (dominated by some other solutions) on time \( t_k + 1 \), where the current \( D_{i,j} \) is marked as time 0. During the process, we order the dominant list decreasingly by \( s_k \), thus the first solution \( e_1 \in D_{i,j} \), denoted as \( D_{i,j,1} \), is always with the highest score. For \( D_{i,j,1} \), we use \( t_{i,j,1} \) to denote its survival time. The function for computing the survival time for two solutions \( e_{k_1} \) and \( e_{k_2} \) is denoted as \( \text{INTERSECT}(e_{k_1}, e_{k_2}) \), which is assumed to take constant time.

The extending functions with survival time are denoted as \( \text{EXT}_1 \) and \( \text{EXT}_2 \). When \( x_i = y_j \), \( \text{EXT}_1 \) extends one solution \( e_k = (s_k, v_k, t_k) \in D_{i-1,j-1} \) to \( e'_k = (s'_k, v'_k, t'_k) \in D_{i,j} \), shown as follows.

\[
\text{EXT}_1(e_k) = (s_k - v_k^2 + (v_k + 1)^2, v_k + 1, t_k - 1). \quad (8)
\]

Similarly, when \( x_i \neq y_j \), \( \text{EXT}_2 \) is defined as follows.

\[
\text{EXT}_2(e_k) = (s_k, 0, \infty). \quad (9)
\]

The new algorithm for constructing the dominant list \( D_{i,j} \) with survival time is shown in Algorithm 2, and the function \( \text{INSERT}(D_{i,j}, e') \) is described by Algorithm 3.

Here we provide some explanations for Algorithms 2 and 3. To construct the dominant list \( D_{i,j} \), extended solutions are collected from \( D_{i-1,j}, D_{i,j-1} \) and \( D_{i-1,j-1} \). For extended solutions from \( D_{i-1,j} \) or \( D_{i,j-1} \), the solution with the highest

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**Algorithm 1** The dynamic programming algorithm with the dominant concept.

\[
D_{i,0} = D_{0,j} = \text{the list of the initial solution.}
\]

for \( i = 1 \to m \) do

for \( j = 1 \to n \) do

if \( x_i = y_j \) then

\[
E = \text{EX1}(D_{i-1,j-1}) \cup \text{EX2}(D_{i-1,j}) \cup \text{EX2}(D_{i,j-1})
\]

else

\[
E = \text{EX2}(D_{i-1,j}) \cup \text{EX2}(D_{i,j-1})
\]

end if

\[
D_{i,j} = \text{DOM}(E)
\]

end for

end for

---

**Algorithm 2** Construction of \( D_{i,j} \) with predicted survival time.

\[
D_{i,0} = D_{0,j} = \{(0,0,\infty)\}
\]

for \( i = 1 \to m \) do

for \( j = 1 \to n \) do

\[
E = \text{EX1}(D_{i-1,j-1}) \cup \text{EX2}(D_{i,j-1})
\]

\[
e' = \text{DOM}(E)
\]

if \( x_i = y_j \) then

\[
D_{i,j} = \text{EX1}(D_{i-1,j-1})
\]

if \( t_{i,j,1} < 0 \), remove \( D_{i,j,1} \) from \( D_{i,j} \)

\[
\text{INSERT}(D_{i,j}, e')
\]

else

\[
D_{i,j} = \{e'\}
\]

end if

end for

end for

---

**Algorithm 3** Insert \( e' \) into \( D_{i,j} \) and maintain elements in the list.

function \( \text{INSERT}(D_{i,j}, e' = (s', v', t')) \)

if \( s' \leq s_1 \), then discard \( e' \) and return

\[
k \leftarrow 1
\]

\[
t' = \text{INTERSECT}(e', e_1)
\]

while \( t' \geq t_k \) do

remove \( e_k \) from \( D_{i,j} \)

\[
k \leftarrow k + 1
\]

\[
t' = \text{INTERSECT}(e', e_k)
\]

end while

push \( e' \) on top of \( D_{i,j} \)

end function
score would dominate all other solutions, because \( v' = 0 \). That is, what we have to compare is the top solutions in \( D_{i-1,j} \) and \( D_{i,j-1} \), which are \( D_{i-1,j-1} \) and \( D_{i,j-2} \). For \( x_i = y_j \), \( D_{i,j} \) can be constructed by extending \( D_{i-1,j-1} \), removing expired solutions, and inserting the solution \( e' = (s', v', t') \) (extended from \( D_{i-1,j} \) or \( D_{i,j-1} \)).

To insert \( e' = (s', v', t') \) into \( D_{i,j} \), and to maintain the dominant solutions, the function \( \text{INSERT} \) is presented in Algorithm 3. First, to ensure that \( e' \) is not dominated by any other solution in \( D_{i,j} \), since for each \( e_k \in D_{i,j} \) we have \( v_k \geq v' = 0 \), \( s' > s_1 \) should be satisfied. Otherwise \( e' \) is dominated and it must be discarded. Next, if \( t' \) should be updated by intersecting \( e' \) with existing solutions in \( D_{i,j} \). For some solution \( e_k = (s_k, v_k, t_k) \), if \( t' \geq t_k \), then \( e_k \) is dominated by \( e', \) which means \( e_k \) should be removed. Note that even when \( t' = t_k \), \( e_k \) should be removed because \( s' > s_k \). There are up to \( |D_{i,j}| \) intersections to be computed, and finally, \( e' \) should be placed on the top of \( D_{i,j} \).

Figure 3 illustrates an example for \( X = \text{aabbbaaccba} \), \( Y = \text{aabbccba} \), and \( \gamma = 2 \), where the situation \( x_i = y_j \) is specified by grey color. For demonstration, to construct the dominant list of the cell \((5,4)\), solutions from \( D_{4,4} \), \( D_{5,3} \) and \( D_{4,3} \) should be collected. For the solutions from \( D_{4,4} \) and \( D_{5,3} \), the \( e' \) will become 0, the solution with the highest \( s' \) will dominate all the others in \( D_{4,4} \) and \( D_{5,3} \). By comparing the extended score of \((16,4,\infty)\) (the best one in \( D_{4,4} \)) and \((9,0,2)\) (the best one in \( D_{5,3} \)), the solution \( e' = (16,0,\infty) \) (extended from \((16,4,\infty)\)) is retrieved. For the solutions from \( D_{4,3} \), since the \( e' \) will become 0, the solution with the highest \( s' \) will dominate all the others in \( D_{4,3} \). By comparing the extended score of \((10,4,1)\) (the best one in \( D_{4,4} \)) and \((10,1,1)\) (the best one in \( D_{5,3} \)), the solution \( e' = (10,1,1) \) (extended from \((16,4,\infty)\)) is retrieved. For the solutions from \( D_{4,3} \), since the situation has been predicted, the list \( \{ (9,0,2) \} \) is extended as \( \{ (10,1,1), (8,2,\infty) \} \). Since \( t \) values are in increasing order, the first element (with the smallest \( t \) value) in the list should be checked for expiration. The expired solution will be removed. In this example, no solution is expired.

Following Algorithm 3, the detailed steps for inserting \( e' = (16,0,\infty) \) into \( \{ (10,1,1), (8,2,\infty) \} \) are described as follows.

- At first, because \( s' = 16 > s_1 = 10 \), we know that \( e' \) is not dominated by any other solutions in the list, and it should be inserted.
- To update \( t' \) of \( e' \), the intersection of \( e' \) and \( D_{5,4,1} \) is first computed, which is \( t' = 3 \). By \( t' = 3 \geq t_{5,4,1} = 1 \), \( D_{5,4,1} \) is dominated and removed.
- Moving on to the new \( D_{5,4,1} \), the intersection of \( e' \) and \( D_{5,4,1} \) is \( t' = 2 \). Because \( t' = 2 < t_{5,4,1} = \infty \) is satisfied, no more domination will occur, and the loop terminates.

- To keep the list in order, \( e' \) is placed on the top of \( D_{5,4} \).
- Finally, the process of \( D_{5,4} \) finishes. We have \( D_{5,4} = \{ (16,0,2), (8,2,\infty) \} \).

For \( D_{i,j} \) with \( x_i \neq y_j \), the dominant list can be constructed in constant time. However, for the case \( x_i = y_j \), both the extension from \( D_{i-1,j-1} \) and the function \( \text{INSERT} \) require \( O(r) \) time, where \( r \) is the maximal size of the dominant list. Thus, the time complexity of this improved algorithm is \( O(n^2 \gamma) \), which is bounded by \( O(n^3) \).

### 3.3 The \( O(n^2) \)-time Algorithm

After analyzing the time complexity of the algorithm in Section 3.2, the bottlenecks are the task for extending solutions from \( D_{i-1,j-1} \) to \( D_{i,j} \), and the function \( \text{INSERT} \) for inserting \( e' \) into \( D_{i,j} \). In this section, we define a new solution representation, by which the one-by-one extension from \( D_{i-1,j-1} \) to \( D_{i,j} \) can be avoided, and the information required by the algorithm is still retrievable. After that, by amortizing the time required for checking and deleting elements in \( \text{INSERT} \), we show that the optimal FLCS can be determined in \( O(n^2) \) time.

Instead of the representation \( e_k = (s_k, v_k, t_k) \), we use \( \sigma_k \) to represent the score that \( e_k \) has accumulated from the beginning till the last mismatched pair, but without the last matched segment. In other words, \( \sigma_k = \sigma_k + v_k^\gamma \). In addition, \( v_k \) denotes the row index of the lattice cell where the last mismatched pair of \( e_k \) is. Hence, for a solution \( e_k \) in \( D_{i,j} \), we have \( v_k = i = v_k \). Finally, let \( \tau_k \) be the largest row index of the lattice cell where \( e_k \) still survives. That is, we have \( t_k = \tau_k - i \) for each \( e_k \in D_{i,j} \).

These two kinds of representations \( e_k \) and \( \sigma_k \) can be converted to each other in constant time. Clearly, \( e_k = (s_k, v_k, t_k) = (s_k + (i - v_k)^\gamma, i - v_k, \tau_k - i) \), and \( \sigma_k = (\sigma_k, v_k, \tau_k) = (s_k - v_k^\gamma, i - v_k, t_k + i) \). As \( e_k = (0,0,\infty) \) in \( D_{0,j} \) or \( D_{i,0} \) for an initial solution, the corresponding initial solution is now set as \( e_k = (0 - 0, i - 0, \infty + i) = (0, i, \infty) \). Therefore, with this new representation, a solution \( e_k \) can be extended to \( e_k' \) by \( \text{EXTD}_1 \) or \( \text{EXTD}_2 \) with Equation 10.

\[ t' = 2 < t_{5,4,1} = \infty \] is satisfied, no more domination will occur, and the loop terminates.
The computation for \( t_k \) and \( v_k \) is transformed to the row indexes of lattice cells, which need not be updated cell by cell. The score \( \sigma_k \) is updated only when the last matched segment terminates with \( x_i \neq y_j \). Note that \( v_k = i - \nu_k \), where \( i \) is the row index of the cell that \( \epsilon_k \) comes from. Since only one solution is concerned for a mismatch pair, we update variables only when a mismatch occurs. Hence, we have a representation that reduces the one-by-one extension from \( D_{i-1,j-1} \).

We apply the new representation \( \epsilon_k \) based the algorithm in Section 3.2, obtaining a new modified algorithm. By \( \epsilon_k = (s_k, v_k, t_k) = (\sigma_k + (i - \nu_k)^y, i - \nu_k, \tau_k - i) \), we have the following transformation. First, the score predicting function \( f_k(\tau) \) is shown as Equation 11.

\[
\begin{align*}
f_k(t) &= s_k - v_k^y + (v_k + t)^y \\
&= \sigma_k + (i + t - \nu_k)^y \\
f_k(\tau) &= \sigma_k + (\tau - v_k)^y
\end{align*}
\]

Next, the function \( \text{intersect}(\epsilon_{k1}, \epsilon_{k2}) \) returns the integer-precise point \( \tau \) that makes \( f_{k1}(\tau) = f_{k2}(\tau) \). The modified algorithm to construct dominant lists is shown in Algorithm 4, where \( \tau_{i,j-1} \) is the counterpart of \( l_{i,j-1} \), and the modified insertion function is shown in Algorithm 5.

Note that the original bottleneck of extending solutions from \( D_{i-1,j-1} \) to \( D_{i,j} \) is now drastically reduced with the simple operation \( D_{i,j} =

\[
\text{Algorithm 4 Construction of } D_{i,j} \text{ for each lattice cell } (i,j).
\]

\[
D_{i,0} = D_{0,j} = \{(0, i, \infty)\}
\]

for \( i = 1 \rightarrow m \) do

for \( j = 1 \rightarrow n \) do

\( E = \text{EXTD}(D_{i-1,j-1}) \cup \text{EXTD}(D_{i,j-1}) \)

\( \epsilon' = \text{DOM}(E) \)

if \( x_i = y_j \) then

\( D_{i,j} = D_{i-1,j} \)

if \( \tau_{i,j-1} < i \), remove \( D_{i,j} \) from \( D_{i,j} \)

INST\( (D_{i,j}, \epsilon') \)

else

\( D_{i,j} = \{\epsilon'\} \)

end if

end for

end for

\[
\text{Algorithm 5 Insert } \epsilon' \text{ into } D_{i,j} \text{ and maintain elements in the list.}
\]

function \( \text{INST}(D_{i,j}, \epsilon' = (\sigma', v', \tau')) \)

if \( \sigma' \leq \sigma_1 + (i - v_1)^y \), then discard \( \epsilon' \) and return

\( k \leftarrow 1 \)

\( \tau' = \text{INTERSECT}(\epsilon', \epsilon_1) \)

while \( \tau' \geq \tau_k \) do

remove \( \epsilon_k \) from \( D_{i,j} \)

\( k \leftarrow k + 1 \)

\( \tau' = \text{INTERSECT}(\epsilon', \epsilon_k) \)

end while

push \( \epsilon' \) on top of \( D_{i,j} \)

end function

---

\[\text{Figure 3: The 2D lattice of the dynamic programming with the efficient dominant method.}\]
\(D_{i-1,j-1}\). It can be done in constant time by shallow copying from \(D_{i-1,j-1}\) to \(D_{i,j}\). In other words, the address of \(D_{i-1,j-1}\) is copied as the address of \(D_{i,j}\), and only a minor part of data are updated.

Here we again take the same example \(X = aabbcabba, Y = aabbcab\), and \(\gamma = 2\) for comparison. The computation of the previous algorithm is shown in Figure 3, and the computation with the new representation is shown in Figure 4. The dominant lists are constructed row by row as the progressing score of dynamic programming. Since the address of \(D_{i-1,j-1}\) is copied to \(D_{i,j}\), the memory of \(D_{i-1,j-1}\) is reused in \(D_{i,j}\). Thus, only the dominant lists in the current row are recorded, and the lists in previous rows are simply moved and slightly modified. In Figure 4, one can find that lists \(D_{i-1,j-1}\) and \(D_{i,j}\) are almost the same if \(x_i = y_j\).

For demonstration, to construct the dominant list \(D_{5,4}\), solutions from \(D_{4,4}, D_{5,3}\) and \(D_{4,3}\) are collected. For the solutions from \(D_{4,4}\) and \(D_{5,3}\), the solution \(e'\) with the highest score \(\sigma_k + (i - \gamma k)\) will dominate all the others in \(D_{4,4}\) and \(D_{5,3}\). By comparing \((0, 0, \infty)\) (the best one in \(D_{4,4}\) with extended score \(0 + (4 - 5) = 16\) and \((9, 5, 7)\) (the best one in \(D_{5,3}\) with extended score \(9 + (5 - 5) = 9\), the solution \(e' = (16, 5, \infty)\) (extended from \((0, 0, \infty)\) in \(D_{4,4}\) is retrieved. For the solutions from \(D_{4,3}\), the whole list \{(9, 4, 6), (4, 3, \infty)\} is kept for \(D_{5,4}\). However, the first element in the list should be checked for expiration. Because \(\tau_{5,4,1} = 6 > i = 5\), no solution is expired. Following Algorithm 5, the detailed steps for inserting \(e' = (16, 5, \infty)\) into \{(9, 4, 6), (4, 3, \infty)\} are described as follows.

- At first, because \(\sigma' = 16 > \sigma_1 + (i - \nu_1)\) = 10, we know that \(e'\) is not dominated by any other solutions in the list, and it should be inserted.

- To update \(\tau'\) of \(e'\), the intersection of \(e'\) and \(D_{5,4,1} = (9, 4, 6)\) is first computed, which is 8. By \(\tau' = 8 \geq \tau_{5,4,1} = 6\), \(D_{5,4,1}\) is dominated and removed.

- Moving on to the new \(D_{5,4,1} = (4, 3, \infty)\), the intersection of \(e'\) and \(D_{5,4,1}\) is 7. Because \(\tau' = 7 < \tau_{5,4,1} = \infty\) is satisfied, no more domination will occur, and the loop stops.

- To keep the list in order, \(e'\) is placed on the top of \(D_{5,4}\).

- Finally, the maintenance process of \(D_{5,4}\) finishes. We have \(D_{5,4} = \{(16, 5, 7), (4, 3, \infty)\}.

To end this section, we give the analysis of the amortized time complexity. By Algorithm 4, at most one solution is newly generated and inserted into the list of each \(D_{i,j}\). The number of all new solutions is therefore bounded by \(O(n^2)\), which is also the maximal number of times to remove a solution. Though the function \(\text{INST}\) may take linear time for processing some \(D_{i,j}\), the overall time is still bounded by \(O(n^2)\). Note that by applying Hirschberg’s divide-and-conquer strategy [7], the optimal solution can also be obtained in \(O(n^2)\) time, which completes our algorithm.

4 Conclusion

In this paper, we first define the flexible longest common subsequence (FLCS) problem with the concept of matching continuity. To find the optimal solution of FLCS, an \(O(n^2 r \log r)\)-time algorithm is first developed, which is further improved to \(O(n^2 r)\) and \(O(n^2 r)\), where \(n\) denotes the length of the two input sequences, and \(r\) is the size of dominant list. In real applications, the FLCS score is useful for estimating the similarity with emphasized continuity. In the future, we hope to design a more general function that simultaneously considers the penalty of gaps, providing a more flexible tool for sequence analysis.

References


Figure 4: The 2D lattice for the dynamic programming with the new representation.

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