# Some Results on the Incidence Coloring Number of Chordal Rings*

Kung-Fu Ding[1], Kung-Jui Pai[1,+] and Ro-Yu Wu[2]

[1] Department of Industrial Engineering and Management, Ming Chi University of Technology, New Taipei City, Taiwan

[2] Department of Industrial Management, Lunghwa University of Technology, Taoyuan, Taiwan

## Abstract

*In a graph G, an incidence is a pair (u, e) which u is a vertex of G and e is an edge of G incident with vertex u. Two incidences (v, e) and (u, f) are adjacent if v = u, or e = f, or the edge vu = e or f. The incidence coloring number of G, denoted as $\chi_i(G)$, is the smallest k for which there exists a mapping from the set of incidences of G to a set of k colors that adjacent incidences are assigned distinct colors. In this paper, we study the incidence coloring on chordal rings, abbreviated as CR(n, d), and provide the following results: (i) $\chi_i(CR(N, d)) = 5$, if $N \equiv 0 \pmod 5$ and d = 2 or 3, (ii) $\chi_i(CR(N, 2)) = 6$, if N mod 5 ≠ 0, and (iii) $\chi_i(CR(N, 3)) = 6$, if $N \equiv 2 \pmod 5$.*

*Keywords:* Incidence Coloring; Chordal Rings.

## 1. Introduction

Let $G = (V(G), E(G))$ be a finite, simple and undirected graph, where $V(G)$ and $E(G)$ are vertex and edge sets of $G$, respectively. An incidence of $G$ is a pair $(v, e)$ where $v \in V(G)$ is a vertex and $e \in E(G)$ is an edge incident with $v$. Two incidences $(v, e)$ and $(u, f)$ are adjacent if one of the following holds: (a) $v = u$, (b) $e = f$ or (c) $vu = e$ or $vu = f$. See Figure 1 for the three configurations of adjacent incidences, where a '*' attached on an edge $e$ and was near to a vertex $v$ represents an incidence $(v, e)$.

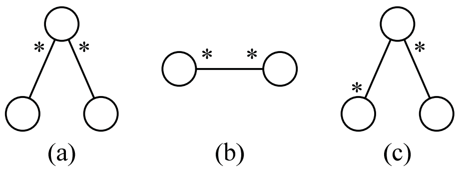


Figure 1: 3 configurations of adjacent incidences.

We denote by $I(G)$ the set consisting of all incidences of a graph G. A proper incidence coloring of $G$ is a mapping from $I(G)$ to a set of $k$ colors such that adjacent incidences are assigned distinct colors. A $k$-incidence coloring of $G$ is a proper incidence coloring such that the size of the color set is $k$. The smallest $k$ required for such a coloring is called the incidence coloring number (or incidence chromatic number) of $G$, and is denoted by $\chi_i(G)$.

The incidence coloring problem was introduced by Brualdi and Massey [3] and they further conjectured the upper bound $\chi_i(G) \le \Delta(G) + 2$ for every graph $G$, abbreviated as IC conjecture, where $\Delta(G)$ is the maximum degree of $G$. Later on, Guiduli [7] gave a counterexample to disprove the IC conjecture. However, some authors showed that the IC conjecture holds for special classes of graphs. Incidence coloring on various classes of graphs has been studied in [4-9, 12-21]. Li and Tu [10] showed that the problem of determining whether a graph $G$ has incidence coloring number at most $k$ or not is NP-complete.

In this paper, we study the incidence coloring on chordal rings, abbreviated as $CR(n, d)$, and provide the following results: (i) $\chi_i(CR(N, d)) = 5$, if $d \in \{2, 3\}$ and $N \equiv 0 \pmod 5$, (ii) $\chi_i(CR(N, 2)) = 6$, if $N$ mod 5 ≠ 0, and (iii) $\chi_i(CR(N, 3)) = 6$, if $N \equiv 2 \pmod 5$.

## 2. Premilirary

Chordal rings (also called distributed loop networks) are a variation of ring networks. By adding two extra links at each vertex in a ring network, the reliability and fault-tolerance of the network are enhanced [2, 3]. A chordal ring $CR(n, d)$ is a graph with vertex set $V(G) = \{v_0, v_1, \dots, v_{N-1}\}$ and edge set $E(G) = \{v_iv_j \mid [i - j]_N = 1 \text{ or } d\}$, where $[x]_y$ denotes $x$ modulo $y$. To ensure that every vertex has four adjacent vertices, we assume $d < N/2$. An example of a $CR(10, 4)$ is shown in Figure 2.

Let $w$ be a vertex with the maximum degree in a graph $G$. By the condition (a) of adjacency, any proper incidence coloring of G needs $\Delta(G)$ colors to assign to $(w, e)$ for all $e \in E(G)$ which is incident with $w$. Also, by conditions (b) and (c) of adjacency, at least one additional color is required to assign to all $(u, e)$ for every $e \in E(G)$ which is incident with $w$.

*This research was partially supported by Ministry of Science and Technology of Taiwan under Grants MOST103-2221-E-131-030 and MOST103-2221-E-262-023.
+Corresponding author: poter@mail.mcut.edu.tw

Thus, $\chi_i(G) \geq \Delta(G) + 1$. Since maximum degree of a chordal ring is four, we have the following lemma.

**Lemma 1.** $\chi_i(CR(N, d)) \geq 5$.

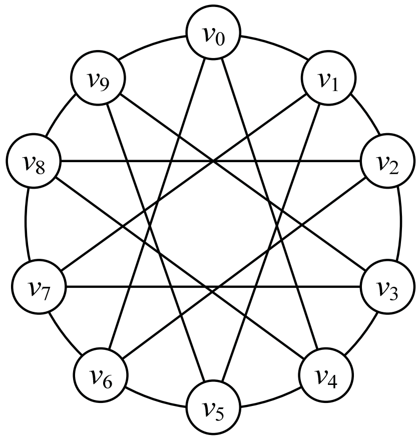



Figure 2: CR(10, 4).

For a vertex $v \in CR(N, d)$, we denote by $I_n(v)$ the set of incidences of the form $(v, vu)$ and by $I_f(v)$ the set of incidence of the form $(u, vu)$. Clearly, both $I_n(v)$ and $I_f(v)$ contain exactly 4 incidences in $CR(N, d)$. Wu [10] gave the following property for regular graphs:

**Lemma 2.** [20] Let $G$ be a regular graph $G$. If $\chi_i(G) = \Delta(G) + 1$, then $|V(G)|$ is a multiple of $\Delta(G) + 1$.

For $CR(N, d)$, it is clear that $|V(G)| = N$, and $N$ is a multiple of $\Delta(G) + 1 = 5$ if and only if $N \equiv 0 \pmod 5$. By Lemma 1 and 2, the following is an immediate corollary.

**Corollary 3.** $\chi_i(CR(N, d)) \geq \begin{cases} 5, & \text{if } N \equiv 0 \pmod 5 \\ 6, & \text{otherwise.} \end{cases}$

### 3. Incidence Coloring on *CR*(*N*, 2)

In this section, we first present an algorithm that gives a 5-incidence coloring of $CR(N, 2)$ when $N \equiv 0 \pmod 5$ and 6-incidence coloring of $CR(N, 2)$ when $N \equiv 1 \pmod 5$.

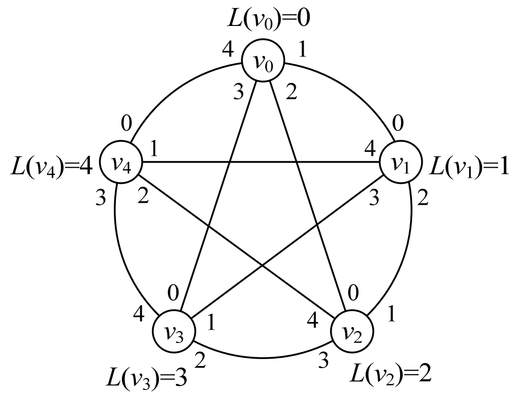
**Procedure A**

Input: A $CR(N, 2)$ and a integer $t$.
Output: A incidence coloring $\psi$ of $I_f(v_i)$ which $i = 0, 1, 2, \ldots, t-1$.
For $i \leftarrow 0$ to $t-1$ do
  $L(v_i) \leftarrow i \bmod 5$;
  For each incidence $j \in I_f(v_i)$ do
    $\psi(j) \leftarrow L(v_i)$ ;
End for

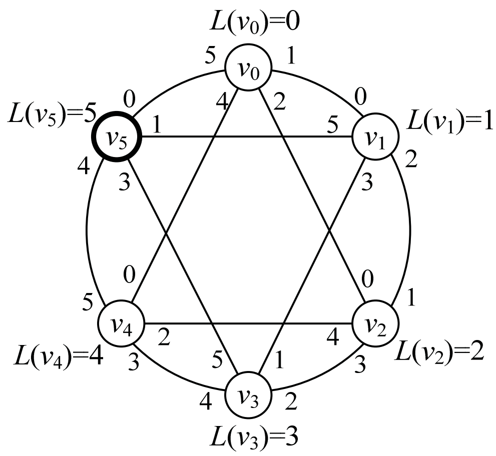
**Algorithm IC2A**

Input: A $CR(N, 2)$ which $N \equiv 0$ or $N \equiv 1 \pmod 5$
Output: A incidence coloring $\psi$ for $CR(N, 2)$.
If $N \bmod 5 = 0$ then
  Call Procedure A with $t = N$;
Else
  Call Procedure A with $t = N - 1$;
  $L(v_{N-1}) \leftarrow 5$;
  For each incidence $j \in I_f(v_{N-1})$ do
    $\psi(j) \leftarrow L(v_{N-1})$ ;
End if

In Algorithm IC2A, we partition $V(CR(N, 2))$ into five (respectively, six) color sets when $N \bmod 5 = 0$ (respectively, $N \bmod 5 = 1$), and $L(v)$ be the membership function to confirm that which color set does the vertex $v$ belong to. All incidences of $I_f(v)$ are assigned color $L(v)$. For example, Figure 3(a) and 3(b) shows a 5 and 6-incidence coloring of $CR(5, 2)$ and $CR(6, 2)$, respectively.




(a) A 5-incidence coloring on $CR(5, 2)$.




(b) A 6-incidence coloring on $CR(6, 2)$.

Figure 2: Incidence colorings on $CR(5, 2)$ and $CR(6, 2)$.

**Lemma 4.** $\chi_i(CR(N, 2)) \leq 5$ for $N \equiv 0 \pmod 5$, and $\chi_i(CR(N, 2)) \leq 6$ for $N \equiv 1 \pmod 5$.

**Proof.** We first show that Algorithm IC2A produces a 5-incidence coloring of $CR(N, 2)$ for $N \equiv 0 \pmod 5$.

For any vertex $v_i$ in $V(CR(N, 2))$, we have $\psi(v_i, v_i v_{i+1}) = (i + 1) \bmod 5$, $\psi(v_i, v_i v_{i+2}) = (i + 2) \bmod 5$, $\psi(v_i, v_i v_{i-1}) = (i - 1) \bmod 5 = (i + 4) \bmod 5$, and $\psi(v_i, v_i v_{i-2}) = (i - 2) \bmod 5 = (i + 3) \bmod 5$. Also, all incidences of $I_f(v_i)$ are assigned the same color ($i \bmod 5$). We can verify that all incidences of $I_n(v_i)$ and $I_f(v_i)$ result in a proper incidence coloring without the three configurations of adjacent incidences.

When $N \equiv 1 \pmod 5$, the proof is similar as the case $N \equiv 0 \pmod 5$. Since the last vertex $v_{N\text{-}1}$ is a neighbor of vertex $v_0$, we assign a new color 5 to all incidences of $I_f(v_{N\text{-}1})$. Again, all incidences of $I_n(v_i)$ and $I_f(v_i)$ result in a proper incidence coloring. Algorithm IC2A produces a 6-incidence coloring of $CR(N, 2)$ for $N \equiv 1 \pmod 5$ □

Next, we gave Algorithm IC2B to provide a 6-incidence coloring of $CR(N, 2)$ for $N \bmod 5 \in \{2, 3, 4\}$. Algorithm IC2B partition all vertices of $CR(N, 2)$ into six color sets by function $L(v)$ except the last vertex $v_{N\text{-}1}$. All incidences of $I_f(v)$ are assigned color $L(v)$. Finally, we assign colors on all incidences in $I_f(v_{N\text{-}1})$ and modified the color on some conflicted incidences, i.e., adjacent incidences with same color.

---

**Algorithm IC2B**

---

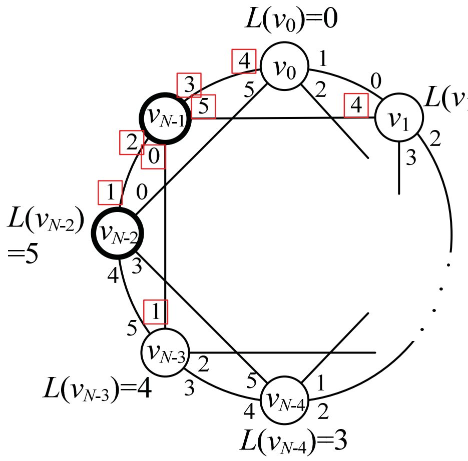
Input: A $CR(N, 2)$ which $N \bmod 5 \in \{2, 3, 4\}$
Output: A 6-incidence coloring $\psi$ for $CR(N, 2)$.
Call Procedure A with $t = N - 2$;
$L(v_{N\text{-}2}) \leftarrow 5$;
For each incidence $j \in I_f(v_{N\text{-}2})$ do
  $\psi(j) \leftarrow L(v_{N\text{-}2})$;
If $N \bmod 5 = 2$ then
  $\psi((v_0, v_0v_{N\text{-}1})) \leftarrow 4$;  $\psi((v_1, v_1v_{N\text{-}1})) \leftarrow 4$;
  $\psi((v_{N\text{-}3}, v_{N\text{-}3}v_{N\text{-}1})) \leftarrow 1$; $\psi((v_{N\text{-}2}, v_{N\text{-}2}v_{N\text{-}1})) \leftarrow 1$;
  $\psi((v_{N\text{-}1}, v_{N\text{-}3}v_{N\text{-}1})) \leftarrow 0$; $\psi((v_{N\text{-}1}, v_{N\text{-}2}v_{N\text{-}1})) \leftarrow 2$;
  $\psi((v_{N\text{-}1}, v_0v_{N\text{-}1})) \leftarrow 3$;  $\psi((v_{N\text{-}1}, v_1v_{N\text{-}1})) \leftarrow 5$;
Else if $N \bmod 5 = 3$ then
  $\psi((v_0, v_0v_{N\text{-}1})) \leftarrow 4$;  $\psi((v_1, v_1v_{N\text{-}1})) \leftarrow 4$;
  $\psi((v_{N\text{-}3}, v_{N\text{-}3}v_{N\text{-}1})) \leftarrow 1$; $\psi((v_{N\text{-}2}, v_{N\text{-}2}v_{N\text{-}1})) \leftarrow 1$;
  $\psi((v_{N\text{-}2}, v_0v_{N\text{-}2})) \leftarrow 3$;  $\psi((v_{N\text{-}1}, v_{N\text{-}3}v_{N\text{-}1})) \leftarrow 2$;
  $\psi((v_{N\text{-}1}, v_0v_{N\text{-}1})) \leftarrow 3$;  $\psi((v_{N\text{-}1}, v_1v_{N\text{-}1})) \leftarrow 5$;
Else  // $N \bmod 5 = 4$
  $\psi((v_0, v_0v_{N\text{-}1})) \leftarrow 4$;  $\psi((v_1, v_1v_{N\text{-}1})) \leftarrow 4$;
  $\psi((v_{N\text{-}3}, v_{N\text{-}3}v_{N\text{-}1})) \leftarrow 3$; $\psi((v_{N\text{-}2}, v_{N\text{-}2}v_{N\text{-}1})) \leftarrow 4$;
  $\psi((v_{N\text{-}2}, v_0v_{N\text{-}2})) \leftarrow 3$;  $\psi((v_{N\text{-}1}, v_{N\text{-}2}v_{N\text{-}1})) \leftarrow 2$;
  $\psi((v_{N\text{-}1}, v_1v_{N\text{-}1})) \leftarrow 5$;
End if

---

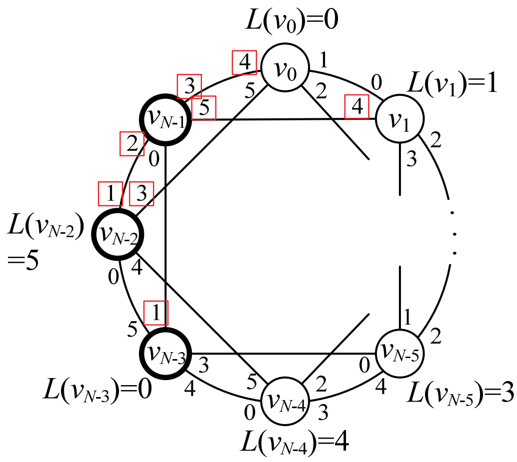
**Lemma 5.** $\chi_i(CR(N, 2)) \leq 6$ for $N \bmod 5 \in \{2, 3, 4\}$.

**Proof.** Since Algorithm IC2B call the same Procedure A with Algorithm IC2A, there exist a proper coloring on all incidences in $I_f(v_i)$ for $i = 2, 3, 4, \ldots, N - 4$. We only need to check colors on incidences which incident with five vertices $v_0$, $v_1$, $v_{N\text{-}3}$, $v_{N\text{-}2}$ and $v_{N\text{-}1}$, because the last vertex $v_{N\text{-}1}$ is not partitioned by membership function $L(v)$ and $v_{N\text{-}1}$ connects to $v_0$, $v_1$, $v_{N\text{-}3}$ and $v_{N\text{-}2}$ in $CR(N, 2)$.

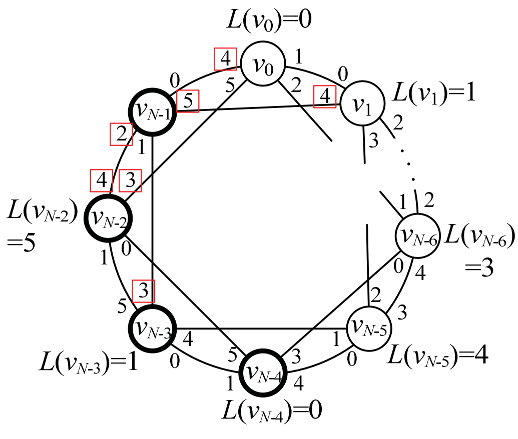
In the remaining part of Algorithm IC2B, it assigns colors on four incidences in $I_f(v_{N\text{-}1})$ and modifies the colors of conflicted incidences for three cases, i.e. $N \bmod 5 \in \{2, 3, 4\}$, respectively. It is easy to check that the coloring is proper by seeing Figure 4 for illustration. □




(a) A 6-incidence coloring on CR($N$, 2) with $N$ mod 5 = 2.




(b) A 6-incidence coloring on CR($N$, 2) with $N$ mod 5 = 3.




(c) A 6-incidence coloring on CR($N$, 2) with $N$ mod 5 = 4.

Figure 4: Three illustrations of Lemma 5. The color numbers which are surrounded with red line are

assigned in the remaining part of Algorithm IC2B.

We summarize Corollary 3 and Lemmas 4, 5 as follows.

**Theorem 6.** $\chi_i(CR(N, 2)) = \begin{cases} 5, \text{ if } N \equiv 0 \pmod 5 \\ 6, \text{ otherwise.} \end{cases}$

## 4. Incidence Coloring on *CR*(*N*, 3)

In fact, Procedure A can be applied in Algorithm IC3A to provide a 5-incidence coloring of $CR(N, 3)$ when $N \equiv 0 \pmod 5$.

**Algorithm IC3A**

Input: A $CR(N, 3)$ which $N \equiv 0 \pmod 5$
Output: A 5-incidence coloring $\psi$ for $CR(N, 3)$.
Call Procedure A with $t = N$;

Since Algorithm IC3A is similar to Algorithm IC2A, it is easy to check the following lemma.

**Lemma 7.** $\chi_i(CR(N, 3)) \leq 5$ for $N \equiv 0 \pmod 5$.

For example, Figure 5 shows a 5-incidence coloring of $CR(10, 3)$. Next, we gave Algorithm IC3B to provide a 6-incidence coloring of $CR(N, 3)$ when $N \equiv 2 \pmod 5$ and $N \geq 11$.

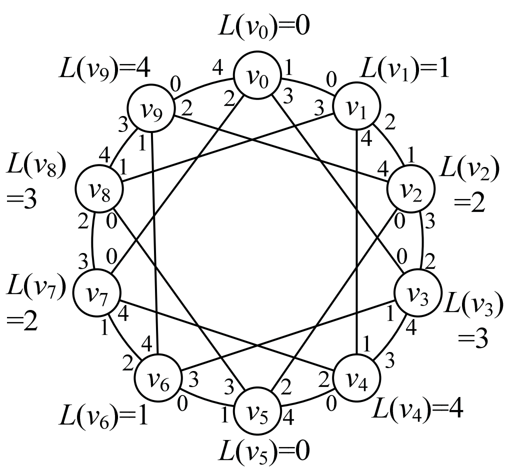



Figure 5: 5-incidence coloring of *CR*(10, 3).

**Algorithm IC3B**

Input: A $CR(N, 3)$ which $N \equiv 2 \pmod 5$ and $N \geq 11$.
Output: A 6-incidence coloring $\psi$ for $CR(N, 3)$.
Call Procedure A with $t = N - 2$;
$L(v_{N-2}) \leftarrow 5$;
$L(v_{N-1}) \leftarrow 5$;
For each incidence $j \in I_f(v_{N-2}) \cup I_f(v_{N-1})$ do
    $\psi(j) \leftarrow 5$;
$\psi((v_{N-2}, v_{N-2}v_{N-3})) \leftarrow 2$;  $\psi((v_{N-2}, v_{N-2}v_{N-5})) \leftarrow 0$;
$\psi((v_{N-2}, v_{N-2}v_{N-1})) \leftarrow 4$; $\psi((v_{N-1}, v_{N-1}v_{N-4})) \leftarrow 1$;
$\psi((v_{N-1}, v_{N-1}v_{N-2})) \leftarrow 3$;

**Lemma 8.** $CR(N, d)$ is isomorphic to $CR(N, d')$ if and only if $dd' \bmod N = 1$. [11]

**Lemma 9.** $\chi_i(CR(N, 3)) \leq 6$ for $N \equiv 2 \pmod 5$.

**Proof.** We first show that $\chi_i(CR(N, 3)) \leq 6$ when $N = 7$. By Lemma 8, $CR(7, 3) \cong CR(7, 2)$. Then, $\chi_i(CR(7, 3)) = \chi_i(CR(7, 2)) \leq 6$ by Lemma 5. When $N \geq 11$, Algorithm IC3B provide a 6-incidence coloring on $CR(N, 3)$. The coloring is proper by seeing Figure 6 for illustration. □

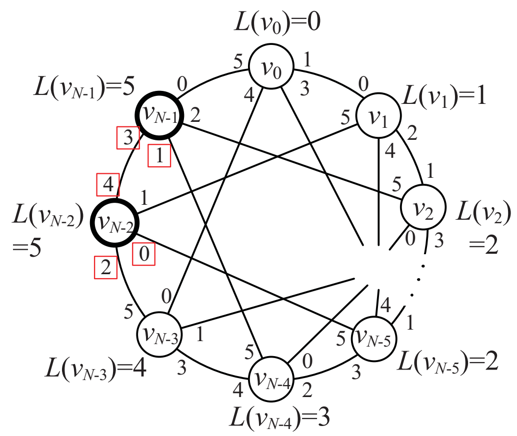



Figure 6: Illustrations of Lemma 9. The color numbers which are surrounded with red line are assigned in the remaining part of Algorithm IC3B.

We summarize Corollary 3 and Lemmas 7, 9 as follows.

**Theorem 10.** $\chi_i(CR(N, 3)) = \begin{cases} 5, \text{ if } N \equiv 0 \pmod 5 \\ 6, \text{ if } N \equiv 2 \pmod 5. \end{cases}$

## 5. Conclusion and Future Work

In this paper, we study the incidence coloring problem on chordal rings, and show some exact values of $\chi_i(CR(N, d))$. In our ongoing research, we find a interesting result that $\chi_i(CR(N, 4)) \neq \chi_i(CR(N, 3)) = \chi_i(CR(N, 2)) = 5$ with $N \equiv 0 \pmod 5$. For a direction of future research, it would be an interesting question to determine $\chi_i(CR(N, d))$ for the remaining cases.

## References


[1] B.W. Arden and H. Lee, Analysis of Chordal Ring Network, *IEEE Transactions on Computers* 30 (1981) 291-295.

[2] J.C. Bermond, F. Comellas, and D.F. Hsu, Distributed Loop Computer Networks: A Survey, *Journal of Parallel and Distributed Computing* 24 (1995) 2-10.

[3] R.A. Brualdi and J.Q. Massey, Incidence and strong edge colorings of graphs, *Discrete Mathematics* 122 (1993) 51-58.

[4] M.H. Dolama and É. Sopena, On the maximum average degree and the incidence chromatic number of a graph, *Discrete Mathematics and Theoretical Computer Science* 7 (2005) 203-216.

[5] M.H. Dolama, É. Sopena and X. Zhu, Incidence coloring of k-degenerated graphs, *Discrete Mathematics* 283 (2004) 121-128.

[6] K.F. Ding, K.J. Pai, J.M. Chang and R.H. Tsaur, Some Results of Incidence Coloring on Generalized Petersen Graphs, *International Computer Symposium* (ICS2014), Dec. 12-14, Taichung, Taiwan, (2014) 83-89.

[7] B. Guiduli, On incidence coloring and star arboricity of graphs, *Discrete Mathematics* 163 (1997) 275-278.

[8] C.I. Huang, Y.L. Wang and S.S. Chung, The incidence coloring numbers of meshes, *Computers & Mathematics with Applications* 48 (2004) 1643-1649.

[9] D. Li and M. Liu, Incidence coloring of the squares of some graphs, *Discrete Mathematics* 308 (2008) 6569-6574.

[10] X. Li and J. Tu, NP-completeness of 4-incidence colorability of semi-cubic graphs. *Discrete Mathematics* 308 (2008) 1334-1340.

[11] B. Mans, On the interval routing of chordal rings of degree 4, Tech. Rep. C/TR98-09, Macquarie University, Dept. of Computing, School of MPCE, Macquarie University, Sydney NSW2109, Australia, July 1998.

[12] M. Maydanskiy, The incidence coloring conjecture for graphs of maximum degree three, *Discrete Mathematics* 292 (2005) 131-141.

[13] X. Meng, J. Guo and B. Su, Incidence coloring of pseudo-Halin graphs, *Discrete Mathematics* 312 (2012) 3276-3282.

[14] K. Nakprasit and K. Nakprasit, Incidence colorings of the powers of cycles, *International Journal of Pure and Applied Mathematics* 76 (2012) 143-148.

[15] K.J. Pai, J.M. Chang, J.S. Yang and R.Y. Wu, Incidence coloring on hypercubes, *Theoretical Computer Science* 557 (2014) 59-65.

[16] K.J. Pai, J.M. Chang, J.S. Yang, and R.Y. Wu, On the Incidence Coloring Number of Folded Hypercubes, *Proceeding of the 18th International Computer Science and Engineering Conference* (ICSEC 2014), July 30 - August 1, Khon Kaen, Thailand (2014) 7-11.

[17] W.C. Shiu, P.C.B. Lam and D.L. Chen, On incidence coloring for some cubic graphs, *Discrete Mathematics* 252 (2002) 259-266.

[18] É. Sopena and J. Wu, The incidence chromatic number of toroidal grids, *Discussiones Mathematicae Graph Theory* 33 (2013) 315-327.

[19] P.K. Sun, Incidence coloring of regular graphs and complement graphs, *Taiwanese Journal of Mathematics* 16 (2012) 2289-2295.

[20] S.D. Wang, D.L. Chen and S.C. Pang, The incidence coloring number of Halin graphs and outerplanar graphs, *Discrete Mathematics* 256 (2002) 397-405.

[21] J. Wu, Some results on the incidence coloring number of a graph, *Discrete Mathematics* 309 (2009) 3866-3870.